Solutions Manual to Accompany:

Volume II: Intermediate Probability: A Computational Approach

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(Currently, references to chapters 1-9 refer to Fundamental Probability, while references to chapters 10-19 refer to 1-10 in Intermediate Probability. Need some Latex tricks...)

Contents

Solutions to Chapter 1: Generating Functions

Solution to Problem 1.1: Define $j_i = j_i(s) = \mathbb{E}\left[X^i e^{sX}\right]$ for $i \geq 0$ and observe that $j_i\,|_{s=0} = \mathbb{E}\left[X^i\right] = \mu_i'$, where $\mu_0' := 1$ and $dj_i/ds = j_{i+1}$. Then

$$\frac{d\mathbb{K}_X(s)}{ds} = \frac{1}{\mathbb{E}\left[e^{sX}\right]}\frac{d}{ds}\mathbb{E}\left[e^{sX}\right] = \frac{j_1}{j_0},$$

which, evaluated at $s = 0$ is $\mu_1' = \mu = \mathbb{E}[X]$. Next,

$$\frac{d^2\mathbb{K}_X(s)}{ds^2} = \frac{d}{ds}\frac{j_1}{j_0} = \frac{j_0 j_2 - j_1^2}{j_0^2},$$

which, at $s = 0$, is $\mu_2' - \mu^2 = \mu_2 = \mathbb{V}(X)$. Similarly,

$$\frac{d^3\mathbb{K}_X(s)}{ds^3} = \frac{j_0^2\left(j_0 j_3 + j_2 j_1 - 2 j_1 j_2\right) - \left(j_0 j_2 - j_1^2\right) 2 j_0 j_1}{j_0^4}, \tag{S-1.1}$$

which, at $s = 0$, simplifies to $\mu_3' - 3\mu\mu_2' + 2\mu^3$. From (4.49), this is precisely μ_3. Finally,

$$\frac{d^4\mathbb{K}_X(s)}{ds^4} = \frac{j_0^4\left(dN/ds\right) - 4 j_0^3 j_1 N}{j_0^8},$$

where

$$N = j_0^2\left(j_0 j_3 + j_2 j_1 - 2 j_1 j_2\right) - 2\left(j_0 j_2 - j_1^2\right) j_0 j_1$$

is the numerator in (S-1.1) and

$$\begin{aligned}
\frac{dN}{ds} &= j_0^2\left(j_0 j_4 + j_3 j_1 + j_2 j_2 + j_1 j_3 - 2\left(j_1 j_3 + j_2 j_2\right)\right) + \left(j_0 j_3 + j_2 j_1 - 2 j_1 j_2\right) 2 j_0 j_1 \\
&\quad -2\left(\left(j_0 j_2 - j_1^2\right)\left(j_0 j_2 + j_1 j_1\right) + j_0 j_1\left(j_0 j_3 + j_2 j_1 - 2 j_1 j_2\right)\right).
\end{aligned}$$

Setting j_0 to one and simplifying (using Maple),

$$\begin{aligned}
N\big|_{j_0=1} &= \left(j_3 + j_2 j_1 - 2 j_1 j_2\right) - 2\left(j_2 - j_1^2\right) j_1 = j_3 - 3 j_2 j_1 + 2 j_1^3 \\
\left.\frac{dN}{ds}\right|_{j_0=1} &= j_4 - 3 j_2^2 + 2 j_1^4,
\end{aligned}$$

so that

$$\begin{aligned}
\left.\frac{d^4\mathbb{K}_X(s)}{ds^4}\right|_{j_0=1} &= \left(j_4 - 3 j_2^2 + 2 j_1^4\right) - 4 j_1\left(j_3 - 3 j_2 j_1 + 2 j_1^3\right) \\
&= j_4 - 3 j_2^2 - 6 j_1^4 - 4 j_3 j_1 + 12 j_2 j_1^2.
\end{aligned}$$

Thus, using (4.49),

$$\begin{aligned}\left.\frac{d^4\mathbb{K}_X(s)}{ds^4}\right|_{s=0} &= \mu_4' - 3\mu_2'^2 - 6\mu^4 - 4\mu_3'\mu + 12\mu_2'\mu^2 \\ &= \left(\mu_4 + 4\mu_3\mu + 6\mu_2\mu^2 + \mu^4\right) - 3\left(\mu_2^2 + 2\mu^2\mu_2 + \mu^4\right) \\ &\quad -6\mu^4 - 4\left(\mu_3 + 3\mu_2\mu + \mu^3\right)\mu + 12\left(\mu_2 + \mu^2\right)\mu^2 \\ &= \mu_4 - 3\mu_2^2.\end{aligned}$$

Solution to Problem 1.2: For $Y \sim \text{Lap}(0,1)$, it is clear from symmetry that $\mathbb{E}[Y] = \mu_3 = 0$. A straightforward calculation (or see §14.1.3) shows that the mgf of Y is $\mathbb{M}_Y(s) = (1-s^2)^{-1}$, $|s| < 1$. Then, with the generous help of Maple,

$$\begin{aligned}\mathbb{M}_Y'(s) &= \frac{2s}{(1-s^2)^2}, \quad \mathbb{M}_Y''(s) = \frac{6s^2+2}{(1-s^2)^3}, \quad \mathbb{M}_Y'''(s) = 24\frac{s^3+s}{(1-s^2)^4} \\ \mathbb{M}_Y^{(4)}(s) &= \frac{240s^2+120s^4+24}{(1-s^2)^5}, \quad \mathbb{M}_Y^{(5)}(s) = \frac{720s+2400s^3+720s^5}{(1-s^2)^6} \\ \mathbb{M}_Y^{(6)}(s) &= \frac{15\,120s^2+25\,200s^4+5040s^6+720}{(1-s^2)^7}.\end{aligned}$$

Clearly, for $0 \le r \le 6$, $\left.\mathbb{M}_Y^{(r)}(s)\right|_{s=0}$ yields $\mathbb{E}[Y] = 0$ for r odd and $r!$ for r even. From the mgf, $\mathbb{K}_Y(s) = -\ln(1-s^2)$ and

$$\mathbb{K}_Y'(s) = \frac{2s}{1-s^2}, \qquad \mathbb{K}_Y''(s) = \frac{2(1+s^2)}{(1-s^2)^2},$$

from which it follows that $\mathbb{V}(Y) = 2$, recalling (10.9). Furthermore,

$$\mathbb{K}_Y^{(3)}(s) = \frac{4s(3+s^2)}{(1-s^2)^3}, \qquad \mathbb{K}_Y^{(4)}(s) = \frac{12(1+6s^2+s^4)}{(1-s^2)^4},$$

so that $\kappa_4 = 12$ and $\mu_4 = \kappa_4 + 3\mu_2^2 = 12 + 3(2)^2 = 24$. Thus, the kurtosis is $\mu_4/\mu_2^2 = 6$.

Solution to Problem 1.3: The calculations are almost the same as in Example 10.8: The mgf of Z is

$$\mathbb{M}_Z(t) = \mathbb{E}\left[e^{tZ}\right] = \mathbb{E}\left[X^t\right] = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{t+\alpha-1}e^{-\beta x}dx = \beta^{-t}\frac{\Gamma(t+\alpha)}{\Gamma(\alpha)}$$

using substitution $y = \beta x$. Then, with $d\beta^{-t}/dt = -\beta^{-t}\ln\beta$,

$$\frac{d\mathbb{M}_Z(t)}{dt} = \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\left(\beta^{-t}\Gamma(t+\alpha)\right) = \frac{1}{\Gamma(\alpha)}\left(\beta^{-t}\Gamma'(t+\alpha) - \beta^{-t}\ln(\beta)\,\Gamma(t+\alpha)\right)$$

and

$$\mathbb{E}[Z] = \left|\frac{d}{dt}\mathbb{M}_Z(t)\right|_{t=0} = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \ln\beta = \psi(\alpha) - \ln\beta.$$

The second derivative simplifies to

$$\frac{d^2\mathbb{M}_Z(t)}{dt^2} = \frac{1}{\Gamma(\alpha)}\left(\beta^{-t}\Gamma''(t+\alpha) - 2\beta^{-t}\ln(\beta)\,\Gamma'(t+\alpha) + \ln^2(\beta)\,\beta^{-t}\,\Gamma(t+\alpha)\right),$$

so that

$$\mathbb{E}\left[Z^2\right] = \left|\frac{d^2\mathbb{M}_Z(t)}{dt^2}\right|_{t=0} = \frac{\Gamma''(\alpha)}{\Gamma(\alpha)} - \frac{2\ln(\beta)\,\Gamma'(\alpha)}{\Gamma(\alpha)} + \ln^2(\beta)$$

and

$$\mathbb{V}(Z) = \frac{\Gamma''(\alpha)}{\Gamma(\alpha)} - \left(\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\right)^2.$$

For the covariance, use of the above result for $\mathbb{E}[Z]$ gives

$$\begin{aligned}\mathbb{E}[X\ln X] &= \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^\alpha e^{-\beta x}\ln x\,dx = \frac{\alpha}{\beta}\frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^\infty x^\alpha e^{-\beta x}\ln x\,dx \\ &= \frac{\alpha}{\beta}\left(\psi(\alpha+1) - \ln\beta\right),\end{aligned}$$

so that

$$\begin{aligned}\mathrm{Cov}(Z,X) &= \mathbb{E}[X\ln X] - \mathbb{E}[X]\,\mathbb{E}[\ln X] \\ &= \frac{\alpha}{\beta}\left(\psi(\alpha+1) - \ln\beta\right) - \frac{\alpha}{\beta}\left(\psi(\alpha) - \ln\beta\right) \\ &= \frac{\alpha}{\beta}\left(\psi(\alpha+1) - \psi(\alpha)\right) = \beta^{-1}.\end{aligned}$$

Solution to Problem 1.4: Let X be binomial distributed. From the binomial theorem

$$\sum_{x=0}^{n}\binom{n}{x}a^x b^{n-x} = (a+b)^n$$

we have, defining $q = 1-p$,

$$\begin{aligned}\mathbb{M}_X(t) &= \sum_{x=0}^{n} e^{tx}\binom{n}{x}p^x(1-p)^{n-x} \\ &= \sum_{x=0}^{n}\binom{n}{x}\left(pe^t\right)^x(1-p)^{n-x} \\ &= \left(pe^t+q\right)^n \qquad \text{(S-1.2)}\end{aligned}$$

and $\mathbb{K}_X(t) = n\log\left(pe^t+q\right)$. Thus,

$$\frac{d}{dt}\mathbb{K}_X(t) = \frac{npe^t}{pe^t+q} \Rightarrow \mu = np, \qquad \frac{d^2}{dt^2}\mathbb{K}_X(t) = \frac{npqe^t}{\left(pe^t+q\right)^2} \Rightarrow \mu_2 = npq,$$

$$\frac{d^3}{dt^3}\mathbb{K}_X(t) = -npq\frac{pe^{2t} - e^t + pe^t}{\left(pe^t+q\right)^3} \Rightarrow \mu_3 = np(p-1)(2p-1) = npq(q-p)$$

and

$$\frac{d^4}{dt^4}\mathbb{K}_X(t) = npq\frac{p^2e^{3t} - 4pe^{2t}q + e^tq^2}{\left(pe^t+q\right)^4} \Rightarrow \kappa_4 = npq\left(p^2 - 4pq + q^2\right)$$

so that, from (10.9),

$$\mu_4 = \kappa_4 + 3\kappa_2^2 = npq\left(6p^2 - 6p + 1\right) + 3n^2p^2q^2$$
$$= npq\,(1 - 6pq) + 3n^2p^2q^2.$$

Finally,

$$\frac{\mu_3}{\mu_2^{3/2}} = \frac{1-2p}{\sqrt{npq}} \qquad \text{and} \qquad \frac{\mu_4}{\mu_2^2} = 3 + \frac{1}{npq} - \frac{6}{n}$$

which converge to 0 and 3, respectively, as $n \to \infty$ for any p, $0 < p < 1$.

Solution to Problem 1.5:

a) Because the mass function sums to one,

$$\sum_{x=0}^{\infty} \binom{r+x-1}{x} (1-p)^x = p^{-r},$$

so that, with $q = 1 - p$,

$$\mathbb{M}_X(t) = p^r \sum_{x=0}^{\infty} \binom{r+x-1}{x} \left(qe^t\right)^x = p^r \left(1 - qe^t\right)^{-r}, \quad t < -\log(1-p)$$

and $\mathbb{K}_X(t) = r\ln p - r\ln(1 - qe^t)$.

b) From the cgf $\mathbb{K}_X(t)$,

$$\frac{d}{dt}\mathbb{K}_X(t) = \frac{rqe^t}{1-qe^t} \Rightarrow \mu = \frac{rq}{p}, \qquad \frac{d^2}{dt^2}\mathbb{K}_X(t) = \frac{rqe^t}{\left(1-qe^t\right)^2} \Rightarrow \mu_2 = \frac{rq}{p^2},$$

$$\frac{d^3}{dt^3}\mathbb{K}_X(t) = rq\frac{e^t + qe^{2t}}{\left(1-qe^t\right)^3} \Rightarrow \mu_3 = rq\frac{1+q}{p^3}$$

and

$$\frac{d^4}{dt^4}\mathbb{K}_X(t) = rq\frac{e^t + 4qe^{2t} + q^2e^{3t}}{\left(1-qe^t\right)^4} \Rightarrow \kappa_4 = rq\frac{1+4q+q^2}{p^4}$$

so that, from (10.9),

$$\mu_4 = \kappa_4 + 3\kappa_2^2 = rq\frac{6q + p^2 + 3rq}{p^4}.$$

Finally,

$$\frac{\mu_3}{\mu_2^{3/2}} = \frac{1+q}{\sqrt{rq}} \qquad \text{and} \qquad \frac{\mu_4}{\mu_2^2} = 3 + \frac{p^2}{rq} + \frac{6}{r}$$

which converge to 0 and 3, respectively, as $r \to \infty$ for any p, $0 < p < 1$.

c) The geometric mass function which counts the success is given by $f_Y(y;p) = p\,(1-p)^{y-1}\,\mathbb{I}_{\{1,2,\ldots\}}(y)$, and, using the above result for X, its mgf is just

$$\mathbb{M}_{X+1}(t) = \mathbb{E}\left[e^{(X+1)t}\right] = e^t\mathbb{E}\left[e^{Xt}\right] = pe^t\left(1 - qe^t\right)^{-1}.$$

Then

$$\mathbb{K}_{X+1}(t) = \ln \mathbb{M}_{X+1}(t) = \ln p + t - \ln\left(1 - qe^t\right)$$

and

$$\mathbb{K}'_{X+1}(t) = \frac{qe^t}{1 - qe^t} + 1$$

so that

$$\mathbb{K}'_{X+1}(0) = \frac{1-p}{p} + 1 = \frac{1}{p}$$

as in (4.33).

As location shifts do not affect the variance of any random variable, $\mathbb{V}(X+1) = \mathbb{V}(X)$. Going through the calculations just to check,

$$\mathbb{K}''_{X+1}(t) = \frac{(1 - qe^t)\, qe^t - qe^t\,(-qe^t)}{(1 - qe^t)^2}$$

so that

$$\mathbb{K}''_{X+1}(0) = \frac{(1-q)\, q - q\,(-q)}{(1-q)^2} = \frac{q}{(1-q)^2} = \frac{1-p}{p^2},$$

as in (4.51).

Now let $Y = X + r$, so that Y is negative binomial for the total number of trials required to obtain r successes. Then

$$\mathbb{M}_Y(t) = \mathbb{E}\left[e^{tY}\right] = \mathbb{E}\left[e^{t(X+r)}\right] = e^{rt}\mathbb{E}\left[e^{tX}\right] = e^{rt}\mathbb{M}_X(t)$$

and

$$\mathbb{K}_Y(t) = rt + \mathbb{K}_X(t)\,, \qquad \mathbb{K}'_Y(t) = r + \mathbb{K}'_X(t)\,, \qquad \mathbb{K}'_Y(0) = r + \frac{r\,(1-p)}{p} = \frac{r}{p}.$$

Solution to Problem 1.6: Differentiating the mgf given in (10.11) gives

$$\mathbb{M}'_X(t) = \lambda e^{-\lambda} e^t e^{\lambda e^t},$$

so that $\mathbb{E}[X] = \mathbb{M}'_X(0) = \lambda$. Likewise, direct calculation shows $\mathbb{M}''_X(0) = \lambda^2 + \lambda$ so that $\mathbb{V}(X) = \lambda^2 + \lambda - (\mathbb{E}[X])^2 = \lambda$.

Solution to Problem 1.7:

a) The mgf of Y is

$$\mathbb{M}_Y(t) = \mathbb{E}\left\{\exp\left(t\sum_{i=1}^{k+1} X_i\right)\right\} = \prod_{i=1}^{k+1} \mathbb{M}_{X_i}(t) = \prod_{i=1}^{k+1} \exp\left\{\lambda_i\left(e^t - 1\right)\right\}$$
$$= \exp\left\{\sum_{i=1}^{k+1} \lambda_i\left(e^t - 1\right)\right\}$$

so that $Y \sim \text{Pois}(\lambda)$, where $\lambda = \sum_{i=1}^{k+1} \lambda_i$.

b) To simplify notation, define $\mathbf{X}_j$ to denote vector $X_1, \ldots, X_j$, $j = k, k+1$. As $X_{k+1} = n - \sum_{i=1}^{k} X_i$,

$$f_{\mathbf{X}_k,Y}(\mathbf{x}_k, y) = \Pr(\mathbf{X}_k = \mathbf{x}_k, Y = y) = \Pr(\mathbf{X}_k = \mathbf{x}_{k+1}) = f_{\mathbf{X}_{k+1}}(\mathbf{x}_{k+1}),$$

so that

$$f_{\mathbf{X}_k|Y=n}(\mathbf{x}_k \mid y = n) = \frac{f_{\mathbf{X}_k,Y}(\mathbf{x}_k, y)}{f_Y(y)} = \frac{f_{\mathbf{X}_{k+1}}(\mathbf{x}_{k+1})}{f_Y(y)}.$$

From the independence of the X_i,

$$f_{\mathbf{X}_{k+1}}(\mathbf{x}_{k+1}) = \prod_{i=1}^{k+1} f_{X_i}(x_i) = \prod_{i=1}^{k+1} \frac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i!},$$

so that $f_{\mathbf{X}_k|Y=n}(\mathbf{x}_k \mid y = n)$ is given by

$$\frac{\dfrac{e^{-\sum_{i=1}^{k+1}\lambda_i}\prod_{i=1}^{k+1}\lambda_i^{x_i}}{\prod_{i=1}^{k+1} x_i!}}{\dfrac{e^{-\lambda}\lambda^n}{n!}} = \frac{n!\prod_{i=1}^{k+1}\lambda_i^{x_i}}{\prod_{i=1}^{k+1} x_i!\prod_{i=1}^{k+1}\lambda^{x_i}} = \frac{n!\prod_{i=1}^{k+1}\left(\frac{\lambda_i}{\lambda}\right)^{x_i}}{\prod_{i=1}^{k+1} x_i!},$$

which is the multinomial distribution with parameters $\left(n, \frac{\lambda_1}{\lambda}, \frac{\lambda_2}{\lambda}, \ldots, \frac{\lambda_k}{\lambda}\right)$. As $\lambda_i > 0$ and all $\lambda_i < \lambda$ for $i \geq 2$, it follows that $0 < \lambda_i/\lambda < 1$.

Solution to Problem 1.8:

a) Directly,

$$\mathbb{E}\left[Z^{-c}\right] = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1-c}e^{-\beta x}dx = \frac{\beta^\alpha}{\Gamma(\alpha)}\Gamma(\alpha-c)\,\beta^{-(\alpha-c)} = \beta^c\frac{\Gamma(\alpha-c)}{\Gamma(\alpha)}.$$

From (10.99) with $\mathbb{M}_Z(t) = \left(\frac{\beta}{\beta-t}\right)^\alpha$, the substitution $u = t/(\beta+t)$, $t = \beta u/(1-u)$, $dt = \beta(1-u)^{-2}du$ yields

$$\begin{aligned}
\mathbb{E}\left[Z^{-c}\right] &= \frac{1}{\Gamma(c)}\int_0^\infty \frac{t^{c-1}\beta^\alpha dt}{(\beta+t)^\alpha} \\
&= \frac{1}{\Gamma(c)}\int_0^1 \beta^{c-1}u^{c-1}(1-u)^{-(c-1)}\beta^\alpha\left[\beta\left(1+\frac{u}{1-u}\right)\right]^{-\alpha}\beta(1-u)^{-2}du \\
&= \frac{1}{\Gamma(c)}\beta^c\int_0^1 u^{c-1}(1-u)^{\alpha-c-1}du \\
&= \frac{1}{\Gamma(c)}\beta^c B(c, \alpha-c) = \frac{\beta^c\Gamma(\alpha-c)}{\Gamma(\alpha)}
\end{aligned}$$

which exists only when $\alpha > c$.

b) The gamma equality follows directly from inspection of the gamma density function, or perform the substitution $u = tx$ and integrate. Using this and assuming we can exchange the order of integration,

$$\begin{aligned}\mathbb{E}\left[X^{-c}\right] &= \int_0^\infty x^{-c} f_X(x)\,dx = \frac{1}{\Gamma(c)}\int_0^\infty \int_{-\infty}^0 (-t)^{c-1} e^{tx} dt\, f_X(x)\,dx \\ &= \frac{1}{\Gamma(c)}\int_{-\infty}^0 \int_0^\infty (-t)^{c-1} e^{tx} f_X(x)\,dx\,dt = \frac{1}{\Gamma(c)}\int_{-\infty}^0 (-t)^{c-1}\,\mathbb{M}_X(t)\,dt \\ &= \frac{1}{\Gamma(c)}\int_0^\infty t^{c-1}\mathbb{M}_X(-t)\,dt,\end{aligned}$$

assuming that $\mathbb{M}_X(t)$ exists for $-\infty < t \leq 0$.

Solution to Problem 1.9: The moment generating function $\mathbb{M}_N(t)$ of K, where N denotes the number of matches initially in the box, is given by

$$\begin{aligned}&\sum_{k=0}^{N}\binom{2N-k}{N}e^{tk}\left(\frac{1}{2}\right)^{2N-k} \\ =\ & \binom{2N}{N}\left(\frac{1}{2}\right)^{2N} + \sum_{k=0}^{N-1}\binom{2N-k-1}{N}e^{t(k+1)}\left(\frac{1}{2}\right)^{2N-k-1} \\ =\ & \binom{2N}{N}\left(\frac{1}{2}\right)^{2N} + \sum_{k=0}^{N-1}\binom{2N-k-2}{N-1}e^{t(k+1)}\left(\frac{1}{2}\right)^{2N-k-1} \\ & +\binom{2N-k-2}{N}e^{t(k+1)}\left(\frac{1}{2}\right)^{2N-k-1} \\ =\ & \binom{2N}{N}\left(\frac{1}{2}\right)^{2N} + \frac{e^t}{2}\sum_{k=0}^{N-1}\binom{2N-k-2}{N-1}e^{tk}\left(\frac{1}{2}\right)^{2N-k-2} \\ & +\sum_{k=2}^{N}\binom{2N-k}{N}e^{t(k-1)}\left(\frac{1}{2}\right)^{2N-k+1} \\ =\ & \binom{2N}{N}\left(\frac{1}{2}\right)^{2N} + \frac{e^t}{2}\mathbb{M}_{N-1}(t) + C,\end{aligned}$$

where, with

$$\binom{2N-1}{N}\left(\frac{1}{2}\right)^{2N-1} = \binom{2N}{N}\left(\frac{1}{2}\right)^{2N},$$

$$\begin{aligned}C &= \frac{e^{-t}}{2}\left(\sum_{k=0}^{N}\binom{2N-k}{N}e^{tk}\left(\frac{1}{2}\right)^{2N-k} - \binom{2N}{N}\left(\frac{1}{2}\right)^{2N} - \binom{2N}{N}\left(\frac{1}{2}\right)^{2N}e^t\right) \\ &= \frac{e^{-t}}{2}\mathbb{M}_N(t) - \binom{2N}{N}\left(\frac{1}{2}\right)^{2N}\frac{e^{-t}+1}{2}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{M}_N(t) &= \binom{2N}{N}\left(\frac{1}{2}\right)^{2N} + \frac{e^t}{2}\mathbb{M}_{N-1}(t) + \frac{e^{-t}}{2}\mathbb{M}_N(t) - \binom{2N}{N}\left(\frac{1}{2}\right)^{2N}\frac{e^{-t}+1}{2} \\ &= \frac{e^t}{2}\mathbb{M}_{N-1}(t) + \frac{e^{-t}}{2}\mathbb{M}_N(t) + \frac{1-e^{-t}}{2}\binom{N-\frac{1}{2}}{N}\end{aligned}$$

or

$$\mathbb{M}_N(t) = \frac{e^{2t}}{2e^t-1}\mathbb{M}_{N-1}(t) + \frac{e^t-1}{2e^t-1}\binom{N-\frac{1}{2}}{N}, \quad \mathbb{M}_1(t) = \frac{e^t+1}{2}.$$

Observe that

$$\lim_{t\to 0}\mathbb{M}_N(t) = \lim_{t\to 0}\mathbb{M}_{N-1}(t), \quad \lim_{t\to 0}\mathbb{M}_1(t) = 1,$$

while

$$\begin{aligned}\mathbb{M}_N'(t) &= \frac{2e^{2t}(2e^t-1) - e^{2t}2e^t}{(2e^t-1)^2}\mathbb{M}_{N-1}(t) \\ &\quad + \frac{e^{2t}}{2e^t-1}\mathbb{M}_{N-1}'(t) + \frac{e^t(2e^t-1)-2e^t(e^t-1)}{(2e^t-1)^2}\binom{N-\frac{1}{2}}{N} \\ &= \frac{2(e^{3t}-e^{2t})}{(2e^t-1)^2}\mathbb{M}_{N-1}(t) + \frac{e^{2t}}{2e^t-1}\mathbb{M}_{N-1}'(t) + \frac{e^t}{(2e^t-1)^2}\binom{N-\frac{1}{2}}{N}.\end{aligned}$$

For example, letting $t \to 0$ yields

$$\mathbb{E}_N[K] = \mathbb{E}_{N-1}[K] + \binom{N-\frac{1}{2}}{N},$$

which agrees with the result in Example 13.

Solution to Problem 1.10: Computing $\mathbb{M}_m'(t)$ from (10.15) as

$$\frac{(1-q\mathbb{M}_{m-1}(t)\,e^t)\,p\left(e^t\mathbb{M}_{m-1}'(t) + e^t\mathbb{M}_{m-1}(t)\right) + pe^t\mathbb{M}_{m-1}(t)\,q\left(\mathbb{M}_{m-1}(t)\,e^t + \mathbb{M}_{m-1}'(t)\,e^t\right)}{(1-q\mathbb{M}_{m-1}(t)\,e^t)^2},$$

with $e^0 = 1$ and $\mathbb{M}_1(0) = \mathbb{M}_m(0) = 1$ gives

$$\mathbb{M}_m'(t)\big|_{t=0} = \frac{p^2\left(\mathbb{M}_{m-1}'(0)+1\right) + pq\left(1+\mathbb{M}_{m-1}'(0)\right)}{p^2} = \frac{1}{p}\left(\mathbb{M}_{m-1}'(0)+1\right),$$

so that

$$\mathbb{E}\left[N_m\right] = \frac{1}{p}\mathbb{E}\left[N_{m-1}\right] + \frac{1}{p},$$

as was found directly in Example 8.13.

Solution to Problem 1.11:

a) The cf is

$$\varphi_X(t) = \mathbb{E}\left[e^{itX}\right] = \int_0^1 e^{itx}dx = \frac{e^{it}-1}{it}.$$

b) For $F_X = 0.5$, it is clear that $x = 0.5$. The integrand in (10.70) is zero, seen by using (10.32), so that

$$z(t) = e^{-\frac{1}{2}it}\frac{e^{it}-1}{it} = \frac{2}{t}\sin\frac{1}{2}t$$

is real for all $t \in (0,\infty)$, and $g(t) = 0$.

c) With $z(t)$ and $g(t)$ as defined in (10.70), for $x \in (0,1)$,

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} e^{-itx} \lim_{t\to\infty} \left(\frac{e^{it}-1}{it}\right) = 0,$$

so that

$$\lim_{t\to\infty} g(t) = \frac{\lim_{t\to\infty} \operatorname{Im}(z(t))}{\lim_{t\to\infty} t} = \frac{\operatorname{Im}(\lim_{t\to\infty} z(t))}{\lim_{t\to\infty} t} = 0,$$

using the continuity of $z(\cdot)$, $\operatorname{Im}(\cdot)$ and $g(\cdot)$. For the lower limit,

$$\lim_{t\to 0} z(t) = \lim_{t\to 0} e^{-itx} \lim_{t\to 0} \left(\frac{e^{it}-1}{it}\right) = 1,$$

where the last result follows from the Taylor series

$$\frac{e^{it}-1}{it} = 1 + \frac{1}{2}it - \frac{1}{6}t^2 + O\left(t^3\right).$$

But

$$\lim_{t\to 0} g(t) = \frac{\operatorname{Im}(\lim_{t\to\infty} z(t))}{\lim_{t\to\infty} t}$$

is problematic because both numerator and denominator approach zero. Instead, from the linear part of the Taylor series expansion (easily computed using Maple),

$$\frac{z(t)}{t} = t^{-1} + \left(\frac{1}{2}i - ix\right) + \left(-\frac{1}{6} + \frac{1}{2}x - \frac{1}{2}x^2\right)t + O\left(t^2\right),$$

we have for $t \approx 0$ that

$$\lim_{t\to 0} g(t) = \frac{1}{2} - x,$$

so that the integrand in (10.70) is well–behaved at both extremes.

d) For this cf, (10.70) should be used; Figure S-1.1 plots the integrand using (10.71) for $x = 0.2$. (The same graph, but flipped over the $u = 1/2$ axis, is obtained by using (10.72) instead). The increasing oscillatory behavior as $t \to 0$ renders numeric integration difficult.

Solution to Problem 1.12:

a) To derive (10.56), substitute $s = it$ in (10.54), and recall that, if the mgf exists, then $\varphi_X(t) = \mathbb{M}_X(it)$, so that $\varphi_X(s/i) = \mathbb{M}_X(s)$.

b) By definition,

$$\bar{F}_X(x) = 1 - F_X(x) = \int_x^\infty f_X(y)\,dy = \frac{1}{2\pi i}\int_x^\infty \int_{c-i\infty}^{c+i\infty} \exp\{\mathbb{K}_X(s) - sy\}ds\,dy.$$

Exchanging the order of integration,

$$\begin{aligned}
\bar{F}_X(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_x^\infty \exp\{\mathbb{K}_X(s) - sy\}dy\,ds \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\frac{\exp\{\mathbb{K}_X(s) - sy\}}{-s}\right]_x^\infty ds \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp\{\mathbb{K}_X(s) - sx\}\,\frac{ds}{s}, \qquad \text{(S-1.3)}
\end{aligned}$$

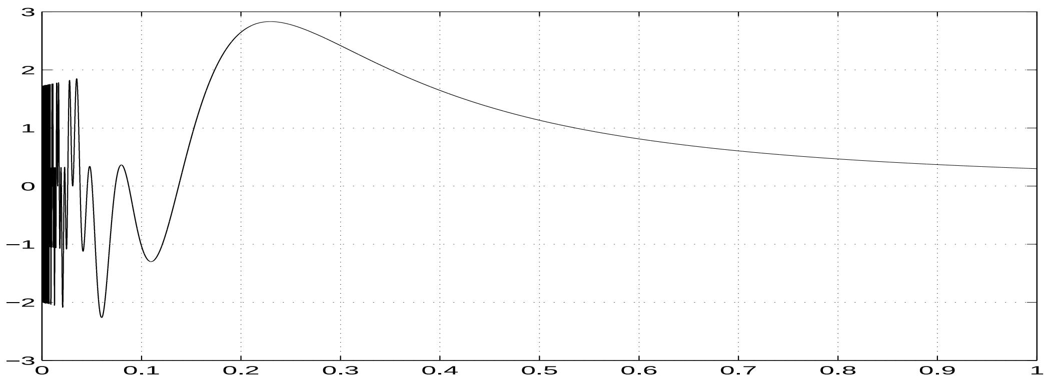

Figure S-1.1: Integrand of (10.71) for $X \sim \text{Unif}\,(0,1)$ and $x = 0.2$

where the last equality holds as long as $\text{Re}(s) = c > 0$. If instead we choose $c < 0$, we can integrate over $(-\infty, x]$ to obtain

$$\begin{aligned} F_X(x) &= \frac{1}{2\pi i}\int_{-\infty}^{x}\int_{c-i\infty}^{c+i\infty} \exp\{\mathbb{K}_X(s) - sy\}ds\;dy \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_{-\infty}^{x} \exp\{\mathbb{K}_X(s) - sy\}dy\;ds \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\frac{\exp\{\mathbb{K}_X(s) - sy\}}{-s}\right]_{-\infty}^{x} ds \\ &= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp\{\mathbb{K}_X(s) - sx\}\;\frac{ds}{s}. \end{aligned} \tag{S-1.4}$$

c) See the program in Listing S-1.1. It works for either negative or positive values of c. Running it for a grid of values between zero and one indeed produces the cdf of the uniform.

```
function F=unifcdf_inv(xvec)
c=-0.1; % any nonzero number (in the conv. strip of K)
M=1e+4; % pick large enough (should be infinity)
F=xvec*0;
for loop=1:length(xvec)
  x=xvec(loop);
  F(loop)=(c>0) -real(quadl(@integrand,c-i*M,c+i*M,[],[],x));
end

function I=integrand(t,x)
K=log(exp(t)-1)-log(t); % this is the cgf
I=exp(K-t*x)./t./(2*pi*i);
```

Program Listing S-1.1: Computes the cdf of the uniform distribution via (S-1.3) or (S-1.4), depending on the sign of c chosen in the first line of the function.

Solution to Problem 1.13: The pdf of $X \sim \text{Lap}(0,1)$ is $f_X(x) = 2^{-1}\exp(-|x|)$, so that the m.g.f. of X is

$$\begin{aligned}\mathbb{M}_X(t) &= \mathbb{E}\left[\mathrm{e}^{tX}\right] = \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{e}^{tx}\mathrm{e}^{-|x|}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{0}\mathrm{e}^{x(t+1)}\,\mathrm{d}x + \frac{1}{2}\int_{0}^{\infty}\mathrm{e}^{-x(-t+1)}dx \\ &= \frac{1}{2}\left(\frac{1}{t+1} - \frac{1}{t-1}\right) = \frac{1}{1-t^2}, \quad |t| < 1.\end{aligned}$$

As an aside, it follows that $\mathbb{K}_X(t) = -\ln(1-t^2)$ for $|t| < 1$ and

$$\begin{aligned}\mathbb{K}'_X(t) &= \frac{2t}{1-t^2}, \qquad \mathbb{E}[X] = \mathbb{K}'_X(0) = 0, \\ \mathbb{K}''_X(t) &= \frac{2(1-t^2) - 2t(-2t)}{(1-t^2)^2}, \qquad \text{Var}(X) = \mathbb{K}''_X(0) = 2.\end{aligned}$$

If we calculate the characteristic function of X by using the relation $\varphi_X(t) = \mathbb{M}_X(it)$, we get $\varphi_X(t) = 1/\left(1-(it)^2\right) = 1/(1+t^2)$.

To verify this, we compute $\varphi_X(t)$ as an integral which treats the real and imaginary parts separately. We have, from Euler's formula, and similar to Example 10.22,

$$\begin{aligned}\varphi_X(t) &= \mathbb{E}\left[e^{itX}\right] = \int_{-\infty}^{\infty} e^{itx} f_X(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{itx}e^{-|x|}dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\cos(tx)\,e^{-|x|}dx + \frac{i}{2}\int_{-\infty}^{\infty}\sin(tx)\,e^{-|x|}dx \\ &= \frac{1}{2}\int_{-\infty}^{0}\cos(tx)\,e^{x}dx + \frac{1}{2}\int_{0}^{\infty}\cos(tx)\,e^{-x}dx \\ &\quad + \frac{i}{2}\int_{-\infty}^{0}\sin(tx)\,e^{x}dx + \frac{i}{2}\int_{0}^{\infty}\sin(tx)\,e^{-x}dx.\end{aligned}$$

Then substituting $u = -x$ in the first and third integrals, this is

$$\begin{aligned}\varphi_X(t) &= \frac{1}{2}\int_{0}^{\infty}\cos(-tu)\,e^{-u}du + \frac{1}{2}\int_{0}^{\infty}\cos(tx)\,e^{-x}dx \\ &\quad + \frac{i}{2}\int_{0}^{\infty}\sin(-tu)\,e^{-u}du + \frac{i}{2}\int_{0}^{\infty}\sin(tx)\,e^{-x}dx \\ &= \int_{0}^{\infty}\cos(tu)\,e^{-u}du.\end{aligned}$$

With $s = tu$ (and note we don't have to worry about the sign of t because $\cos(-t) = \cos(t)$, so that $\varphi_X(t) = \varphi_X(-t)$) and using (10.42), for $t \neq 0$,

$$\varphi_X(t) = \frac{1}{t}\int_{0}^{\infty}\cos(s)\,e^{-s/t}ds = \frac{1}{t}\frac{1/t}{1/t^2+1} = \frac{1}{1+t^2},$$

which agrees with the use of $\mathbb{M}_X(it)$.

Furthermore,

$$\varphi_X(t) = \left(1+t^2\right)^{-1} = \left[\left(1+t^2\right)^{-1/n}\right]^n = \left[\varphi_Z(t)\right]^n$$

and $\varphi_Z(t)$ is the characteristic function of random variable Z. Example 11.10 demonstrates that Z is the difference of two iid, scale–one gamma r.v.s with shape parameter $1/n$.

Solution to Problem 1.14: Let $x = a - kh$. Consider the first case in (10.68). For $k > 1$, $x < a - h$ and

$$m(x, a, T, h) = -\frac{1}{\pi}\int_0^T \frac{\sin(h(k-1))t}{t}dt + \frac{1}{\pi}\int_0^T \frac{\sin(h(k+1))t}{t}dt = 0$$

from (10.55), as both $(k-1)$ and $(k+1)$ are > 0. Similarly, for $k < -1$, $x > a + h$ and

$$m(x, a, T, h) = \frac{1}{\pi}\int_0^T \frac{\sin(h(-k+1))t}{t}dt - \frac{1}{\pi}\int_0^T \frac{\sin(-h(k+1))t}{t}dt = 0,$$

as both $(-k+1)$ and $-(k+1)$ are > 0. For the second case, if $x = a - h$, then

$$\begin{aligned} m(x, a, T, h) &= \frac{1}{\pi}\int_0^T \frac{\sin 0}{t}dt - \frac{1}{\pi}\int_0^T \frac{\sin(-2h)t}{t}dt \\ &= 0 + \frac{1}{\pi}\int_0^T \frac{\sin(2h)t}{t}dt \\ &= \frac{1}{2}. \end{aligned}$$

Similarly, $m = 1/2$ if $x = a + h$. For the third case, let $x = a - kh$ with $|k| < 1$ so that $x \in (a - h, a + h)$. Then

$$\begin{aligned} m(x, a, T, h) &= \frac{1}{\pi}\int_0^T \frac{\sin(-h(k-1))t}{t}dt - (-1)\frac{1}{\pi}\int_0^T \frac{\sin(h(k+1))t}{t}dt \\ &= \frac{1}{2} + \frac{1}{2}, \end{aligned}$$

as $-(k-1) > 0$.

Solution to Problem 1.15: Dividing both sides of (10.65) by $2h$ and taking limits (assuming the exchange of integral and limit),

$$\lim_{h\to 0} \frac{F_X(a+h) - F_X(a-h)}{2h} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{h\to 0}\left(\frac{\sin(ht)}{ht}\right) e^{-ita}\varphi_X(t)\, dt \qquad \text{(S-1.5)}$$

or, assuming F_X has a derivative at x, denoted $f_X(x)$, and using the well known fact that

$$\lim_{h\to 0} \frac{\sin h}{h} = 1,$$

it follows that

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ita}\varphi_X(t)\, dt.$$

Notice that, as $h^{-1}\sin(h) < 1$ for $h \neq 0$,[24]

$$\left|\frac{\sin(ht)}{ht}e^{-ita}\varphi_X(t)\right| = \left|\frac{\sin(ht)}{ht}\right|\left|e^{-ita}\right|\left|\varphi_X(t)\right| < \left|\varphi_X(t)\right|,$$

so that the integral on the rhs of (S-1.5) exists if $\int_{-\infty}^{\infty}|\varphi_X(t)|\,dt < \infty$.

Solution to Problem 1.16: With $t = \pi/2$, Euler's formula implies $\text{cis}\,(\pi/2) = \cos(\pi/2) + i\sin(\pi/2) = i$ or $e^{i\pi/2} = i$. Raising both sides to the power i, $\left(e^{i\pi/2}\right)^i = e^{i^2\pi/2} = e^{-\pi/2}$, so that $i^i = e^{-\pi/2} \approx 0.2079$.

Solution to Problem 1.17:

a) We need to show that the two integrals

$$\int_0^{t_0}\left|e^{-st} - e^{-s_0t}\right|\ |f(t)|\,dt \quad \text{and} \left|\int_{t_0}^{\infty}\left(e^{-st} - e^{-s_0t}\right)\ |f(t)|\,dt\right|$$

are arbitrarily small. For the first, because f is (piecewise) continuous, it is bounded on $[0, t_0]$ by, say, M, so that

$$\int_0^{t_0}\left|e^{-st} - e^{-s_0t}\right|\ |f(t)|\,dt \leq M\int_0^{t_0}\left|e^{-st} - e^{-s_0t}\right|dt,$$

and, for $t \in [0, t_0]$, quantity $|e^{-st} - e^{-s_0t}|$ can be made arbitrarily small by choosing s_0 sufficiently close to s. For the second integral,

$$\left|\int_{t_0}^{\infty}\left(e^{-st} - e^{-s_0t}\right)\ |f(t)|\,dt\right| \leq \left|\int_{t_0}^{\infty}e^{-st}\ |f(t)|\,dt\right| + \left|\int_{t_0}^{\infty}e^{-s_0t}\ |f(t)|\,dt\right|,$$

each of which are, from (10.103), arbitrarily small for any s and s_0 in D, for t_0 sufficiently large.

b) Because $\int_0^{\infty}D_1f(x,t)\,dt$ converges uniformly for $a \leq x \leq b$, (10.104) implies that

$$G(u) = \int_0^{\infty}D_1f(u,t)\,dt \quad \text{is continuous,} \tag{S-1.6}$$

and (A.52) then implies that $\int_a^x G$ exists. Now (10.106), the fundamental theorem of calculus (FTC), and the assumption that $F(x) = \int_0^{\infty}f(x,t)\,dt$ exists imply

$$\begin{aligned}\int_a^x G &= \int_a^x\int_0^{\infty}D_1f(u,t)\,dtdu = \int_0^{\infty}\int_a^x D_1f(u,t)\,dudt \\ &= \int_0^{\infty}\left[f(x,t) - f(a,t)\right]dt = F(x) - F(a),\end{aligned}$$

i.e., from the FTC and (S-1.6),

$$\frac{d}{dx}F(x) = \frac{d}{dx}\left[F(a) + \int_a^x G\right] = G(x) = \int_0^{\infty}\frac{d}{dx}f(x,t)\,dt,$$

as was to be shown.

[24] This is clear from the Taylor series $\sin(h) = h - \frac{1}{6}h^3 + \frac{1}{120}h^5 + O\left(h^6\right)$, and h close to zero.

c) If f is of exponential order α on on $[0,\infty)$, then $|f(t)| \le Me^{-\alpha t}$ for $t \ge t_0$ for some M, α and t_0, so that $|(-1)^n t^n f(t)| = t^n |f(t)| \le t^n M e^{-\alpha t}$. From (A.44), $\lim_{t\to\infty} e^{-t} t^k = 0$, which implies that there exists an $M_1$and t_1 such that $t^n M e^{-\alpha t} \le M_1 e^{-\alpha t}$ for $t \ge t_1$, i.e., that $(-1)^n t^n f(t)$ is of exponential order α, and thus that $\mathcal{L}((-1)^n t^n f(t))$ exists and converges uniformly. As such, we can apply (10.108) to get

$$\begin{aligned}\frac{d}{ds}F(s) &= \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt \\ &= \int_0^\infty \frac{d}{ds} e^{-st} f(t)\,dt = \int_0^\infty (-t)\,e^{-st} f(t)\,dt = \mathcal{L}\{-tf(t)\},\end{aligned}$$

for $s > \alpha$.

d) Integrating both sides of $F(x) = \int_0^\infty e^{-xt} f(t)\,dt$, $x \in \mathbb{R}$, gives

$$\int_s^\infty F(x)\,dx = \lim_{u\to\infty}\int_s^u \int_0^\infty e^{-xt} f(t)\,dtdx,$$

and as $F(x)$ is uniformly convergent for $x > s > \alpha$, (10.106) is applicable, so that

$$\begin{aligned}\int_s^\infty F(x)\,dx &= \lim_{u\to\infty}\int_0^\infty f(t)\int_s^u e^{-xt}dxdt \\ &= \int_0^\infty e^{-st}\frac{f(t)}{t}dt - \lim_{u\to\infty}\int_0^\infty e^{-ut}\frac{f(t)}{t}dt.\end{aligned}$$

For the latter term, the condition that $f(t)/t$ has a finite limit as $t \to 0$ implies that $f(t)/t$ is (piecewise) continuous, and it is of exponential order from the previous exercise, so that, from (10.40),

$$\int_0^\infty e^{-ut}\frac{f(t)}{t}dt \le \frac{M}{u-\alpha}.$$

Taking the limit as $u \to \infty$ of both sides shows that it is zero. Thus,

$$\int_s^\infty F(x)\,dx = \mathcal{L}\left\{\frac{f(t)}{t}\right\}.$$

For $f(t) = \sin(t)$, (10.110), (10.42) and (A.33) give

$$\begin{aligned}\int_0^\infty e^{-st}\frac{\sin(t)}{t}dt &= \mathcal{L}\left\{\frac{\sin(t)}{t}\right\} = \int_s^\infty \frac{1}{x^2+1}dx \\ &= \frac{\pi}{2} - \arctan(s),\end{aligned}$$

which was also confirmed in (A.124).

e) Let $u = e^{-st}$ and $dv = f'(t)\,dt$, so that $du = -se^{-st}dt$ and $v = f(t)$, and integrating by parts,

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t)\,dt = uv|_0^\infty - \int_0^\infty vdu \\ &= e^{-st} f(t)\big|_0^\infty + \int_0^\infty f(t)\,se^{-st}dt \\ &= -f(0) + s\mathcal{L}\{f(t)\},\end{aligned}$$

which is (10.111).

f) First consider the hint regarding reversing the order of integration. From the definitions

$$a_{mn} = \int_n^{n+1}\int_m^{m+1} |h(x,y)|\, dxdy \quad \text{and} \quad b_{mn} = \int_n^{n+1}\int_m^{m+1} h(x,y)\, dxdy,$$

it is clear that $|b_{mn}| \leq a_{mn}$, and from the assumption

$$\int_0^{\infty}\int_0^{\infty} |h(x,y)|\, dxdy < \infty,$$

it follows that $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn} < \infty$, which implies that $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |b_{mn}| < \infty$. Thus, (A.97) justifies the exchange of sums, i.e.,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_{mn} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{mn},$$

so that

$$\int_0^{\infty}\int_0^{\infty} h(x,y)\, dxdy = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_{mn} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{mn} = \int_0^{\infty}\int_0^{\infty} h(x,y)\, dydx.$$

For the convolution result, observe that

$$\begin{aligned}
\mathcal{L}\{f\}\mathcal{L}\{g\} &= \left(\int_0^{\infty} e^{-sv} f(v)\, dv\right)\left(\int_0^{\infty} e^{-su} g(u)\, du\right) \\
&= \int_0^{\infty} e^{-sv} f(v) \left(\int_0^{\infty} e^{-su} g(u)\, du\right) dv \\
&= \int_0^{\infty} \left(\int_0^{\infty} e^{-s(v+u)} f(v)\, g(u)\, du\right) dv,
\end{aligned}$$

and setting $t = v + u$, $u = t - v$, $du = dt$ (because v is a fixed quantity in the inner integral) gives

$$\mathcal{L}\{f\}\mathcal{L}\{g\} = \int_0^{\infty} \left(\int_v^{\infty} e^{-st} f(v)\, g(t-v)\, dt\right) dv,$$

and extending g by setting $g(t) = 0$ for $t < 0$, this can be written as

$$\mathcal{L}\{f\}\mathcal{L}\{g\} = \int_0^{\infty}\int_0^{\infty} e^{-st} f(v)\, g(t-v)\, dtdv. \tag{S-1.7}$$

As both $\mathcal{L}\{f\}$ and $\mathcal{L}\{g\}$ are absolutely convergent, the equality (S-1.7) implies that the rhs is also absolutely convergent, so that the order of integration can be reversed, i.e.,

$$\begin{aligned}
\mathcal{L}\{f\}\mathcal{L}\{g\} &= \int_0^{\infty}\int_0^{\infty} e^{-st} f(v)\, g(t-v)\, dvdt \\
&= \int_0^{\infty} e^{-st} \left(\int_0^{t} f(v)\, g(t-v)\, dv\right) dt \qquad g(t-v) = 0 \text{ for } v > t \\
&= \int_0^{\infty} e^{-st} \left((f * g)(t)\right) dt = \mathcal{L}\{f * g\}.
\end{aligned}$$

Solution to Problem 1.18: With $z_T := e^{2\pi i/T}$, the n^{th} element of $\mathcal{F}(\mathbf{c}) = \mathcal{F}(\mathbf{g} * \mathbf{h})$, i.e., the forward discrete Fourier transform (10.88) for $\mathbf{c} = (c_0, \ldots, c_{T-1})$, is

$$G_n = \frac{1}{T}\sum_{t=0}^{T-1} c_t e^{-2\pi i n t/T} = \frac{1}{T}\sum_{t=0}^{T-1} c_t z_T^{nt}, \quad n = 0, \ldots, T-1.$$

Using the convolution definition (10.114) and setting $k = t - j$,

$$G_n = \frac{1}{T}\sum_{t=0}^{T-1}\left(\sum_{j=0}^{T-1} g_{t-j} h_j\right) z_T^{nj} z_T^{n(t-j)} = \frac{1}{T}\left(\sum_{j=0}^{T-1} h_j z_T^{nj}\right)\left(\sum_{k=-j}^{T-1-j} g_k z_T^{nk}\right).$$

For $-(T-1) \leq k \leq -1$, $g_k = g_{k+T}$ from (10.114), and $z_T^{nk} = z_T^{n(T+k)}$ from the results in Example 10.16), so that

$$\begin{aligned}
\sum_{k=-j}^{T-1-j} g_k z_T^{nk} &= g_{-j} z_T^{n(-j)} + \cdots + g_{-1} z_T^{n(-1)} \quad + g_0 z_T^0 + \cdots + g_{T-1-j} z_T^{n(T-1-j)} \\
&= g_{T-j} z_T^{n(T-j)} + \cdots + g_{T-1} z_T^{n(T-1)} + g_0 z_T^0 + \cdots + g_{T-1-j} z_T^{n(T-1-j)} \\
&= \sum_{w=0}^{T-1} g_w z_T^{nw},
\end{aligned}$$

i.e.,

$$G_n = \frac{1}{T}\left(\sum_{j=0}^{T-1} h_j z_T^{nj}\right)\left(\sum_{w=0}^{T-1} g_w z_T^{nw}\right) = T\mathcal{F}(\mathbf{h})\,\mathcal{F}(\mathbf{g}).$$

Using the programs in Listings 10.4, 10.5 and S-1.2, the following code verifies the convolution result:

```
T=9; g=randn(T,1)+i*randn(T,1); h=randn(T,1)+i*randn(T,1);
fgfh = dft(g) .* dft(h);
c=convolutiondiscrete(g,h); fgh = dft(c);
T*fgfh - fgh % should be zero, except for roundoff error
```

Solution to Problem 1.19: From (10.115),

$$G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t}\, dt = a\int_{-T}^{T} e^{-i\omega t}\, dt = -\frac{a}{i\omega} \cdot e^{-i\omega t}\Big|_{-T}^{T}$$

where, from (10.32),

$$e^{-i\omega t}\Big|_{-T}^{T} = e^{-i\omega T} - e^{i\omega T} = -2i\sin(\omega T),$$

so that $G(\omega) = 2a\sin(\omega T)/\omega$.

For the inverse, the hint suggests first showing that

$$\sin(a-b) + \sin(a+b) = 2\sin(a)\cos(b). \qquad \text{(S-1.8)}$$

```
function c=convolutiondiscrete(g,h)

T=length(g); c=zeros(T,1);
for t=0:(T-1)
  s=0;
  for j=0:(T-1)
    if t-j < 0, u=g(t-j+T+1); else u=g(t-j+1); end
    s=s+u*h(j+1);
  end
  c(t+1) = s;
end
```

Program Listing S-1.2: The circular convolution of two T–length vectors

From (A.23a) and the fact that $\sin(-b) = -\sin(b)$, we get

$$\begin{aligned} \sin(a+b) &= \sin a \cos b + \cos a \sin b, \\ \sin(a-b) &= \sin a \cos b - \cos a \sin b, \end{aligned}$$

and adding yields (S-1.8). So, from (10.115), the results in Example A.21, and identity (S-1.8),

$$\begin{aligned} h(t) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t} d\omega = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{1}{\omega}\sin(\omega T)\, e^{i\omega t} d\omega \\ &= \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{1}{\omega}\sin(\omega T)\left[\cos(\omega t) + i\sin(\omega t)\right] d\omega \\ &= \frac{2a}{\pi}\lim_{u\to\infty}\int_{0}^{u} \frac{1}{\omega}\sin(\omega T)\cos(\omega t)\ d\omega \\ &= \frac{a}{\pi}\lim_{u\to\infty}\int_{0}^{u} \frac{1}{\omega}\left[\sin((T-t)\,\omega) + \sin((T+t)\,\omega)\right]\ d\omega. \end{aligned}$$

Note that

$$|t| \le T \iff -T \le t \le T \iff 0 \le T - t \le 2T.$$

So, if $|t| \le T$, then $K_1 := T - t \ge 0$, $K_2 := T + t \ge 0$, and using the result mentioned at the end of Example A.30, namely

$$\int_0^{\infty} \frac{\sin(ax)}{x}\, dx = \operatorname{sgn}(a)\frac{\pi}{2},$$

(see also Abramowitz and Stegun, 1972, Eq. 4.3.142), and the fact that $\omega > 0$,

$$v(K_1) := \lim_{u\to\infty}\int_0^u \frac{1}{\omega}\sin(K_1\omega)\ d\omega = \frac{\pi}{2}, \quad v(K_2) = \lim_{u\to\infty}\int_0^u \frac{1}{\omega}\sin(K_2\omega)\ d\omega = \frac{\pi}{2},$$

where v is the so-defined function which we re-use below. Thus, if $|t| \le T$, then

$$h(t) = \frac{a}{\pi}\left(\frac{\pi}{2} + \frac{\pi}{2}\right)\mathbb{I}(|t| \le T) = a\mathbb{I}(|t| \le T) = g(t),$$

as was to be shown. Now consider the case when $|t| > T$, which implies that either $t > T$ or $t < -T$. If $t > T$, then $K_1 = T - t < 0$, so $v(K_1) = -\pi/2$, and $K_2 = T + t > 0$, so $v(K_2) = \pi/2$, and $h(t) = 0$. Likewise, if $t < -T$, then $K_1 = T - t > 0$, so $v(K_1) = \pi/2$, and $K_2 = T + t < 0$, so $v(K_2) = -\pi/2$.

Thus, if $|t| > T$, then $h(t) = 0$, and we have shown that, for all $t \in \mathbb{R}$, $h(t) = g(t)$, i.e., that the inverse Fourier transform of $G(\omega)$ indeed yields $g(t)$.

Solution to Problem 1.20:

a) Let $x \in \mathbb{R}$. Then $f(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x) \geq 0$, as $f_n(x) \geq 0$. Moreover,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} 2^{-n} f_n(x)dx = \sum_{n=1}^{\infty} 2^{-n} \int_{-\infty}^{\infty} f_n(x)dx = \sum_{n=1}^{\infty} 2^{-n} 1 = 1.$$

So f is a proper density function.

b) The density f_n is given by

$$f_n(x) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left\{-\frac{(x-\mu_n)^2}{2\sigma_n^2}\right\}, \quad x \in \mathbb{R}.$$

The maximum is attained at μ_n, where the exponential term is unity, so the maximum is $1/(\sqrt{2\pi}\sigma_n)$.

c) The maximum of $2^{-n} f_n$ is

$$2^{-n} \frac{1}{\sqrt{2\pi}\sigma_n} = \frac{1}{\sqrt{2\pi}\sigma_n 2^n}.$$

This will be an increasing sequence if σ_n grows faster than 2^n; take for example $\sigma_n := 3^n$. As the maximum of $2^{-n} f_n$ is attained at μ_n, the choice $\mu_n := n$ guarantees that the maximum of f_n will become larger and larger as n increases.

d) The following Matlab program returns the density of our f:

```
function f=NormalSum(tvec,N)
f=zeros(length(tvec),1);
n=1:N; mu=n; sigma=3.^(-n);
for tloop=1:length(tvec)
  t=tvec(tloop);
  f(tloop)=sum(2.^(-n) .* normpdf(t,mu,sigma));
end
```

Invoke it with

```
tv=-1:0.02:10; f=NormalSum(tv,100); plot(tv,f)
```

Solution to Problem 1.21: Matlab code to evaluate (10.96) is given in Listing S-1.4. It requires the cf of the bivariate normal distribution, given in Listing S-1.3. For example, running the following code:

```
mu1=1; mu2=2; s1squared=3; s2squared=4; x=1.5; y=2.5; rho=0.5;
shephardd2(x,y,@bvnjcf,mu1,mu2,s1squared,s2squared,rho)
%% We can standardize the bivariate normal as follows:
A=(x-mu1)/sqrt(s1squared); B=(y-mu2)/sqrt(s2squared);
shephardd2(A,B,@bvnjcf,0,0,1,1,rho)
```

will produce 0.445912, which agrees with the result shown in §12.4.

```
function phi=bvnjcf(t1,t2,m1,m2,ss1,ss2,r)
phi = exp(i*(t1*m1+t2*m2) - 0.5*(ss1*t1^2 + ss2*t2^2 + ...
      2*r*sqrt(ss1)*sqrt(ss2)*t1*t2));
```

Program Listing S-1.3: The cf of the bivariate normal distribution (10.117)

Solution to Problem 1.22:

a) Defining $X_3 := X_1 - rX_2$, we can write

$$\begin{aligned}\Pr\left(\frac{X_1}{X_2} < r\right) &= \Pr\left(\frac{X_1}{X_2} < r \wedge X_2 < 0\right) + \Pr\left(\frac{X_1}{X_2} < r \wedge X_2 > 0\right) \\ &= \Pr\left(X_1 > rX_2 \wedge X_2 < 0\right) + \Pr\left(X_1 < rX_2 \wedge X_2 > 0\right) \\ &= \Pr\left(X_3 > 0 \wedge X_2 < 0\right) + \Pr\left(X_3 < 0 \wedge X_2 > 0\right).\end{aligned}$$

A bit of thought (or a Venn diagram) yields (10.119). Next, from (10.96),

$$F_{X_3,X_2}(0,0) = -\frac{1}{4} + \frac{1}{2}\big[F_{X_3}(0) + F_{X_2}(0)\big] - \frac{1}{2\pi^2}\int_0^\infty \int_0^\infty g(0,0,t_1,t_2)dt_1\, dt_2,$$

where

$$g(0,0,t_1,t_2) = \operatorname{Re}\left[\frac{\varphi_{X_3,X_2}(t_1,t_2)}{t_1 t_2} - \frac{\varphi_{X_3,X_2}(t_1,-t_2)}{t_1 t_2}\right].$$

Rearranging, we find that

$$F_{X_2}(0) + F_{X_3}(0) - 2F_{X_2,X_3}(0,0) = \frac{1}{2} + \frac{1}{\pi^2}\int_0^\infty \int_0^\infty g(0,0,t_1,t_2)dt_1\, dt_2,$$

the left hand side of which has the same form as the right hand side of (10.119). Hence,

$$\Pr(R < r) = \frac{1}{2} + \frac{1}{\pi^2}\int_0^\infty \int_0^\infty \operatorname{Re}\left[\frac{\varphi_{X_3,X_2}(t_1,t_2) - \varphi_{X_3,X_2}(t_1,-t_2)}{t_1 t_2}\right] dt_1\, dt_2,$$

or, as $\varphi_{X_3,X_2}(t_1,t_2) = \varphi_{X_1,X_2}(t_1, t_2 - rt_1)$,

$$F_R(r) = \frac{1}{2}+\frac{1}{\pi^2}\int_0^\infty \int_0^\infty \operatorname{Re}\left[\frac{\varphi_{X_1,X_2}(t_1,t_2 - rt_1) - \varphi_{X_1,X_2}(t_1,-t_2 - rt_1)}{t_1 t_2}\right] dt_1\, dt_2. \tag{S-1.9}$$

```
function F=shephardd2(x1vec,x2vec,jointcf,varargin);
lower=1e-8; upper=40; % integration range; change as needed.
F=zeros(length(x1vec),length(x2vec));
for x1loop=1:length(x1vec)
  x1=x1vec(x1loop);
  for x2loop=1:length(x2vec)
    x2=x2vec(x2loop);
    I1=-2/pi^2*dblquad(@integrand,lower,upper,lower,upper, ...
                      [],@quadl,x1,x2,jointcf,varargin{:});
    I2=-1/pi*quad(@integrand1,lower,upper,[],[],x1,jointcf,varargin{:});
    I3=-1/pi*quad(@integrand2,lower,upper,[],[],x2,jointcf,varargin{:});
    F(x1loop,x2loop)=(I1+I2+I3+1)/4;
  end
end
function I=integrand(t1vec,t2,x1,x2,jointcf,varargin)
I=zeros(size(t1vec));
for loop=1:length(t1vec)
  t1=t1vec(loop);
  K1 = feval(jointcf,t1, t2,varargin{:})*exp(-i*x1*t1-i*x2*t2)/t1/t2;
  K2 = feval(jointcf,t1,-t2,varargin{:})*exp(-i*x1*t1+i*x2*t2)/t1/t2;
  I(loop)=real(K1-K2);
end
function I=integrand1(t1vec,x1,jointcf,varargin)
I=zeros(size(t1vec));
for loop=1:length(t1vec)
  t1=t1vec(loop);
  I(loop)=real(feval(jointcf,t1,0,varargin{:})*exp(-i*x1*t1)/i/t1...
             -feval(jointcf,-t1,0,varargin{:})*exp(i*x1*t1)/t1/i);
end
function I=integrand2(t2vec,x2,jointcf,varargin)
I=zeros(size(t2vec));
for loop=1:length(t2vec)
  t2=t2vec(loop);
  I(loop)=real(feval(jointcf,0,t2,varargin{:})*exp(-i*x2*t2)/t2/i ...
             -feval(jointcf,0,-t2,varargin{:})*exp(i*x2*t2)/t2/i);
end
```

Program Listing S-1.4: Computes the bivariate inversion formula (10.96) at the values in $x1$, $x2$, based on the cf specified by the function passed as `jointcf`

For the density, differentiating with respect to r yields

$$\begin{aligned}
f_R(r) &= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty \operatorname{Re}\left[\frac{\varphi_2(t_1,-t_2-rt_1)-\varphi_2(t_1,t_2-rt_1)}{t_2}\right]dt_1\,dt_2 \\
&= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty \operatorname{Re}\left[\frac{\varphi_2(t_1,-t_2-rt_1)+\varphi_2(-t_1,-t_2+rt_1)}{t_2}\right]dt_1\,dt_2 \\
&= \frac{1}{\pi^2}\int_0^\infty\int_{-\infty}^\infty \operatorname{Re}\left[\frac{\varphi_2(t_1,-t_2-rt_1)}{t_2}\right]dt_1\,dt_2,
\end{aligned}$$

where the first equality follows because $\varphi_{X_1,X_2}(t_1,t_2) = \bar{\varphi}_{X_1,X_2}(-t_1,-t_2)$, whence

$$\operatorname{Re}\frac{\partial}{\partial t_2}\varphi_{X_1,X_2}(t_1,t_2) = \operatorname{Re}\frac{\partial}{\partial t_2}\varphi_{X_1,X_2}(-t_1,-t_2) = -\operatorname{Re}\varphi_2(-t_1,-t_2).$$

b) Matlab code to evaluate (10.118), (10.62) and (S-1.9) is given in Listing (S-1.5). The partial derivative of the joint cf with respect to t_2 is

$$\varphi_2(t_1,t_2) = \frac{\partial\varphi(t_1,t_2)}{\partial t_2} = \left[i\mu_2 - \sigma_2^2 t_2 - \rho\sigma_1\sigma_2 t_1\right]\varphi(t_1,t_2)$$

and can be evaluated by means of Program S-1.6. The error incurred when using (10.62) over (10.118) is shown in Figure S-1.2. Clearly, the approximation improves as $F_{X_2}(0)$ becomes smaller. The deviation of the pdf computed from (10.118) from the true pdf is due to numerical error.

Solution to Problem 1.23: With $a := z_n^k \neq 1$, $\sum_{j=0}^{n-1} a^j = (1-a^n)/(1-a)$ and, for integer k, $a^n = \left\{[\exp(2\pi i/n)]^k\right\}^n = e^{2i\pi k} = 1$, while for $k \in \{1,\ldots,n-1\}$, $a \neq 1$, so the ratio $(1-a^n)/(1-a)$ is zero.

For the graphic, see the program in Listing S-1.7 and use the command line

```
n=6; for k=0:(n-1), subplot(2,3,k+1), spokesnk(n,k), end
```

to generate Figure S-1.3.

```
function [pdf,gearypdf,cdf]= ratioinv(rvec,jointcf,jointcfp,varargin);
% Compute pdf and cdf of the ratio x1/x2, where x1,x2 have joint cf
% 'jointcf', at the values in rvec. jointcfp = d jointcf / dt2

lower=1e-10; upper=50; % integration range; change as you see fit.
F=zeros(size(rvec));
for rloop=1:length(rvec),  r=rvec(rloop);
    pdf(rloop)=1/pi^2*dblquad(@integrand,-upper,upper,lower,upper,...
      [],@quadl,r,jointcfp,varargin{:});
    if nargout>=2
      gearypdf(rloop)=1/2/pi*quad(@gearyintegrand,-upper,upper,...
          [],[],r,jointcfp,varargin{:});
    end
    if nargout==3
      cdf(rloop)=.5+1/pi^2*dblquad(@Fintegrand,lower,upper,lower, ...
        upper,[],@quadl,r,jointcf,varargin{:})
    end
end

function I=integrand(t1vec,t2,r,jointcfp,varargin)
I=zeros(size(t1vec));
for loop=1:length(t1vec),   t1=t1vec(loop);
    I(loop)=real((feval(jointcfp,t1,-t2-r*t1,varargin{:})/t2));
end

function I=Fintegrand(t1vec,t2,r,jointcf,varargin)
I=zeros(size(t1vec));
for loop=1:length(t1vec),   t1=t1vec(loop);
    I(loop)=real((feval(jointcf,t1,t2-r*t1,varargin{:})...
        -feval(jointcf,t1,-t2-r*t1,varargin{:}))/t1/t2);
end

function I=gearyintegrand(t1vec,r,jointcfp,varargin)
I=zeros(size(t1vec));
for loop=1:length(t1vec),   t1=t1vec(loop);
    I(loop)=real((feval(jointcfp,t1,-r*t1,varargin{:})/i));
end
```

Program Listing S-1.5: Matlab code for the inversion formula for ratios.

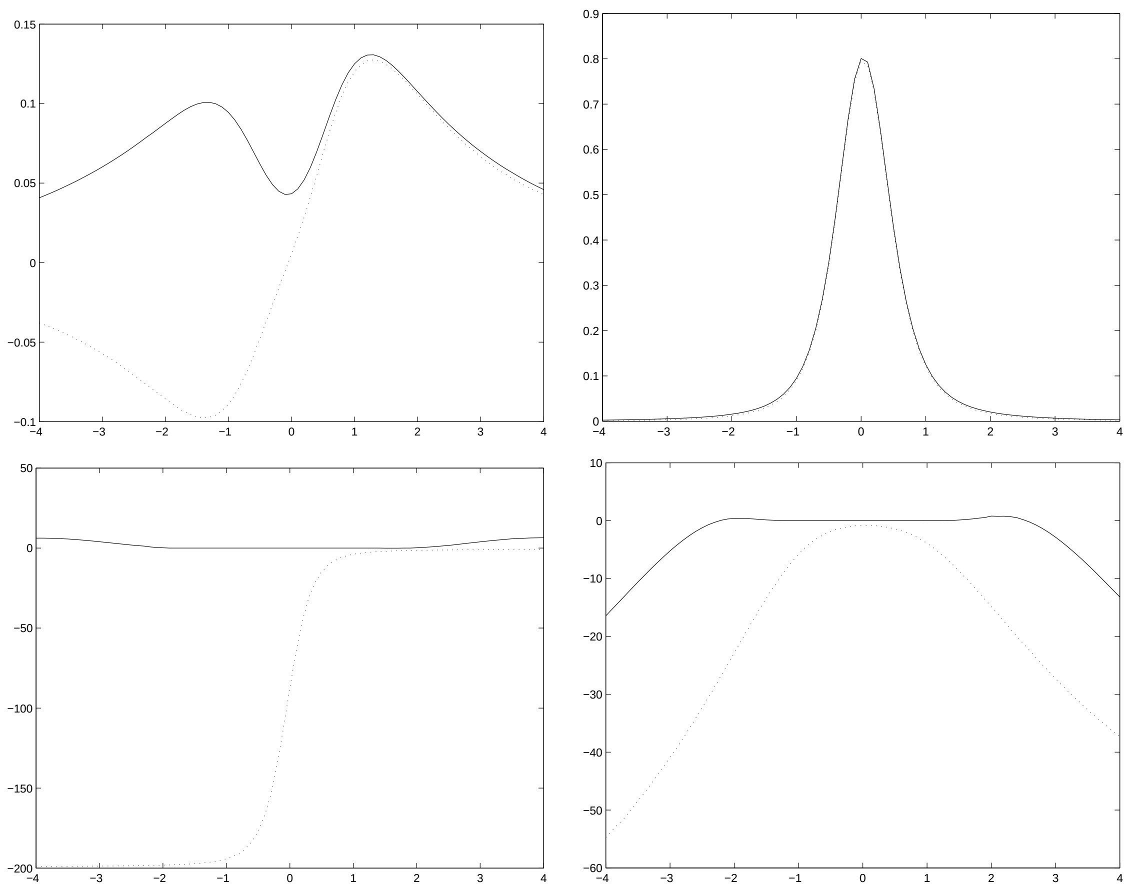

Figure S-1.2: The pdf (top panels) and its relative percentage error 100 (Approx − Exact) /Exact (bottom panels) of $R_1 = X_1/X_2$ (left) and $R_2 = X_2/X_1$ (right), $\mu_1 = 2$, $\mu_2 = 0.1$, $\sigma_1 = \sigma_2 = 1$, $\rho = 0$, when computed via (10.118) (solid) and (10.62) (dotted). To compute the numbers for the left panel, run `[pdf,pdfgeary]=ratioinv(-4:.1:4,@bvnjcf,@bvnjcfp,2,.1,1,1,0)`

```
function phi=bvnjcfp(t1,t2,m1,m2,ss1,ss2,r)
% partial deriv. of the biv. normal cf. w.r.t. t2
phi=(i*m2-ss2*t2-r*sqrt(ss1)*sqrt(ss2)*t1)*exp(i*(t1*m1+t2*m2) - ...
    .5*(ss1*t1^2+ss2*t2^2+2*r*sqrt(ss1)*sqrt(ss2)*t1*t2));
```

Program Listing S-1.6: Matlab code for the partial derivative of the normal cf (used with Program S-1.5).

```
function spokesnk(n,k)
theta = linspace(0,2*pi,50); x = cos(theta); y = sin(theta);
h=plot(x,y,'r-'); axis('equal'), axis off
z=exp(2*pi*i/n); z=z^k;
for j=0:(n-1)
  x=z^j; c=real(x); s=imag(x); rr=(rand-0.5)/8;
  h=line([0+rr c+rr],[0+rr s+rr],'linewidth',1);
end
text(-0.5,1.2,['n=',int2str(n),'   k=',int2str(k)],'fontsize',18)
```

Program Listing S-1.7: Used for creating Figure S-1.3

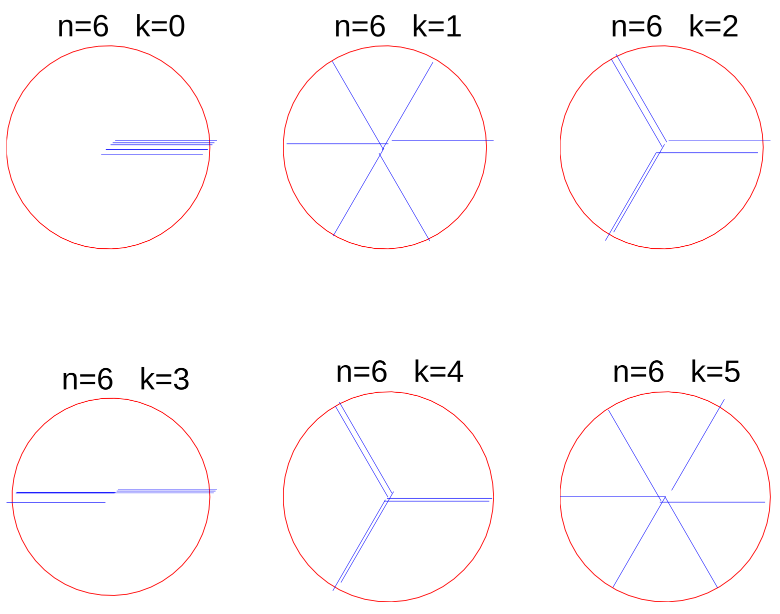

Figure S-1.3: Each plot shows $(z_n^k)^j$, $j = 0, \ldots, n-1$, for $n = 6$, for $k = 0, \ldots, n-1$. The $(z_n^k)^j$ are "perturbed" by a small random amount so that duplicates can be seen.

Solutions to Chapter 2: Sums and Other Functions of Several Random Variables

Solution to Problem 2.1:

a) Recalling Example 8.18, R is Cauchy distributed.

b) S follows an F distribution with one numerator and one denominator degree of freedom.

c) Random variable E is a location scale transformation of a normal random variable, i.e., $E \sim \mathrm{N}(c, s^2 n)$.

d) As $\sum_{i=1}^{n} X_i^2 \sim \chi_n^2$, Z is a location c, scale s transformed χ_n^2 random variable.

e) The ratio X_1/X_2 is Cauchy, and as the X_i are symmetric about zero, allowing only X_1 to influence the sign of the ratio does not change its distribution, i.e., it is also Cauchy.

f) Let C be a Cauchy random variable with density

$$f_C(c) = \frac{1}{\pi} \frac{1}{1+c^2}.$$

A univariate transformation similar to that in the second part of Example 7.13 shows that $Y = |C|$ has density

$$f_Y(y) = 2 f_C(y) \mathbb{I}_{(0,\infty)}(y) = \frac{2}{\pi} \frac{1}{1+c^2} \mathbb{I}_{(0,\infty)}(y).$$

g) The numerator is χ_1^2, independent of the denominator, which is χ_{n-1}^2. Recalling how an F r.v. is constructed, it follows that $(n-1)Q \sim \mathrm{F}(1, n-1)$ so that Q is a scaled F.

h) G is $F(m, n-m)$ and B can be written as $U/(U+D)$, where U and D are independent χ^2, which is a special case of gamma random variables (with the same scale parameter). Thus, B is a beta random variable. See (9.10) for the exact derivation, showing that $B \sim \mathrm{Beta}(m/2, (n-m/2))$.

Solution to Problem 2.2:

a) As $\bar{X} \sim \mathrm{N}(0, n^{-1})$, $\sqrt{n}\bar{X} \sim \mathrm{N}(0,1)$ so that $n\bar{X}^2 \sim \chi^2_1$. Recalling (12.29), $(n-1) S_n^2 \sim \chi^2_{n-1}$.

b) From the discussion in §12.7, $\bar{X}$ and S^2 are independent, so that

$$\frac{n\bar{X}_n^2 \,/\, 1}{(n-1) S_n^2 \,/\, (n-1)} = \frac{n\bar{X}^2}{S_n^2} = A \sim \mathrm{F}(1, n-1).$$

c) From (9.10), $V \sim \mathrm{Beta}\left(\frac{1}{2}, \frac{n-1}{2}\right)$.

d) To show (11.38), use the fact that that

$$(n-1) S^2 = \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2 = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2$$

to get

$$(n-1)^{-1} A = (n-1)^{-1} n\bar{X}^2 / S^2 = \frac{n\bar{X}^2}{\sum_{i=1}^n X_i^2 - n\bar{X}^2}$$

so that

$$\begin{aligned} V &= \frac{(n-1)^{-1} A}{1 + (n-1)^{-1} A} = \frac{\dfrac{n\bar{X}^2}{\sum_{i=1}^n X_i^2 - n\bar{X}^2}}{\dfrac{\sum_{i=1}^n X_i^2 - n\bar{X}^2}{\sum_{i=1}^n X_i^2 - n\bar{X}^2} + \dfrac{n\bar{X}^2}{\sum_{i=1}^n X_i^2 - n\bar{X}^2}} \\ &= \frac{n\bar{X}^2}{\sum_{i=1}^n X_i^2} \\ &= \frac{\left(\sum_{i=1}^n X_i\right)^2}{n \sum_{i=1}^n X_i^2}. \end{aligned}$$

e) H is just a scaled beta. In particular,

$$f_H(h) = \frac{1}{nB\left(\frac{1}{2}, \frac{n-1}{2}\right)} \left(\frac{h}{n}\right)^{-1/2} \left(1 - \frac{h}{n}\right)^{(n-3)/2} \mathbb{I}_{(0,n)}(h).$$

f) Figure S-2.1 shows the plot. Indeed, the kernel density (based on 10,000 replications) and the true density are virtually identical. Using Cauchy data, the density has a peak at 1, and then less mass in the right tail, while, as expected, the density corresponding to Student's t data with 2 degrees of freedom lies between the Cauchy– and normal–based densities.

Solution to Problem 2.3:

a) Let

$$Z = \frac{S - \sum_{i=1}^n a_i}{\sum_{i=1}^n b_i - \sum_{i=1}^n a_i} \quad \text{and} \quad z = \frac{s - \sum_{i=1}^n a_i}{\sum_{i=1}^n b_i - \sum_{i=1}^n a_i}$$

so that $\Pr(S \le s) = \Pr(Z \le z)$. Then

$$\Pr(Z \le z) = \Pr(1 - Z \ge 1 - z)$$

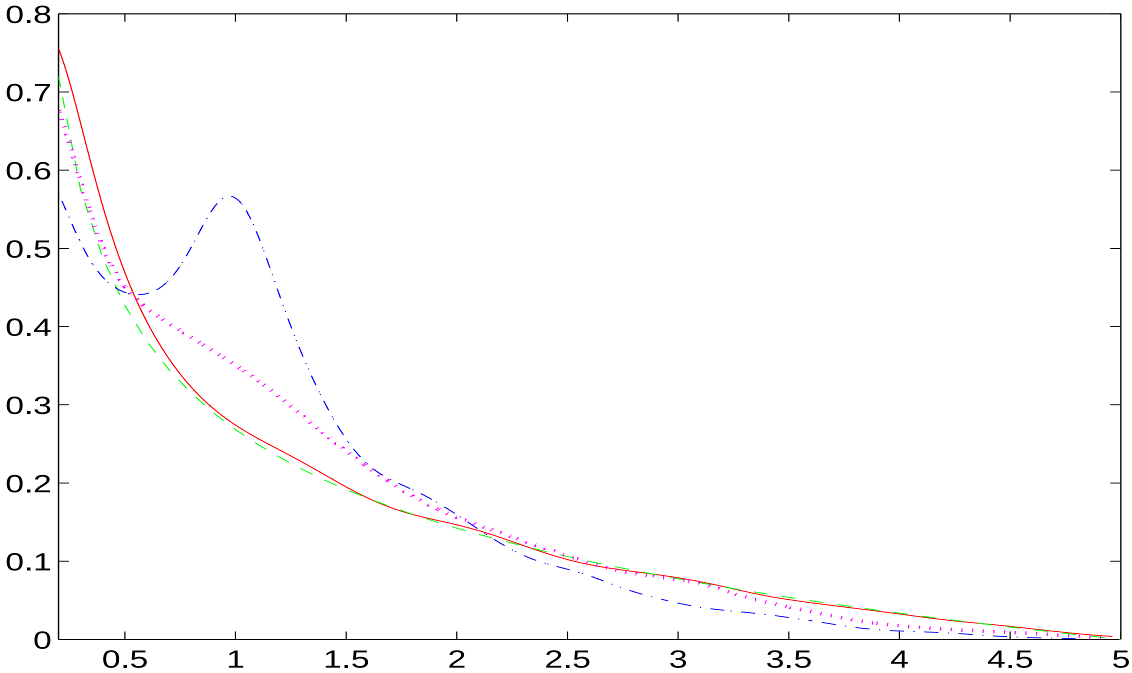

Figure S-2.1: Solid line is kernel density of 10,000 replications of H, and dashed is the pdf f_H. The dash-dot line is the kernel density of H but using Cauchy r.v.s, and the dotted line is the kernel density of H but using Student's t r.v.s with 2 degrees of freedom.

and $1 - Z$ can be written as

$$\begin{aligned} 1 - Z &= \frac{\sum_{i=1}^n b_i - S}{\sum_{i=1}^n b_i - \sum_{i=1}^n a_i} = \frac{\sum_{i=1}^n b_i - S + \sum_{i=1}^n a_i - \sum_{i=1}^n a_i}{\sum_{i=1}^n b_i - \sum_{i=1}^n a_i} \\ &= \frac{\sum_{i=1}^n (b_i - X_i + a_i) - \sum_{i=1}^n a_i}{\sum_{i=1}^n b_i - \sum_{i=1}^n a_i}. \end{aligned}$$

But, because X_i is uniform, $b_i - X_i + a_i$ has the same distribution as X_i, implying that $\sum_{i=1}^n (b_i - X_i + a_i)$ has the same distribution as $\sum_{i=1}^n X_i$ and, finally, that $1 - Z$ has the same distribution as Z. Thus

$$\Pr(Z \le z) = \Pr(1 - Z \ge 1 - z) = \Pr(Z \ge 1 - z),$$

i.e., Z is symmetric. As S is a linear transformation of Z, it too must be symmetric.

b) Listings S-2.1 and S-2.2 show possible programs. The pdf and cdf for the suggested parameter values are shown in Figure S-2.2.

Solution to Problem 2.4: For $n = 1$, a simple scale transformation shows that the density of $S = X/\sqrt{3}$ is

$$f_S(s) = \sqrt{3} f_X(x) = \sqrt{3}\, \frac{3^{-1/2}}{B(3/2, 1/2)} \left(1 + \left(\sqrt{3}s\right)^2 /3\right)^{-(3+1)/2} = \frac{2}{\pi \left(1 + s^2\right)^2}.$$

From (13.70), the cf of $S(1)$ is clearly just $(1 + t)\, e^{-t}$, so that (10.59) gives

$$\frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[(1 + t)\, e^{-t} e^{-its}\right] dt = \frac{1}{\pi} \int_0^\infty (1 + t)\, e^{-t} \cos(st)\, dt.$$

```
function [f,F] = invunicf(xvec,a,b)
% inverts the cf at each ordinate in xvec of a sum of n independent
% Unif(a_i,b_i) r.v.s, where a=(a_1,...,a_n) and b=(b_1,...,b_n)

global a_ b_ x_ dopdf_   % Declare these variables to have global scope

bordertol=1e-6; lo=bordertol; hi=1-bordertol; tol=1e-6;
a_=a; b_=b; xl=length(xvec); F=zeros(xl,1); f=F;
for loop=1:length(xvec)
  x_=xvec(loop);
   dopdf_=1; f(loop)= quadl(@invunicf_,lo,hi,tol) / pi;
%  dopdf_=1; f(loop)=d01ajf('invunicf_',lo,hi,tol,tol) / pi;
  if nargout>1
    dopdf_=0;
    F(loop)=0.5-(1/pi)* quadl(@invunicf_,lo,hi,tol);
 %  F(loop)=0.5-(1/pi)*d01ajf('invunicf_',lo,hi,tol,tol);
  end
end
```

Program Listing S-2.1: Characteristic function inversion for sum of independent uniform r.v.s. Global variables were used (instead of passing them via `quadl`) so that the NAG integration routines could be used instead. The calls to the appropriate NAG integration routines are commented out.

The rhs integral was considered in Example A.26, and is also an integral which Maple recognizes. Either way, it yields exactly $f_S(s)$.

Solution to Problem 2.5: Splitting up the range of y and integrating over the region $x/y \leq r$ gives

$$\begin{aligned} f_R(r) &= \frac{d}{dr} \overbrace{\int_{ry}^{\infty} \int_{-\infty}^{0} f_{X,Y}(x,y)\,dydx}^{y<0} + \frac{d}{dr} \overbrace{\int_{-\infty}^{ry} \int_{0}^{\infty} f_{X,Y}(x,y)\,dydx}^{y>0} \\ &= -\int_{-\infty}^{0} y f_{X,Y}(ry,y)\,dy + \int_{0}^{\infty} y f_{X,Y}(ry,y)\,dy \end{aligned}$$

from Leibniz' rule. Combining the two integrals gives the second expression in (11.10). The region of integration can also be written $y/x \geq 1/r$ or $y \leq x/r$ for $x < 0$ and $y \geq x/r$ for $x > 0$. This gives

$$\begin{aligned} f_R(r) &= \frac{d}{dr} \int_{-\infty}^{x/r} \int_{-\infty}^{0} f_{X,Y}(x,y)\,dxdy + \frac{d}{dr} \int_{x/r}^{\infty} \int_{0}^{\infty} f_{X,Y}(x,y)\,dxdy \\ &= \int_{-\infty}^{0} \left(-\frac{x}{r^2}\right) f_{X,Y}\left(x, \frac{x}{r}\right) dx + (-1) \int_{0}^{\infty} \left(-\frac{x}{r^2}\right) f_{X,Y}\left(x, \frac{x}{r}\right) dx, \end{aligned}$$

or the first expression in (11.10).

```
function I=invunicf_(uvec);

global a_ b_ x_ dopdf_  % Variables must be declared here as well.

x=x_; a=a_; b=b_;
I=zeros(size(uvec));
for loop=1:length(uvec)
  u=uvec(loop); t=(1-u)/u;
  cf = prod( (exp(i*t*b) - exp(i*t*a))./(b-a) / (i*t) );
  z = exp(-i*t*x) * cf;
  if dopdf_==1, g=real(z); else g=imag(z)/t; end
  I(loop)= g / u^2;
end
```

Program Listing S-2.2: Used by Program `invunicf` in Listing S-2.1

Solution to Problem 2.6: The analytic derivation is quite similar to that in the previous question. Figure S-2.3 below corresponds to $R = Y/X$, with the area of the shaded regions, when treated as rectangles, given by dx times $|x|\,dr$.

From (11.11) with $R = Y/X$,

$$\begin{aligned} f_R(r) &= \int_{-\infty}^{\infty} |x|\, e^{-rx}\mathbb{I}_{(0,\infty)}(x)\,\mathbb{I}_{(x,\infty)}(rx)\,dx \\ &= \mathbb{I}_{(1,\infty)}(r)\int_0^{\infty} xe^{-rx}dx \\ &= r^{-2}\mathbb{I}_{(1,\infty)}(r), \end{aligned}$$

and $\mathbb{E}[R]$ does not exist.

Now for $R = X/Y$, using (11.10) gives

$$\begin{aligned} f_{X/Y}(r) &= \int_{-\infty}^{\infty} \frac{|x|}{r^2} f_{X,Y}\left(x, \frac{x}{r}\right) dx \\ &= \int_{-\infty}^{\infty} \frac{|x|}{r^2} e^{-x/r}\mathbb{I}_{(0,\infty)}(x)\,\mathbb{I}_{(x,\infty)}(x/r)\,dx \\ &= \frac{1}{r^2}\mathbb{I}_{(0,1)}(r)\int_0^{\infty} xe^{-x/r}dx \\ &= \mathbb{I}_{(0,1)}(r), \end{aligned}$$

i.e., $R = X/Y \sim \text{Unif}(0,1)$ with $\mathbb{E}[R] = 1/2$ and $\mathbb{V}(R) = 1/12$. Of course, the density of X/Y can be directly obtained from that of Y/X via transformation, which involves less work.

Solution to Problem 2.7: From (11.11),

$$f_{R'}(r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |x| \exp\left\{-\frac{1}{2}(x-\mu_1)^2\right\} \exp\left\{-\frac{1}{2}(rx-\mu_2)^2\right\} dx,$$

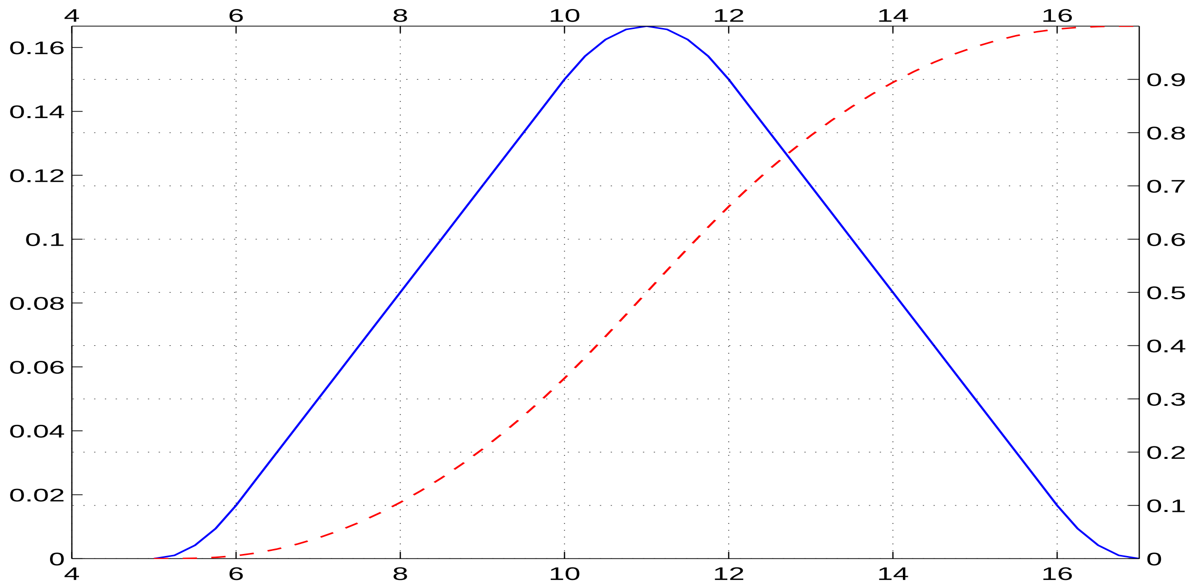

Figure S-2.2: The pdf (solid line and left axis) and cdf (dashed line and right axis) of the sum of three independent uniform r.v.s from Problem 11.3(b).

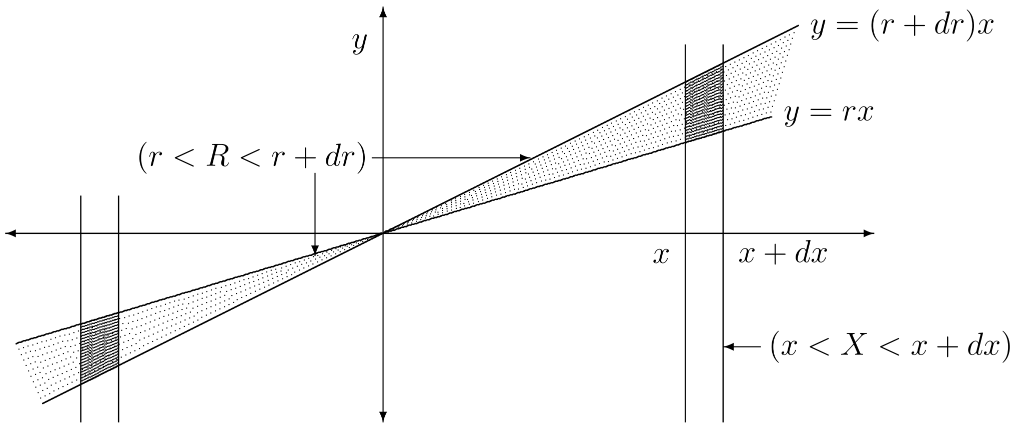

Figure S-2.3: Graphical representation of $\Pr(x < X < x + dx,\, r < R < r + dr)$, $R = Y/X$

and the integral can be split at zero into, say, Q_1 and Q_2, whereby (substituting $y = -x$)

$$Q_1 = \int_0^\infty y \exp\left(-\frac{1}{2}\left(ay^2 + 2by + c\right)\right) dy,$$

$$Q_2 = \int_0^\infty x \exp\left(-\frac{1}{2}\left(ax^2 - 2bx + c\right)\right) dx$$

and $a = 1 + r^2$, $b = \mu_1 + r\mu_2$, $c = \mu_1^2 + \mu_2^2$. For Q_1, completing the square and setting $k = (b/a)^2 - c/a$ gives

$$a\left(y^2 + 2\frac{b}{a}y + \frac{c}{a}\right) = a\left(y^2 + 2\frac{b}{a}y + \left(\frac{b}{a}\right)^2 - \left(\frac{b}{a}\right)^2 + \frac{c}{a}\right) = a\left(\left(y + \frac{b}{a}\right)^2 - k\right)$$

so that, with $u = -b/a$,

$$Q_1 = \exp\left(\frac{b^2/a - c}{2}\right) \int_0^\infty y \exp\left(-\frac{a}{2}\left((y-u)^2\right)\right) dy =: KI(u),$$

where, with $m = y - u$,

$$\begin{aligned} I(u) &= \int_0^\infty y \exp\left(-\frac{a}{2}\left((y-u)^2\right)\right) dy = \int_{-u}^\infty (m+u) \exp\left(-\frac{a}{2}\left(m^2\right)\right) dm \\ &= u \int_{-u}^\infty \exp\left(-\frac{a}{2} m^2\right) dm \qquad \text{(S-2.1)} \\ &\quad + \int_{-u}^0 m \exp\left(-\frac{a}{2} m^2\right) dm \qquad \text{(S-2.2)} \\ &\quad + \int_0^\infty m \exp\left(-\frac{a}{2} m^2\right) dm. \qquad \text{(S-2.3)} \end{aligned}$$

Term (S-2.1) is

$$\begin{aligned} \int_{-u}^\infty \exp\left(-\frac{a}{2} m^2\right) dm &= \sqrt{2\pi/a} \int_{-u}^\infty \sqrt{\frac{1}{2\pi/a}} \exp\left(-\frac{1}{2}\left(\frac{m}{1/\sqrt{a}}\right)^2\right) dm \\ &= \sqrt{2\pi/a}\left(1 - \Phi\left(-u\sqrt{a}\right)\right). \end{aligned}$$

Now let $z = am^2/2$ so that, for (S-2.2),

$$m = -\sqrt{2z/a} \quad \text{and} \quad dm = -z^{-1/2}(2a)^{-1/2} dz$$

and, similarly, for (S-2.3), $m = +\sqrt{2z/a}$. Then

$$\begin{aligned} \int_{-u}^0 m \exp\left(-\frac{a}{2} m^2\right) dm &= -\int_{au^2/2}^0 \left(-\sqrt{2z/a}\right) \exp(-z)\, z^{-1/2}(2a)^{-1/2} dz \\ &= -a^{-1} \int_0^{au^2/2} \exp(-z)\, dz = -a^{-1}\left(1 - \exp\left(-au^2/2\right)\right) \end{aligned}$$

and, similarly,

$$\int_0^\infty m \exp\left(-\frac{a}{2} m^2\right) dm = a^{-1} \int_0^\infty \exp(-z)\, dz = a^{-1},$$

so that

$$I(u) = u\sqrt{2\pi/a}\left(1 - \Phi\left(-u\sqrt{a}\right)\right) + a^{-1}\left(\exp\left(-au^2/2\right)\right)$$

and $Q_1 = KI(-b/a)$. Similarly, $Q_2 = KI(b/a)$, so that, simplifying a bit,

$$\begin{aligned} f_{R'}(r) &= \frac{K}{2\pi}\left(I(b/a) + I(-b/a)\right) \\ &= \exp\left(\frac{b^2/a - c}{2}\right) \frac{1}{2\pi}\left(\frac{b}{a}\sqrt{\frac{2\pi}{a}}\left(1 - 2\Phi\left(-\frac{b}{\sqrt{a}}\right)\right) + 2a^{-1}\exp\left(-\frac{b^2}{2a}\right)\right). \end{aligned}$$

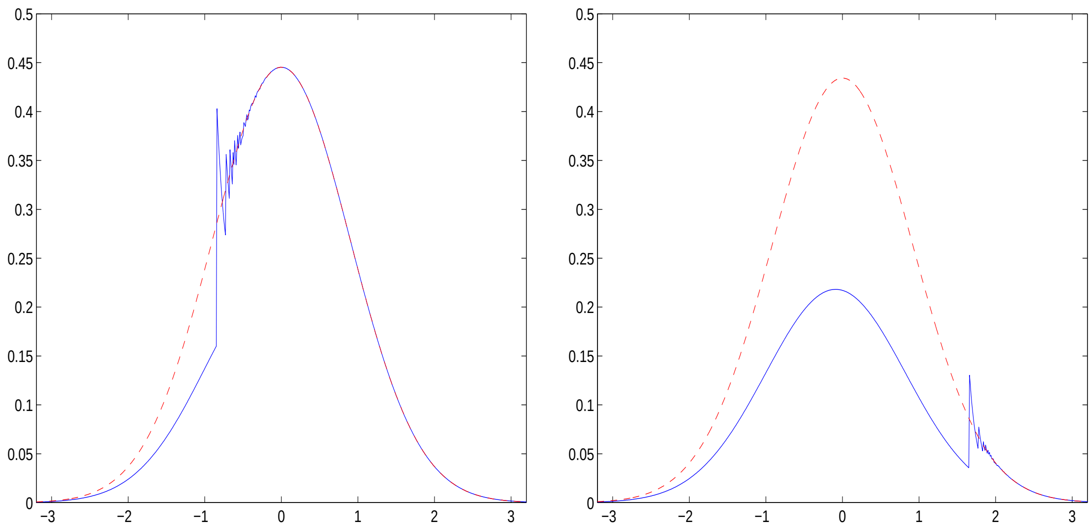

Figure S-2.4: Normal Laplace sum with $c = 0.88$ (left) and $c = 0.91$ (right) using (11.20) and erroneously calculating $\bar{\Phi}(x)$ as $1 - \Phi(x)$ (solid lines) instead of correctly as $\Phi(-x)$ (dashed)

Solution to Problem 2.8:

a) The problem with the evaluation of $1-\Phi(x)$ for x large is that, as x grows, the accuracy of $\Phi(x)$ deteriorates. In particular, the difference between 1 and $\Phi(x)$ is important, but is not revealed in $\Phi(x)$. For example, in Matlab, with $x = 6$, $\Phi(x)$ is computed as 0.99999999901341, which would imply that $1 - \Phi(x)$ is $9.865877004244794 \times 10^{-10}$. But $\Phi(-x) = 9.865876450377014 \times 10^{-10}$, showing the limitations of the former calculation. As x grows, matters worsen. With $x = 7.5$, $\Phi(x)$ is computed as 0.99999999999997, implying that $1 - \Phi(x)$ is $3.186340080674199 \times 10^{-14}$, but is actually $\Phi(-x) = 3.190891672910920 \times 10^{-14}$. For $x \geq 7.8$, Matlab returns $\Phi(x) = 1.00000000000000$ but (apparently) retains more accuracy up to $x = 8.2$. Beyond that, $1 - \Phi(x)$ is computed as zero and "all is lost". Figure S-2.4 demonstrates the effect on the density.

b) Consider $\exp\{(s^2 - z^2)/(2c^2)\}\,\Phi(-s/c)$, the first term in (11.20). For constant z,

$$\lim_{c\to1}\Phi\left(-\frac{s}{c}\right) = \Phi\left(\lim_{c\to1}-\frac{s}{c}\right) = \Phi\left(\lim_{c\to1}-\frac{z}{c} - \frac{c}{1-c}\right) = 0$$

while

$$\frac{s^2 - z^2}{2c^2} = \frac{1}{1-c}z + \frac{c^2}{2(1-c)^2}$$

so that $\exp\{(s^2 - z^2)/(2c^2)\} \to \infty$. Thus, for c near one, a very large number is being multiplied by a very small one. This occurs, in fact, for quite modest values of c. For example, with $c = 0.9$ and $z = 5$, $\Phi(-s/c) = 2.692335 \times 10^{-48}$ and $\exp\{(s^2 - z^2)/(2c^2)\} = 2.01210487 \times 10^{39}$, with product 5.417260×10^{-9}. Because Matlab and other programs use finite precision arithmetic, there will eventually be problems. For $c = 0.973$ and $x = 1$, $\Phi(-s/c) = 5.1893 \times 10^{-301}$

and $\exp\left\{\left(s^2 - z^2\right)/\left(2c^2\right)\right\} = 1.2223 \times 10^{298}$, which can still be successfully multiplied. For $c = 0.975$ and $x = 1$ however, $\Phi\left(-s/c\right)$ is returned in Matlab as zero (resulting from underflow), while the factor $\exp\left\{\left(s^2 - z^2\right)/\left(2c^2\right)\right\}$ is returned as Inf, Matlab's notation for numeric overflow. The product of zero and Inf is undefined, and returned as NaN (not a number) in Matlab.

Matters are similar for the other term in (11.20):

$$\lim_{c\to1} \bar{\Phi}\left(-\frac{d}{c}\right) = \bar{\Phi}\left(\lim_{c\to1} -\frac{d}{c}\right) = 1 - \Phi\left(\lim_{c\to1} -\frac{z}{c} + \frac{c}{1-c}\right) = 0$$

and

$$\frac{d^2 - z^2}{2c^2} = -\frac{1}{1-c}z + \frac{c^2}{2\left(1-c\right)^2} \to \infty.$$

c) One way of integrating an improper integral is by performing an appropriate substitution so that the resulting integral is proper, such as was done for the inversion formulae in §10.2.5 and 10.2.6. In this case, however, the integrands converge to zero so quickly that it is efficient to locate the range for which they are appreciable. For small values of c, the integrands of A and B (as functions of y) will be roughly bell–curve shaped centered around z, while for values of c near one, the term $y/\left(1-c\right)$ will dominate, implying that A and B are roughly exponential shaped with maxima near zero. Thus, for any $c \in (0,1)$, the integrand in A is decreasing for $y < -\operatorname{abs}(z)$, while that for B is decreasing for $y > \operatorname{abs}(z)$. As such, for A, one could repeatedly evaluate its integrand at $y_1 = -\operatorname{abs}(z) - 1$, $y_2 = -\operatorname{abs}(z) - 2$, etc., until a y_i is found such that the integrand is below a certain level, say 10^{-8}. Then A can be evaluated by numerical integration with range y_i to zero. Integral B can be similarly evaluated. A program to compute this is given in Listing S-2.3.

d) From (7.16) and (7.18),

$$F_Y(y) = \frac{1}{2}\exp\left\{\frac{y}{1-c}\right\}\mathbb{I}_{(-\infty,0)}(y) + \left(1 - \frac{1}{2}\exp\left\{-\frac{y}{1-c}\right\}\right)\mathbb{I}_{(0,\infty)}(y). \quad \text{(S-2.4)}$$

Then

$$\begin{aligned} F_Z(z) &= \mathrm{P}(X + Y \le z) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} \frac{1}{2(1-c)}\exp\left\{-\frac{|y|}{1-c}\right\} dy \frac{1}{\sqrt{2\pi}c}\exp\left\{-\frac{x^2}{2c^2}\right\} dx \\ &= A + B, \end{aligned}$$

where, using (S-2.4) and completing the square,

$$\begin{aligned} A &= \int_{-\infty}^{z}\int_{-\infty}^{z-x} \frac{1}{2(1-c)}\exp\left\{-\frac{|y|}{1-c}\right\} dy \frac{1}{\sqrt{2\pi}c}\exp\left\{-\frac{x^2}{2c^2}\right\} dx \\ &= \frac{1}{\sqrt{2\pi}c}\int_{-\infty}^{z}\left(1 - \frac{1}{2}\exp\left\{-\frac{z-x}{1-c}\right\}\right)\exp\left\{-\frac{x^2}{2c^2}\right\} dx \\ &= \Phi\left(\frac{z}{c}\right) - \frac{1}{2}\exp\left\{-\frac{z}{1-c}\right\}\exp\left\{\frac{1}{2}\left(\frac{c}{1-c}\right)^2\right\}\Phi\left(\frac{z}{c} - \frac{c}{1-c}\right) \end{aligned}$$

```
function pdf=normlaptemp(xvec,c)
denom = c * (1-c) * 2 * sqrt(2*pi);
for xloop=1:length(xvec)
  x=xvec(xloop);
  % first the integral from -inf to 0:
    lo=-abs(x); while normlap1_(lo,c,x) > 1e-10, lo=lo-1; end
    hi=0; int1 = quadl(@normlap1_,lo,hi,1e-7,0,c,x);
  % similarly for the integral from 0 to inf:
    hi=abs(x); while normlap1_(hi,c,x) > 1e-10, hi=hi+1; end
    lo=0; int2 = quadl(@normlap2_,lo,hi,1e-7,0,c,x);
  pdf(xloop) = (int1+int2) / denom;
end

function out=normlap1_(y,c,x), out = exp( -0.5 * ((x-y)./c).^2 + y./(1-c));

function out=normlap2_(y,c,x), out = exp( -0.5 * ((x-y)./c).^2 - y./(1-c));
```

Program Listing S-2.3: Computes f_Z using the convolution formula and numerical integration

and

$$\begin{aligned} B &= \int_z^\infty \int_{-\infty}^{z-x} \frac{1}{2(1-c)} \exp\left\{-\frac{|y|}{1-c}\right\} dy \frac{1}{\sqrt{2\pi}c} \exp\left\{-\frac{x^2}{2c^2}\right\} dx \\ &= \frac{1}{2} \int_z^\infty \frac{1}{\sqrt{2\pi}c} \exp\left\{\frac{z-x}{1-c}\right\} \exp\left\{-\frac{x^2}{2c^2}\right\} dx \\ &= \frac{1}{2} \exp\left\{\frac{z}{1-c}\right\} \exp\left\{\frac{1}{2}\left(\frac{c}{1-c}\right)^2\right\} \Phi\left(-\left[\frac{z}{c}+\frac{c}{1-c}\right]\right). \end{aligned}$$

Putting these together,

$$\begin{aligned} F_Z(z) &= \Phi\left(\frac{z}{c}\right) + \frac{1}{2} \exp\left\{\frac{1}{2}\left(\frac{c}{1-c}\right)^2\right\} \\ &\times \left[\exp\left\{\frac{z}{1-c}\right\} \Phi\left(-\frac{z}{c}-\frac{c}{1-c}\right) - \exp\left\{-\frac{z}{1-c}\right\} \Phi\left(\frac{z}{c}-\frac{c}{1-c}\right)\right]. \end{aligned}$$

For the raw moments, using the binomial theorem to get

$$\mathbb{E}\left[Z^m\right] = \mathbb{E}\left[(cX_0 + (1-c)Y_0)^m\right] = \sum_{i=0}^{m} \binom{m}{i} c^i (1-c)^{m-i} \mathbb{E}\left[X_0^i\right] \mathbb{E}\left[Y_0^{m-i}\right].$$

If m is odd, either i is odd or $m-i$ is odd, so that, as all odd moments of a zero–mean normal and a zero–mean Laplace random variable are zero, the odd moments of Z are all zero. For the even moments, we use (7.37), i.e., that

$\mathbb{E}[X_0^{2i}] = 2^{-i}(2i)!/i!$ and that $\mathbb{E}[Y_0^{2i}] = (2i)!$ (see Problem 10.2). This yields

$$\begin{aligned}
\mathbb{E}[Z^m] &= \sum_{i=0}^{m/2} \binom{m}{2i} c^{2i}(1-c)^{m-2i}\mathbb{E}[X_0^{2i}]\mathbb{E}[Y_0^{m-2i}] \\
&= (1-c)^m \sum_{i=0}^{m/2} \left(\frac{c}{1-c}\right)^{2i} \frac{m!}{(2i)!(m-2i)!}\frac{(2i)!}{2^i i!}(m-2i)! \\
&= (1-c)^m m! \sum_{i=0}^{m/2} \frac{1}{i!}\left(\frac{c^2}{2(1-c)^2}\right)^i .
\end{aligned}$$

Solution to Problem 2.9:

a) From (7.65),

$$f_{S_1}(s) = f_{X_1}(x)\left|\frac{dx}{ds}\right| = bx_0^b\left(e^s\right)^{-(b+1)}\mathbb{I}_{[x_0,\infty)}\left(e^s\right)e^s = bx_0^b e^{-sb}\mathbb{I}_{(\ln x_0,\infty)}(s),$$

a truncated exponential.

b) Recall the formula for the density of the product of two random variables in (11.9),

$$f_P(p) = \int_{-\infty}^{\infty}\frac{1}{|x|}f_{X,Y}\left(x,\frac{p}{x}\right)\mathrm{d}x = \int_{-\infty}^{\infty}\frac{1}{|y|}f_{X,Y}\left(\frac{p}{y},y\right)\mathrm{d}y.$$

Using the former, and using P instead of Z, we have

$$\begin{aligned}
f_P(p) &= \int_{-\infty}^{\infty}\frac{1}{|x|}f_{X_1,X_2}\left(x,\frac{p}{x}\right)\mathrm{d}x = \int_{-\infty}^{\infty}\frac{1}{|x|}f_{X_1}(x)\,f_{X_2}\left(\frac{p}{x}\right)\mathrm{d}x \\
&= \int_{-\infty}^{\infty}\frac{1}{|x|}bx_0^b x^{-(b+1)}\mathbb{I}_{[x_0,\infty)}(x)\quad bx_0^b\left(\frac{p}{x}\right)^{-(b+1)}\mathbb{I}_{[x_0,\infty)}\left(\frac{p}{x}\right)\mathrm{d}x \\
&= b^2x_0^{2b}p^{-(b+1)}\int_{-\infty}^{\infty}\frac{1}{|x|}\mathbb{I}_{[x_0,\infty)}(x)\ \mathbb{I}_{[x_0,\infty)}\left(\frac{p}{x}\right)\mathrm{d}x.
\end{aligned}$$

Now, use the indicator functions: $\mathbb{I}_{[x_0,\infty)}(x) \Leftrightarrow x > x_0$; and, as $x_0 > 0$, $\mathbb{I}_{[x_0,\infty)}(p/x) \Leftrightarrow x < p/x_0$ and that $p > xx_0 > x_0^2$. Thus,

$$f_P(p) = b^2x_0^{2b}p^{-(b+1)}\int_{x_0}^{p/x_0}\frac{1}{x}\mathbb{I}_{\left(x_0^2,\infty\right)}(p)\ \mathrm{d}x = b^2x_0^{2b}p^{-(b+1)}\left(\ln p - 2\ln x_0\right)\mathbb{I}_{\left(x_0^2,\infty\right)}(p).$$

c) From (7.65)

$$\begin{aligned}
f_{S_2}(s) &= f_Z\left(e^s\right)e^s = b^2x_0^{2b}\left(e^s\right)^{-(b+1)}\left(\ln\left(e^s\right) - 2\ln x_0\right)\mathbb{I}_{\left(x_0^2,\infty\right)}\left(e^s\right)e^s \\
&= b^2x_0^{2b}e^{-sb}\left(s - 2\ln x_0\right)\mathbb{I}_{(2\ln x_0,\infty)}(s).
\end{aligned}$$

d) From (11.7) with $Y = \ln X_3$,

$$\begin{aligned} f_{S_3}(s) &= \int_{-\infty}^{\infty} f_{S_2}(x) f_Y(s-x)\,dx \\ &= \int_{-\infty}^{\infty} b^2 x_0^{2b} e^{-xb}(x - 2\ln x_0)\,\mathbb{I}_{(2\ln x_0,\infty)}(x) \\ &\quad \times b x_0^b e^{-(s-x)b}\mathbb{I}_{(\ln x_0,\infty)}(s-x)\;dx \\ &= b^3 x_0^{3b} e^{-bs}\int_{2\ln x_0}^{s-\ln x_0}(x - 2\ln x_0)\,dx \\ &= b^3 x_0^{3b} e^{-bs}\frac{1}{2}(s - 3\ln x_0)^2\,\mathbb{I}_{(3\ln x_0,\infty)}(s). \end{aligned}$$

We know that $(S_n \mid x_0 = 1) \sim \mathrm{Gam}(n, b)$, so we might expect that

$$f_{S_n}(s) = \frac{b^n x_0^{nb}}{\Gamma(n)} e^{-bs}(s - n\ln x_0)^{n-1}\,\mathbb{I}_{(n\ln x_0,\infty)}(s).$$

This is correct, as shown by Malik (1966) (see also Johnson *et al.*, 1994, p. 596).

e) As $Z_n = e^{S_n}$, (7.65) gives

$$f_{Z_n}(z) = f_{S_n}(s)\,z^{-1} = \frac{b^n x_0^{nb}}{\Gamma(n)} z^{-(b+1)}\left(\ln\left(\frac{z}{x_0^n}\right)\right)^{n-1}\mathbb{I}_{(x_0^n,\infty)}(z).$$

Solution to Problem 2.10:

a) We have

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z+y) f_Y(y)\,dy = \int_{-\infty}^{\infty} \mathbb{I}_{(0,1)}(z+y)\,\mathbb{I}_{(0,1)}(y)\,dy$$

and

$$\mathbb{I}_{(0,1)}(z+y) \neq 0 \Leftrightarrow 0 < z+y < 1 \Leftrightarrow -z < y < 1-z,$$

so that

$$\mathbb{I}_{(0,1)}(z+y)\,\mathbb{I}_{(0,1)}(y) \neq 0 \Leftrightarrow \begin{cases} 0 < y < 1-z, & \text{for } \; 0 < z < 1, \\ -z < y < 1, & \text{for } \; -1 < z < 0. \end{cases}$$

Thus,

$$\begin{aligned} f_Z(z) &= \int_{-z}^{1} \mathbb{I}_{(-1,0)}(z)\,dy + \int_0^{1-z} \mathbb{I}_{(0,1)}(z)\,dy \\ &= (1+z)\,\mathbb{I}_{(-1,0)}(z) + (1-z)\,\mathbb{I}_{(0,1)}(z) \\ &= (1-|z|)\,\mathbb{I}_{(-1,1)}(z). \end{aligned}$$

b) We have $F_W(w) = \Pr(W \le w) = \Pr(-w \le Z \le w) = F_Z(w) - F_Z(-w)$, using the fact that $0 \le w \le 1$ and differentiating,

$$\begin{aligned} f_W(w) &= f_Z(w) + f_Z(-w) = (1-|w|)\,\mathbb{I}_{(-1,1)}(w) + (1-|-w|)\,\mathbb{I}_{(-1,1)}(w) \\ &= 2(1-w)\,\mathbb{I}_{[0,1]}(w), \end{aligned}$$

a $\mathrm{Beta}(1,2)$ distribution.

c) From the density of $Z = X + Y$ given in (11.16), $A = \frac{Z}{2} \Rightarrow Z = 2A$ and $\frac{dZ}{dA} = 2$, so that

$$\begin{aligned} f_A(a) &= f_Z(z)\frac{dz}{da} = \left[(2a)\,\mathbb{I}_{(0,1)}(2a) + (2-2a)\,\mathbb{I}_{[1,2)}(2a)\right]\cdot 2 \\ &= 4a\mathbb{I}_{\left(0,\frac{1}{2}\right)}(a) + 4(1-a)\,\mathbb{I}_{\left[\frac{1}{2},1\right)}(a). \end{aligned}$$

Solution to Problem 2.11:

a) For $n = 3$, (11.39) simplifies to

$$\begin{aligned} 2f_{S_3}(s) &= s^2\mathbb{I}(s\ge 0) - 3(s-1)^2\,\mathbb{I}(s\ge 1) \\ &\quad +3(s-2)^2\,\mathbb{I}(s\ge 2) - (s-3)^2\,\mathbb{I}(s\ge 3) \\ &= s^2\mathbb{I}(s\ge 0) - 3(s-1)^2\,\mathbb{I}(s\ge 1) + 3(s-2)^2\,\mathbb{I}_{[2,3)}(s), \end{aligned}$$

because $S_3 \le 3$. Splitting the event $\mathbb{I}(s\ge 0)$ into $\mathbb{I}(0\le s<1)$ and $\mathbb{I}(s\ge 1)$ gives

$$2f_{S_3}(s) = s^2\mathbb{I}_{(0,1)}(s) + \left(s^2 - 3(s-1)^2\right)\mathbb{I}(s\ge 1) + 3(s-2)^2\,\mathbb{I}_{[2,3)}(s).$$

Similarly, splitting $\mathbb{I}(s\ge 1)$ into $\mathbb{I}(1\le s<2)$ and $\mathbb{I}(s\ge 2)$ gives

$$\begin{aligned} 2f_{S_3}(s) &= s^2\mathbb{I}_{(0,1)}(s) + \left(s^2 - 3(s-1)^2\right)\mathbb{I}_{[1,2)}(s) \\ &\quad + \left(3(s-2)^2 + \left(s^2 - 3(s-1)^2\right)\right)\mathbb{I}_{[2,3)}(s) \\ &= s^2\mathbb{I}_{(0,1)}(s) + 2\left(3s - s^2 - \frac{3}{2}\right)\mathbb{I}_{[1,2)}(s) + (s-3)^2\,\mathbb{I}_{[2,3)}(s), \end{aligned}$$

which is (11.40).

b) From (11.7),

$$\begin{aligned} f_{S_3}(s) &= \int_{-\infty}^{\infty} f_{S_2}(s-x)\,f_{X_3}(x)\,dx \\ &= \int_{-\infty}^{\infty} (s-x)\,\mathbb{I}_{(0,1)}(s-x)\,\mathbb{I}_{(0,1)}(x)\,dx \\ &\quad + \int_{-\infty}^{\infty} (2-s+x)\,\mathbb{I}_{[1,2)}(s-x)\,\mathbb{I}_{(0,1)}(x)\,dx. \end{aligned}$$

Observe that

$$\begin{aligned} \mathbb{I}_{(0,1)}(s-x)\,\mathbb{I}_{(0,1)}(x) &\Rightarrow 0 < s-x < 1 \Rightarrow s-1 < x < s \\ &\Rightarrow \begin{cases} 0 < x < s, & \text{for} \quad 0 < s < 1, \\ s-1 < x < 1, & \text{for} \quad 1 \le s < 2, \end{cases} \end{aligned}$$

and, if $2 \le s < 3$, $\mathbb{I}_{(0,1)}(x) \equiv 0$. Thus, $\int_{-\infty}^{\infty}(s-x)\,\mathbb{I}_{(0,1)}(s-x)\,\mathbb{I}_{(0,1)}(x)\,dx$

$$\begin{aligned} &= \mathbb{I}_{(0,1)}(s)\int_0^s (s-x)\,dx + \mathbb{I}_{[1,2)}(s)\int_{s-1}^1 (s-x)\,dx \\ &= \frac{1}{2}s^2\mathbb{I}_{(0,1)}(s) + \left(s - \frac{1}{2}s^2\right)\mathbb{I}_{[1,2)}(s). \end{aligned}$$

Similarly,

$$\mathbb{I}_{[1,2)}(s-x)\,\mathbb{I}_{(0,1)}(x) \;\Rightarrow\; 1 < s-x < 2 \Rightarrow s-2 < x < s-1$$
$$\Rightarrow \begin{cases} 0 < x < s-1, & \text{for} \quad 1 < s < 2, \\ s-2 < x < 1, & \text{for} \quad 2 \le s < 3, \end{cases}$$

and, if $0 < s < 1$, $\mathbb{I}_{(0,1)}(x) \equiv 0$. Thus, $\int_{-\infty}^{\infty}(2-s+x)\,\mathbb{I}_{[1,2)}(s-x)\,\mathbb{I}_{(0,1)}(x)\,dx$

$$= \mathbb{I}_{[1,2)}(s)\int_0^{s-1}(2-s+x)\,dx + \mathbb{I}_{[2,3)}(s)\int_{s-2}^{1}(2-s+x)\,dx$$
$$= \left(-\frac{s^2}{2}+2s-\frac{3}{2}\right)\mathbb{I}_{[1,2)}(s) + \left(\frac{s^2}{2}-3s+\frac{9}{2}\right)\mathbb{I}_{[2,3)}(s).$$

Combining the two integrals,

$$f_{S_3}(s) = \frac{1}{2}s^2\mathbb{I}_{(0,1)}(s) + \left(3s-s^2-\frac{3}{2}\right)\mathbb{I}_{[1,2)}(s) + \frac{(s-3)^2}{2}\mathbb{I}_{[2,3)}(s)$$

as in (11.40).

c) With $s_3 = 3a$ and $ds_3/da = 3$, $f_A(a) = f_{S_3}(s)\,ds/da$ is

$$\left[\frac{1}{2}(3a)^2\,\mathbb{I}_{(0,1)}(3a) + \left(3(3a)-(3a)^2-\frac{3}{2}\right)\mathbb{I}_{[1,2)}(3a) + \frac{((3a)-3)^2}{2}\mathbb{I}_{[2,3)}(3a)\right]\cdot 3$$

or

$$\frac{27}{2}a^2\mathbb{I}_{(0,\frac{1}{3})}(a) + \left(27a-27a^2-\frac{9}{2}\right)\mathbb{I}_{[\frac{1}{3},\frac{2}{3})}(a) + \left(\frac{27(a-1)^2}{2}\right)\mathbb{I}_{[\frac{2}{3},1)}(a)$$

or

$$\frac{27}{2}a^2\mathbb{I}_{(0,\frac{1}{3})}(a) + 27\left(\frac{1}{12}-\left(a-\frac{1}{2}\right)^2\right)\mathbb{I}_{[\frac{1}{3},\frac{2}{3})}(a) + \left(\frac{27(a-1)^2}{2}\right)\mathbb{I}_{[\frac{2}{3},1)}(a).$$

d) Let $S = \sum_{i=1}^{n} U_i$ for $U_i \overset{\text{iid}}{\sim} \text{Unif}(0,1)$ and define $Z = (S-\mu)/\sigma$, where $\mu = n/2$ is the mean of S and $\sigma^2 = n/12$ is the variance of S. (The variance of U_i follows from (7.13) with $p = q = 1$.) Then

$$f_Z(z) = f_S(s)\left|\frac{ds}{dz}\right| = \sigma f_S(s)$$

with support $\mathcal{S}_Z = (-\sqrt{3n}, \sqrt{3n})$.

The program in Listing S-2.4 computes (11.39) for a given n over a grid of about 100 points. The commands in Listing S-2.5 were used to evaluate (11.39) for 3 different values of n (3,6 and 12) and plot $f_Z(z)$ along with the standard normal density $\phi(z)$ (dotted line). These are shown in the left panels of Figure S-2.5. The right panels show the relative percentage error.

Because $\mathcal{S}_Z$ is bounded (and from the continuity of f_Z and ϕ), it follows that that the relative percentage error will grow towards 100% as z approaches the extremes in $\mathcal{S}_Z$.

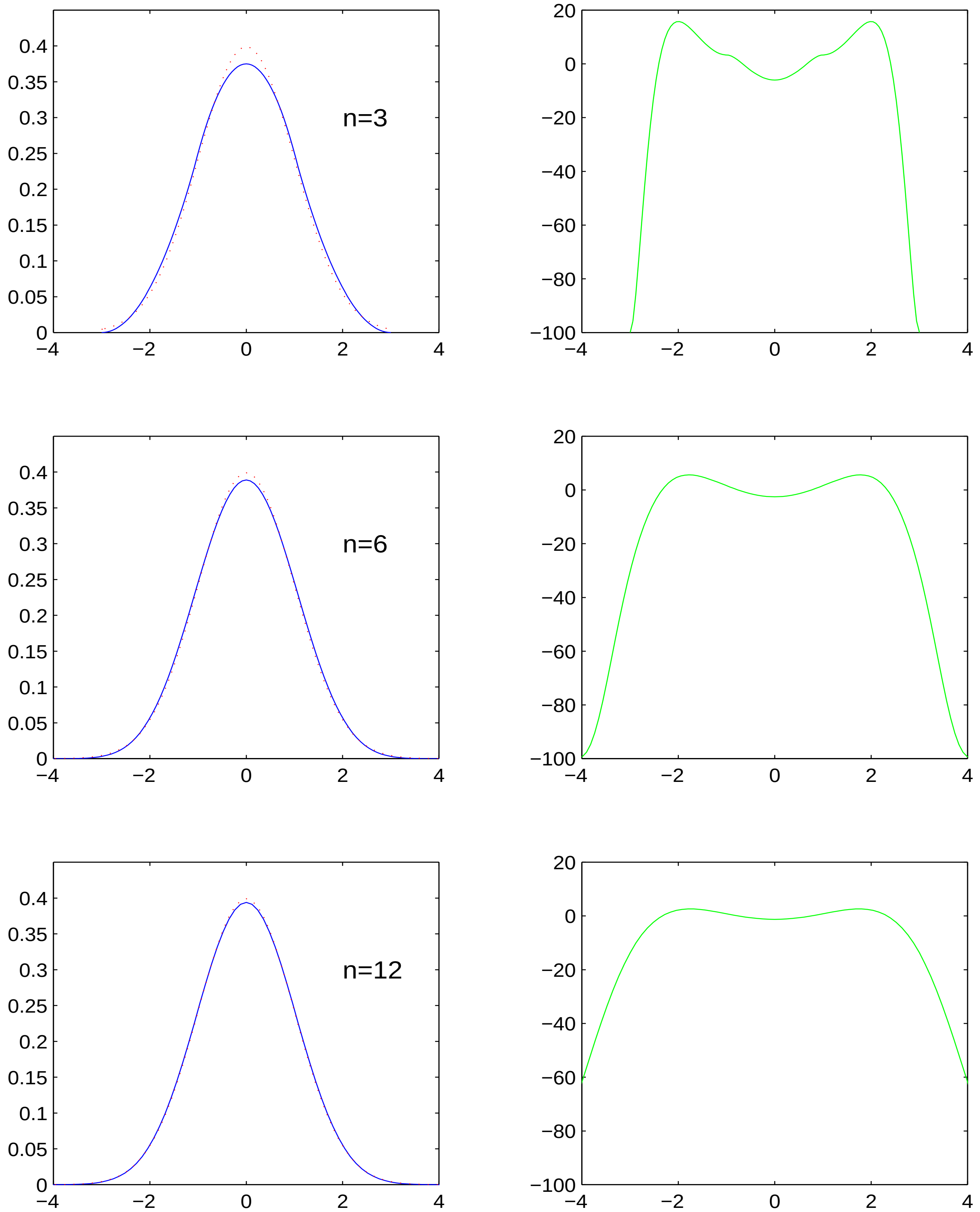

Figure S-2.5: Left panels show density of $Z = (S - \mu)/\sigma$ (solid), where $S = \sum_{i=1}^{n} U_i$ for $U_i \overset{\text{iid}}{\sim} \text{Unif}\,(0, 1)$ and standard normal density (dotted). Right panels are the relative percentage error.

```
function [grd,f] = unifconv(n)
h=100; % approx number of density points
k=0:n;
comb=c(n,k); sign=(-1).^k;
grd=0:(n/h):n;
for i=1:length(grd)
  s=grd(i);
  num=max((s-k),0).^(n-1);
  f(i)=sum(comb .* sign .* num);
end
f=f/gamma(n);
f=max(f,0); % takes care of a bit of round off error.
```

Program Listing S-2.4: Computes the pdf of a sum of iid Unif(0,1) r.v.s, as given in (11.39)

Solution to Problem 2.12:

a) As the density must integrate to one, we see that

$$2 \cdot \frac{1}{2}\left(\frac{b-a}{2}\right) h = 1 \quad \text{or} \quad h = \frac{2}{b-a}.$$

b) For the first half, consider the straight line

$$y = \frac{\triangle y}{\triangle x}(x-a) = \frac{h-0}{c-a}(x-a) = h^2(x-a)$$

and, for the second,

$$y = -\frac{\triangle y}{\triangle x}(x-c) + h = -\frac{h-0}{b-c}(x-c) + h = h - h^2(x-c).$$

c) For $a \le x \le c$,

$$F_X(x) = \int_a^x h^2(t-a)\,dt = h^2\left(\frac{x^2}{2} - xa + \frac{a^2}{2}\right) = 2\frac{(x-a)^2}{(b-a)^2} = \frac{h^2}{2}(x-a)^2.$$

Because of symmetry, when $c \le x \le b$, $F_X(x)$ will be one minus the area to the right of x, or

$$1 - 2\frac{(x-b)^2}{(b-a)^2},$$

so that

$$F_X(x) = \begin{cases} \frac{h^2}{2}(x-a)^2, & \text{if} \quad a \le x \le c, \\ 1 - \frac{h^2}{2}(x-b)^2, & \text{if} \quad c \le x \le b. \end{cases}$$

If the symmetry argument does not appeal to you, then we arrive at $F_X(x)$ for

```
n=3; [s,fs] = unifconv(n);
sigma=sqrt(n/12); fz=fs*sigma; z=(s-n/2)/sigma;
subplot(3,2,1), true=normpdf(z);
plot(z,fz,'b-',z,true,'r:'), axis([-4 4 0 0.45])
text(2,0.3,'n=3','fontsize',13)
subplot(3,2,2)
rpe=100*(fz-true)./true; plot(z,rpe,'g-')
ax=axis; axis([-4 4 ax(3) ax(4)])

n=6; [s,fs] = unifconv(n);
sigma=sqrt(n/12); fz=fs*sigma; z=(s-n/2)/sigma;
subplot(3,2,3), true=normpdf(z);
plot(z,fz,'b-',z,true,'r:'), axis([-4 4 0 0.45])
text(2,0.3,'n=6','fontsize',13)
subplot(3,2,4)
rpe=100*(fz-true)./true; plot(z,rpe,'g-'), axis([-4 4 ax(3) ax(4)])

n=12; [s,fs] = unifconv(n);
sigma=sqrt(n/12); fz=fs*sigma; z=(s-n/2)/sigma;
subplot(3,2,5), true=normpdf(z);
plot(z,fz,'b-',z,true,'r:'), axis([-4 4 0 0.45])
text(2,0.3,'n=12','fontsize',13)
subplot(3,2,6)
rpe=100*(fz-true)./true; plot(z,rpe,'g-'), axis([-4 4 ax(3) ax(4)])
```

Program Listing S-2.5: Code required to construct Figure S-2.5

the interval $c \leq x \leq b$ in the following way. For $c \leq x \leq b$,

$$\begin{aligned}
F_X(x) &= \frac{1}{2} + \int_c^x f_X(t)\,dt = \frac{1}{2} + \frac{2}{b-a}\int_c^x dt - \frac{4}{(b-a)^2}\int_c^x (t-c)\,dt \\
&= \frac{1}{2} + \frac{2(x-c)}{(b-a)} - \frac{4}{(b-a)^2}\left[\left(\frac{x^2}{2} - cx\right) - \left(\frac{c^2}{2} - c^2\right)\right] \\
&= \frac{1}{2} + \frac{2(x-c)}{(b-a)} - \frac{2(x-c)^2}{(b-a)^2} = \frac{1}{2} + \frac{2x-(a+b)}{b-a} - 2\frac{(x-b+b-c)^2}{(b-a)^2} \\
&= \frac{1}{2} + \frac{2x-(a+b)}{b-a} - 2\left[\frac{(x-b)^2}{(b-a)^2} + \frac{(b-c)^2}{(b-a)^2} + \frac{2(x-b)(b-c)}{(b-a)^2}\right].
\end{aligned}$$

Substituting $c = (a+b)/2$ and simplifying, we have

$$F_X(x) = \frac{1}{2} + \frac{2x-(a+b)}{b-a} - 2\left[\frac{(x-b)^2}{(b-a)^2} + \frac{1}{4} + \frac{x-b}{b-a}\right] = 1 - 2\frac{(x-b)^2}{(b-a)^2},$$

as above.

d) Substituting, we have $h = d^{-1}$, $c = 0$ and

$$f_X(x) = \begin{cases} \dfrac{d+x}{d^2}, & \text{if} \quad -d \le x \le 0, \\ \dfrac{d-x}{d^2}, & \text{if} \quad 0 \le x \le d, \end{cases}$$

or simply

$$f_X(x) = \frac{d - |x|}{d^2} \mathbb{I}_{[-d,d]}(x).$$

e) The mean is clearly zero, so that $\mathbb{V}(Y) = \mathbb{E}[Y^2]$ and, using symmetry,

$$\mathbb{E}\left[Y^2\right] = 2\int_0^d \frac{d-t}{d^2} t^2 dt = \frac{d^2}{6}.$$

f) Using the fact that the variance is translation invariant, we may "shift" X so that it is centered about zero. Equating the ranges, $2d = b - a$ or $d = (b-a)/2$ accomplishes this, so that

$$\mathbb{V}(X) = \frac{d^2}{6} = \frac{(b-a)^2}{24}.$$

g)

$$\begin{aligned} \mathbb{M}_Y(s) &= \frac{1}{a^2}\int_{-a}^0 (a+t)\, e^{st} dt + \frac{1}{a^2}\int_0^a (a-t)\, e^{st} dt \\ &= \frac{e^{as} + e^{-as} - 2}{a^2 s^2} = \frac{e^{as} - 1}{as} \cdot \frac{1 - e^{-as}}{as}, \end{aligned}$$

and, from independence,

$$\begin{aligned} \mathbb{M}_{U_1 - U_2}(s) &= \mathbb{M}_{U_1}(s)\, \mathbb{M}_{-U_2}(s) = \mathbb{M}_{U_1}(s)\, \mathbb{M}_{U_2}(-s) \\ &= \frac{e^{s(k+a)} - e^{sk}}{sa} \cdot \frac{-\left(e^{-s(k+a)} - e^{-sk}\right)}{sa} \\ &= \frac{e^{sk}\left(e^{sa} - 1\right)}{sa} \cdot \frac{e^{-sk}\left(1 - e^{-sa}\right)}{sa} \\ &= \frac{e^{as} - 1}{as} \cdot \frac{1 - e^{-as}}{as}, \end{aligned}$$

so that the two expressions are equal, and thus $U_1 - U_2$ and Y have the same distribution.

h) In general,

$$\Pr(Y < y) = [F_X(y)]^n = \begin{cases} \left(2\dfrac{(x-a)^2}{(b-a)^2}\right)^n, & \text{if} \quad a \le y \le c, \\ \left(1 - 2\dfrac{(x-b)^2}{(b-a)^2}\right)^n, & \text{if} \quad c \le y \le b, \end{cases}$$

but, as

$$F_X\left(\frac{a+b}{2}\right)=\frac{1}{2},$$

which is seen either from the symmetry of the distribution or substituting directly into the cdf formula, we have that

$$\Pr\left(Y<\frac{a+b}{2}\right)=2^{-n}.$$

Solution to Problem 2.13:

a) From the iterated expectation and the independence of the X_i and N,

$$\begin{aligned}
\mathbb{M}_S(t) &= \mathbb{E}\left[e^{tS}\right]=\mathbb{E}_N\Big\{\mathbb{E}\left[e^{t\sum_{i=1}^{N}X_i}\mid N=n\right]\Big\}=\mathbb{E}_N\Big\{\mathbb{E}\left[e^{t\sum_{i=1}^{n}X_i}\right]\Big\}\\
&= \mathbb{E}_N\{\mathbb{E}\left[e^{tX_1}\right]\cdots\mathbb{E}\left[e^{tX_n}\right]\}=\mathbb{E}_N\left[\left(\frac{\lambda}{\lambda-t}\right)^n\right]\\
&= \sum_{n=1}^{\infty}\left(\frac{\lambda}{\lambda-t}\right)^n p\,(1-p)^{n-1}=\frac{\lambda p}{\lambda-t}\left(1+\frac{\lambda}{\lambda-t}(1-p)+\dots\right)\\
&= \frac{\lambda p}{\lambda-t}\left(\frac{1}{1-\frac{\lambda}{\lambda-t}(1-p)}\right)=\frac{\lambda p}{\lambda p-t},
\end{aligned}$$

so that $S\sim\exp(\lambda p)$ with expected value $\mathbb{E}[S]=1/(\lambda p)$. One could also derive the density by noting that $S\mid(N=n)\sim\mathrm{Gam}(n,\lambda)$ and using (8.16):

$$\begin{aligned}
f_S(s) &= \sum_{i=1}^{\infty}f_{S|N=n}(s\mid n)\,f_N(n)=\sum_{n=1}^{\infty}\frac{\lambda^n}{\Gamma(n)}s^{n-1}e^{-\lambda s}(1-p)^{n-1}p\\
&= \lambda pe^{-\lambda s}\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(n-1)!}s^{n-1}(1-p)^{n-1}=\lambda pe^{-\lambda s}\sum_{x=0}^{\infty}\frac{[\lambda s(1-p)]^x}{x!}\\
&= \lambda pe^{-\lambda s}e^{\lambda s(1-p)}=(p\lambda)\,e^{-(p\lambda)s}
\end{aligned}$$

with $x=n-1$ and $\exp\{t\}=\sum_{x=0}^{\infty}t^x/x!$.

b) This is

$$\mathbb{E}[S]=\mathbb{E}_N\,\{\mathbb{E}_X[S\mid N=n]\}=\mathbb{E}_N\left[\mathbb{E}_X\left[\sum_{i=1}^{n}X_i\right]\right]=\mathbb{E}_X[X_i]\,\mathbb{E}_N[n]=\frac{1}{\lambda}\frac{1}{p}.$$

Solution to Problem 2.14:

a) A simple integration yields

$$\int_0^1\int_0^x x^a y^b\,dy\,dx=\frac{1}{b+1}\frac{1}{a+b+2}$$

or $c=(b+1)(a+b+2)$.

b) Likewise,

$$f_X(x) = c\int_0^x x^a y^b dy = (a+b+2)\,x^{a+b+1}\mathbb{I}_{(0,1)}(x)$$

and

$$f_Y(y) = c\int_y^1 x^a y^b dx = cy^b \frac{1-y^{a+1}}{a+1}\mathbb{I}_{(0,1)}(y)$$

with $\mathbb{E}[X] = (a+b+2)\,/\,(a+b+3)$ and

$$\begin{aligned}\mathbb{E}[Y] &= \frac{c}{a+1}\int_0^1 y^{b+1}\left(1-y^{a+1}\right)dy \\ &= \frac{c}{a+1}\left(\frac{1}{b+2}-\frac{1}{a+b+3}\right) \\ &= \frac{(b+1)(a+b+2)}{(b+2)(a+b+3)}.\end{aligned}$$

c) Directly,

$$f_{X|Y}(x \mid y) = \frac{(a+1)\,x^a}{1-y^{a+1}}\mathbb{I}_{(y,1)}(x), \quad f_{Y|X}(y \mid x) = \frac{(b+1)\,y^b}{x^{b+1}}\mathbb{I}_{(0,x)}(y)$$

with

$$\mathbb{E}[X \mid Y] = \int_y^1 \frac{(a+1)\,x^{a+1}}{1-y^{a+1}}dx = \frac{(a+1)(1-y^{a+2})}{(a+2)(1-y^{a+1})}$$

and $\mathbb{E}[Y \mid X] = x(b+1)\,/\,(b+2)$. Thus,

$$\begin{aligned}\mathbb{E}[\mathbb{E}[X \mid Y]] &= \int \mathbb{E}[X \mid Y] f_Y(y)\,dy = \int_0^1 \frac{(a+1)(1-y^{a+2})}{(a+2)(1-y^{a+1})}cy^b\frac{1-y^{a+1}}{a+1}dy \\ &= \frac{c}{a+2}\int_0^1\left(1-y^{a+2}\right)y^b dy = \frac{c}{a+2}\left(\frac{1}{b+1}-\frac{1}{a+b+3}\right) \\ &= \frac{a+b+2}{a+b+3} = \mathbb{E}[X]\end{aligned}$$

and

$$\mathbb{E}[\mathbb{E}[Y \mid X]] = \int \mathbb{E}[Y \mid X] f_X(x)\,dx = \frac{b+1}{b+2}\frac{a+b+2}{a+b+3} = \mathbb{E}[Y].$$

d) Because of the finite support, one has to be careful with the range of S. In particular, some thought reveals that, if $0 \le S \le 1$, then $S/2 \le X \le S$, while, if $1 \le S \le 2$, then $S/2 \le X \le 1$. Using (11.7),

$$\begin{aligned}f_S(s) &= \mathbb{I}_{[0,1]}(s)\int_{s/2}^s f_{X,Y}(x, s-x)\,dx + \mathbb{I}_{(1,2]}(s)\int_{s/2}^1 f_{X,Y}(x, s-x)\,dx \\ &= \mathbb{I}_{[0,1]}(s)\,c\int_{s/2}^s x^a(s-x)^b dx + \mathbb{I}_{(1,2]}(s)\,c\int_{s/2}^1 x^a(s-x)^b dx.\end{aligned}$$

To resolve these, use $u = 1 - x/s$, $x = (1-u)\,s$ and $dx = -sdu$ to get

$$\int_{s/2}^{s} x^a (s-x)^b \, dx = s^{a+b+1} \int_0^{1/2} (1-u)^a u^b du = s^{a+b+1} B_{1/2}(b+1, a+1)$$

and, similarly,

$$\begin{aligned}\int_{s/2}^{1} x^a (s-x)^b \, dx &= s^{a+b+1} \int_{(s-1)/s}^{1/2} (1-u)^a u^b du \\ &= s^{a+b+1} \left(B_{1/2}(b+1, a+1) - B_{(s-1)/s}(b+1, a+1)\right).\end{aligned}$$

These general expressions simplify for (low) integer a and b. In particular, if $a = b = 1$, $c = 8$ and

$$f_S(s) = \frac{2}{3} s^3 \mathbb{I}_{[0,1]}(s) + \left(-\frac{2}{3}s^3 + 4s - \frac{8}{3}\right) \mathbb{I}_{(1,2]}(s),$$

while, for $a = 2$ and $b = 3$, $c = 28$ and (using Maple!),

$$f_S(s) = \frac{77}{480} s^6 \mathbb{I}_{[0,1]}(s) + \mathbb{I}_{(1,2]}(s) \left(-\frac{28}{6} + \frac{28}{3}s^3 + \frac{84}{5}s - 21s^2 - \frac{49}{160}s^6\right).$$

These are plotted in Figure S-2.6.

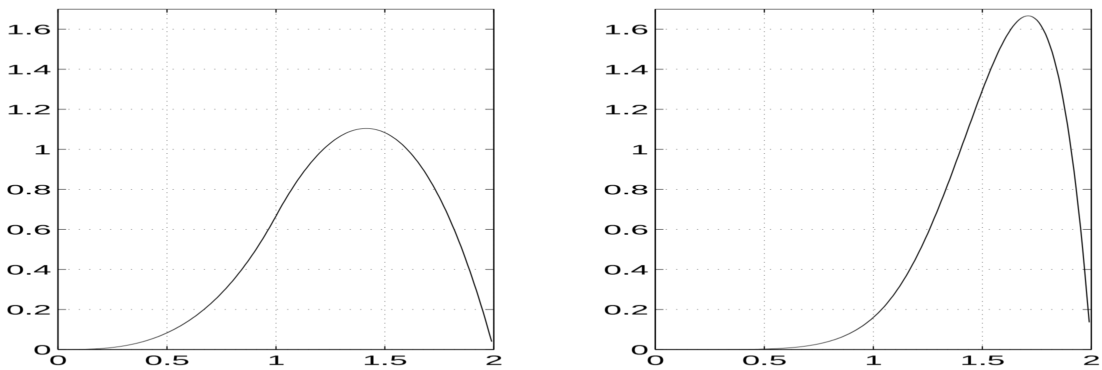

Figure S-2.6: Densities $f_S(s)$ for $a = b = 1$ (left) and $a = 2$, $b = 3$ (right)

Solution to Problem 2.15:

a) These are equal, as $\mathbb{E}\left[\bar{X}_n\right] = n^{-1} \sum_{i=1}^{n} \mathbb{E}[X_i] = \mu = \mathbb{E}[X_1]$.

b) These are not equal for $n > 1$, because

$$\begin{aligned}\mathbb{V}\left(\bar{X}_n\right) &= n^{-2} \left\{ \sum_{i=1}^{n} \mathbb{V}(X_i) + 2 \sum\sum_{i<j} \mathrm{Cov}(X_i, X_j) \right\} \\ &= n^{-2} \left\{ n\sigma_Y^2 + n\sigma_Z^2 + n(n-1)\sigma_Z^2 \right\} \\ &= \frac{1}{n}\sigma_Y^2 + \sigma_Z^2 \\ &\neq \frac{1}{n}\mathbb{V}(X_1) = \frac{1}{n}\left(\sigma_Y^2 + \sigma_Z^2\right).\end{aligned}$$

c) These are not equal. Adding a constant Z to the Y_i should not change their sample variance, so that $S_n^2(X)$ should equal $S_n^2(Y)$. Formally, noting that $\bar{X}_n = \bar{Y}_n + Z$,

$$\begin{aligned} S_n^2(X) &= \frac{1}{n-1}\sum\left(X_i - \bar{X}_n\right)^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left[Y_i + Z - \left(\bar{Y}_n + Z\right)\right]^2 \\ &= \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \bar{Y}_n\right)^2 = S_n^2(Y), \end{aligned}$$

so that $\mathbb{E}\left[S_n^2(X)\right] = \mathbb{E}\left[S_n^2(Y)\right] = \sigma_Y^2 \neq \mathbb{V}(X_1) = \sigma_Y^2 + \sigma_Z^2$.

Solution to Problem 2.16: Using the latter formula in (11.10) with $k = (r+1)/b$ and $z = ky$,

$$\begin{aligned} f_R(r) &= \frac{1}{\Gamma(a)\, b^{a+1}}\int_0^\infty y\,(ry)^{a-1}\exp\left(-\frac{ry+y}{b}\right)dy \\ &= \frac{r^{a-1}}{\Gamma(a)\, b^{a+1}}\int_0^\infty y^a \exp(-ky)\,dy \\ &= \frac{1}{\Gamma(a)}\frac{r^{a-1}}{(r+1)^{a+1}}\int_0^\infty z^a\exp(-z)\,dz \\ &= \frac{\Gamma(a+1)}{\Gamma(a)}\frac{r^{a-1}}{(r+1)^{a+1}} \\ &= \frac{ar^{a-1}}{(r+1)^{a+1}}\mathbb{I}_{(0,\infty)}(r). \end{aligned}$$

Solution to Problem 2.17: Similar to (12.31), from (7.42),

$$\mathbb{E}\left[S^{-2}\right] = \mathbb{E}\left[\left(\frac{\sigma^2}{n-1}\frac{(n-1)\,S^2}{\sigma^2}\right)^{-1}\right] = \frac{n-1}{\sigma^2}\mathbb{E}\left[U^{-1}\right] = \frac{1}{\sigma^2}\frac{n-1}{n-3}, \qquad n > 3.$$

It is straightforward to show that

$$\mathbb{V}\left(S_n^2\right) = n^{-1}\left(3\sigma^4 - \frac{n-3}{n-1}\sigma^4\right) = \frac{2\sigma^4}{n-1}, \tag{S-2.5}$$

so that, from (11.34) and using (S-2.5),

$$\mathbb{E}\left[\frac{1}{S^2}\right] \approx \frac{1}{\mathbb{E}\left[S^2\right]} + \frac{\mathbb{V}\left(S^2\right)}{\left(\mathbb{E}\left[S^2\right]\right)^3} = \frac{1}{\sigma^2} + \frac{2\sigma^4/(n-1)}{\sigma^6} = \frac{1}{\sigma^2}\frac{n+1}{n-1}.$$

The expressions are nearly equal. The relative error of the approximation is

$$\frac{\frac{n+1}{n-1} - \frac{n-1}{n-3}}{\frac{n-1}{n-3}} = -\frac{4}{(n-1)^2},$$

which converges to zero quite fast.

Solution to Problem 2.18: As in Chen and Adatia (1997), straightforward calculation shows that the joint density of U and V is

$$f_{U,V}(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{u^2 - 2\rho u\,|u|\,v + u^2v^2}{2(1-\rho^2)}\right\} |u|,$$

and

$$f_V(v) = \frac{\sqrt{1-\rho^2}}{2\pi}\left\{\frac{1}{v^2-2\rho v+1} + \frac{1}{v^2+2\rho v+1}\right\}.$$

Clearly,

$$f_V(v) = \frac{1}{\pi}\frac{1}{1+v^2} \quad \text{iff} \quad \rho = 0,$$

showing the necessity of independence.

Solution to Problem 2.19: For (11.45), the integral is

$$\int_{N_n+\frac{1}{2}}^{\infty} \frac{dt}{(s+t)^{n+1}} = \int_{s+N_n+\frac{1}{2}}^{\infty} u^{-n-1}du = -\frac{1}{n}u^{-n}\Big|_{s+N_n+\frac{1}{2}}^{\infty} = \frac{1}{n}\left(s+N_n+\tfrac{1}{2}\right)^{-n}.$$

Similarly, for (11.44), substituting $u = 1 + (s-1)/t$ leads to

$$\int_{N_0+\frac{1}{2}}^{\infty} \frac{s-1}{t(t+s-1)}dt = \int_{N_0+\frac{1}{2}}^{\infty} \frac{(s-1)/t^2}{1+(s-1)/t}dt = \ln\left(1+\frac{s-1}{N_0+\frac{1}{2}}\right),$$

so that

$$\psi(s) \approx -\gamma + (s-1)\sum_{k=1}^{N_0} k^{-1}(k+s-1)^{-1} + \ln\left|\frac{N_0+s-0.5}{N_0+0.5}\right|.$$

Solutions to Chapter 3: The Multivariate Normal Distribution

Solution to Problem 3.1: Being linear combinations of normals, both D and S are normally distributed, so that D and S are independent iff $\mathrm{Cov}(D,S)=0$. But

$$\begin{aligned}\mathrm{Cov}(D,S) &= \mathbb{E}\left[(X-Y-\mu_X+\mu_Y)(X+Y-\mu_X-\mu_Y)\right]\\ &= \mathbb{E}\left[X^2-2X\mu_X-Y^2+2Y\mu_Y+\mu_X^2-\mu_Y^2\right]\\ &= \mathbb{E}\left[X^2\right]-\mu_X^2-\mathbb{E}\left[Y^2\right]+\mu_Y^2\\ &= \mathbb{V}(X)-\mathbb{V}(Y),\end{aligned}$$

so that D and S are independent iff $\mathbb{V}(X)=\mathbb{V}(Y)$.

Solution to Problem 3.2: Letting $Z\sim \mathrm{N}(0,1)$, a location–scale transformed normal r.v. is of the form $\sigma Z+\mu$, i.e., the scaling factor σ is applied to a location zero r.v., and then μ is added on. With σY, this is not the case, i.e., Y does not have location zero. The theory of normal r.v.s says that σY is normally distributed with mean $\mathbb{E}[\sigma Y]=\sigma\mathbb{E}[Y]=\sigma\mu$ and variance $\sigma^2\mathbb{V}(Y)=\sigma^2$, so that the mean is indeed $\sigma\mu$ and not μ. This can also be confirmed directly via transformation: with $S=\sigma Y$, $y=s/\sigma$, $dy/ds=1/\sigma$ and

$$f_S(s)=f_Y(y)\frac{dy}{ds}=\frac{1}{\sigma}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}(s/\sigma-\mu)^2\right\}=\frac{1}{\sigma}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{s-\mu\sigma}{\sigma}\right)^2\right\},$$

i.e., $S\sim \mathrm{N}(\mu\sigma,\sigma^2)$.

Solution to Problem 3.3: $\Pr(X_1>X_2)=\Pr(X_1-X_2>0)=\Pr(Y>0)$, where

$$Y=X_1-X_2\sim \mathrm{N}(3-2,1+2-2(-0.4))$$

or $Y\sim \mathrm{N}(1,3.8)$. Thus,

$$\Pr(Y>0)=\Pr\left(Z>\frac{0-1}{\sqrt{3.8}}\right)\approx 1-\Phi(-.513)\approx 0.696.$$

Solution to Problem 3.4: Let $L_i = X_i = \mathbf{a}_i\mathbf{X}$, $i = 1, \ldots, n$, where $\mathbf{a}_i = \mathbf{e}_i$ (i.e., the zero vector with a 1 in the i^{th} place) and $L_{n+1} = \bar{X} = n^{-1}\sum_{i=1}^{n} X_i = \mathbf{a}'_{n+1}\mathbf{X}$, where $\mathbf{a}_{n+1} = (n^{-1}, n^{-1}, \ldots, n^{-1})'$. Then, with $\bar{\mu} = n^{-1}\sum_{i=1}^{n}\mu_i$ and

$$\boldsymbol{\Omega} = \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}', \qquad \mathbf{A} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ n^{-1} & n^{-1} & \cdots & n^{-1} & n^{-1} \end{bmatrix},$$

we have

$$\mathbf{L} = \begin{pmatrix} \mathbf{X} \\ L_{n+1} \end{pmatrix} = \begin{pmatrix} L_1 \\ \vdots \\ L_n \\ L_{n+1} \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \boldsymbol{\mu} \\ \bar{\mu} \end{pmatrix}, \boldsymbol{\Omega} \right),$$

whereby $\mathbf{A}$ and, thus, $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'$ have rank n. Partitioning $\boldsymbol{\Omega}$ as

$$\boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{\Omega}_{11} & \boldsymbol{\Omega}_{12} \\ \boldsymbol{\Omega}_{21} & \omega \end{pmatrix},$$

where $\boldsymbol{\Omega}_{11}$ is $n \times n$, from (12.22), the conditional distribution of $\mathbf{X}$ given $\bar{X}$ is

$$\left(\mathbf{X} \mid \bar{X} = m\right) \sim \mathrm{N}\left(\mathbf{v}, \mathbf{V}\right),$$

where $\mathbf{v} = \boldsymbol{\mu} + \boldsymbol{\Omega}_{12}\left(m - \bar{\mu}\right)/\omega$ and $\mathbf{V} = \boldsymbol{\Omega}_{11} - \boldsymbol{\Omega}_{12}\boldsymbol{\Omega}'_{12}/\omega$.

If the X_i are iid, i.e., $\boldsymbol{\mu} = \mu\mathbf{1}_n$ and $\boldsymbol{\Sigma} = \mathbf{I}$, then

$$\boldsymbol{\Omega} = \mathbf{A}\mathbf{A}' = \begin{pmatrix} \mathbf{I}_n & n^{-1}\mathbf{1}_n \\ n^{-1}\mathbf{1}'_n & n^{-1} \end{pmatrix},$$

$\mathbf{v} = \mu\mathbf{1}_n + \mathbf{1}_n\left(m - \mu\right) = m\mathbf{1}_n$ and $\mathbf{V} = \mathbf{I}_n - n^{-1}\mathbf{1}_{n\times n}$, which is rank $n-1$. That is, conditional on $\bar{X} = m$, $X_i \sim \mathrm{N}\left(m, 1 - 1/n\right)$ and $\mathrm{Cov}\left(X_i, X_j\right) = -n^{-1}$, $i \neq j$.

Solution to Problem 3.5:

a) As X_t is the sum of the constant $c + ax_{t-1}$ and the normal r.v. U_t, it must be normally distributed with mean $c + ax_{t-1}$ plus the mean of U_t, and variance the same as that of U_t, i.e., $X_t \mid \left(X_{t-1} = x_{t-1}\right) \sim \mathrm{N}\left(c + ax_{t-1}, \sigma^2\right)$.

b) Taking expectations of both sides of (12.35) gives $\mu = c + a\mu + 0$, or $\mu = c/\left(1 - a\right)$, which exists when a is constrained as specified in the problem, i.e., $a \in (-1, 1)$. Taking the variance of both sides of (12.35) gives

$$v = \mathbb{V}\left(aX_{t-1} + U_t\right) = a^2 v + \sigma^2$$

or $v = \sigma^2/\left(1 - a^2\right)$.

c) Substituting gives

$$\begin{aligned} X_t &= c + aX_{t-1} + U_t \\ &= c + a\left(c + aX_{t-2} + U_{t-1}\right) + U_t \\ &= c + ac + a^2 X_{t-2} + aU_{t-1} + U_t \\ &= c + ac + a^2\left(c + aX_{t-3} + U_{t-2}\right) + aU_{t-1} + U_t \\ &= c + ac + a^2 c + a^3 X_{t-3} + a^2 U_{t-2} + aU_{t-1} + U_t \\ &\quad \vdots \\ &= \lim_{h\to\infty}\left(c\sum_{i=0}^{h} a^i + \sum_{i=0}^{h} a^i U_{t-i} + a^h X_{t-h}\right) \\ &= \frac{c}{1-a} + \sum_{i=0}^{\infty} a^i U_{t-i}\ . \end{aligned}$$

From this,

$$\mathbb{E}\left[X_t\right] = \frac{c}{1-a} + \sum_{i=0}^{\infty} a^i \mathbb{E}\left[U_{t-i}\right] = \frac{c}{1-a}$$

and, from the iid–ness of the U_t,

$$\mathbb{V}\left(X_t\right) = \mathbb{V}\left(\sum_{i=0}^{\infty} a^i U_{t-i}\right) = \sum_{i=0}^{\infty} a^{2i}\mathbb{V}\left(U_{t-i}\right) = \frac{\sigma^2}{1-a^2},$$

both of which agree with the results obtained above.

d) As the U_t are iid with zero mean and unit variance,

$$\mathbb{E}\left[X_t Y_t\right] = \mathbb{E}\left[\left(\sum_{i=0}^{\infty} a^i U_{t-i}\right)\left(\sum_{j=0}^{\infty} b^j U_{t-j}\right)\right] = \sum_{i=0}^{\infty} (ab)^i = \frac{1}{1-ab}.$$

This is valid only for $|ab| < 1$.

Solution to Problem 3.6: Using the fact that the (multivariate) normal is characterized by its mgf,

$$\begin{aligned} \mathbb{M}_{\mathbf{X}}(\mathbf{t}) &= \mathbb{M}_{\mathbf{a}+\mathbf{B}\mathbf{Y}}(\mathbf{t}) = \mathbb{E}\left[\exp\left\{\mathbf{t}'\left(\mathbf{a} + \mathbf{B}\mathbf{Y}\right)\right\}\right] = e^{\mathbf{t}'\mathbf{a}}\mathbb{M}_{\mathbf{Y}}(\mathbf{B}'\mathbf{t}) \\ &= e^{\mathbf{t}'\mathbf{a}}\exp\left\{\left(\mathbf{B}'\mathbf{t}\right)'\boldsymbol{\mu} + \tfrac{1}{2}\left(\mathbf{B}'\mathbf{t}\right)'\boldsymbol{\Sigma}\left(\mathbf{B}'\mathbf{t}\right)\right\} \\ &= \exp\left\{\mathbf{t}'\boldsymbol{\nu} + \tfrac{1}{2}\mathbf{t}'\boldsymbol{\Omega}\mathbf{t}\right\}, \end{aligned} \qquad \text{(S-3.1)}$$

so that $\mathbf{X} \sim \mathrm{N}\left(\boldsymbol{\nu}, \boldsymbol{\Omega}\right)$.

Solution to Problem 3.7: The given condition on $\boldsymbol{\Sigma}$ implies that there exists $\mathbf{a} \in \mathbb{R}^n$ such that $\mathbf{a} \neq \mathbf{0}$ and $\boldsymbol{\Sigma}\mathbf{a} = \mathbf{0}$. Thus, r.v. $\mathbf{X} = \mathbf{a}'\left(\mathbf{Y} - \boldsymbol{\mu}\right)$ has mean zero and variance $\mathbf{a}'\boldsymbol{\Sigma}\mathbf{a} = 0$, i.e., the *degenerate* r.v. $\mathbf{X}$ is always zero.

Solution to Problem 3.8: Let $\sigma^2 = 1$ without loss of generality. The eigenvalues of $\mathbf{\Sigma}$ are given by the solution to

$$\det(\mathbf{\Sigma} - \lambda \mathbf{I}) = 0.$$

Recall the fact that the determinant does not change by adding a multiple of a column to another column (same for rows), and that multiplying a column or row by a constant C changes the determinant by the same factor. So, (i) adding to the first column each of the remaining columns, (ii) pulling out the factor in the first column, and (iii) subtracting the first row from the remaining rows gives

$$\begin{aligned}
\det(\mathbf{\Sigma} - \lambda \mathbf{I}) &= \begin{vmatrix} 1-\lambda & \rho & \cdots & \rho \\ \rho & 1-\lambda & & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1-\lambda \end{vmatrix} \\
&= \begin{vmatrix} 1-\lambda+(n-1)\rho & \rho & \cdots & \rho \\ 1-\lambda+(n-1)\rho & 1-\lambda & & \rho \\ \vdots & \vdots & \ddots & \vdots \\ 1-\lambda+(n-1)\rho & \rho & \cdots & 1-\lambda \end{vmatrix} \\
&= (1-\lambda+(n-1)\rho) \begin{vmatrix} 1 & \rho & \cdots & \rho \\ 1 & 1-\lambda & & \rho \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \rho & \cdots & 1-\lambda \end{vmatrix} \\
&= (1-\lambda+(n-1)\rho) \begin{vmatrix} 1 & \rho & \cdots & \rho \\ 0 & 1-\lambda-\rho & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-\lambda-\rho \end{vmatrix} \\
&= (1-\lambda+(n-1)\rho)(1-\lambda-\rho)^{n-1}.
\end{aligned}$$

Setting this to zero shows that the eigenvalues of $\mathbf{\Sigma}$ are $1+\rho(n-1)$, of multiplicity one, and $1-\rho$, of multiplicity $n-1$. Thus, $\mathbf{\Sigma}$ is positive definite if $1+\rho(n-1) > 0$ and $1-\rho > 0$, or

$$-\frac{1}{n-1} < \rho < 1.$$

Solution to Problem 3.9:

a) Let $g_1 : \mathbb{R}^2 \to \mathbb{R}$ be given by $g_1(x,y) = (x-\mu_1)/\sigma_1$ and, similarly, $g_2(x,y) = (y-\mu_2)/\sigma_2$. Then $U = g_1(X,Y)$, with $X = g_1^{-1}(U,V) = \sigma_1 U + \mu_1$, and $V = g_2(X,Y)$, with $Y = g_2^{-1}(U,V) = \sigma_2 V + \mu_2$. A multivariate transformation (9.1) involves the Jacobian

$$\mathbf{J} = \begin{bmatrix} \frac{\partial g_1^{-1}(u,v)}{\partial u} & \frac{\partial g_1^{-1}(u,v)}{\partial v} \\ \frac{\partial g_2^{-1}(u,v)}{\partial u} & \frac{\partial g_2^{-1}(u,v)}{\partial v} \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix},$$

with $|\det \mathbf{J}| = \sigma_1\sigma_2$ and $f_{U,V}(u,v) = f_{X,Y}(x,y)\ \ |\det \mathbf{J}|$, which easily leads to the stated result.

b) We have

$$\begin{aligned}\mathbb{M}_{U,V}(s,t) &= E\left[\exp\left(sU+tV\right)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{su+tv} f_{U,V}\left(u,v\right) dudv \\ &= \frac{1}{2\pi\left(1-\rho^2\right)^{1/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{T\left(u,v\right)\right\} dudv,\end{aligned}$$

where the term in the exponent is

$$T\left(u,v\right) = su + tv - \frac{u^2 - 2\rho uv + v^2}{2\left(1-\rho^2\right)} = A\left(u,v\right) + B\left(v\right),$$

where

$$A\left(u,v\right) = -\frac{\left[u - \left(\rho v + s\left(1-\rho^2\right)\right)\right]^2}{2\left(1-\rho^2\right)}$$

and

$$\begin{aligned}B\left(v\right) &= \frac{\left(\rho v + s\left(1-\rho^2\right)\right)^2}{2\left(1-\rho^2\right)} - \frac{v^2}{2\left(1-\rho^2\right)} + tv \\ &= -\frac{v^2}{2} + \rho sv + tv + \frac{s^2\left(1-\rho^2\right)}{2} \\ &= -\frac{1}{2}\left(v - \left(t+\rho s\right)\right)^2 + \frac{\left(t+\rho s\right)^2}{2} + \frac{s^2\left(1-\rho^2\right)}{2}.\end{aligned}$$

Thus,

$$\mathbb{M}_{U,V}(s,t) = \frac{1}{2\pi\left(1-\rho^2\right)^{1/2}}\int_{-\infty}^{\infty} B\left(v\right)\int_{-\infty}^{\infty}\exp\left\{A\left(u,v\right)\right\} dudv$$

and the integral

$$H\left(v\right) = \int_{-\infty}^{\infty}\exp\left\{A\left(u,v\right)\right\} du = \frac{\left(1-\rho^2\right)^{1/2}}{\left(1-\rho^2\right)^{1/2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\left(\frac{u - \left(\rho v + s\left(1-\rho^2\right)\right)}{\left(1-\rho^2\right)^{1/2}}\right)^2\right\} du$$

is, except for the constant of integration, the integral of a $N\left(\left(\rho v + s\left(1-\rho^2\right)\right), \left(1-\rho^2\right)^{1/2}\right)$ random variable. Thus,

$$H\left(v\right) = \sqrt{2\pi}\left(1-\rho^2\right)^{1/2}, \quad \text{i.e.,} \quad M_{U,V}\left(s,t\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} B\left(v\right) dv$$

or

$$\begin{aligned}\mathbb{M}_{U,V}(s,t) &= \exp\left\{\frac{\left(t+\rho s\right)^2}{2} + \frac{s^2\left(1-\rho^2\right)}{2}\right\}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\left(v-\left(t+\rho s\right)\right)^2\right\} dv \\ &= \exp\left\{\frac{\left(t+\rho s\right)^2}{2} + \frac{s^2\left(1-\rho^2\right)}{2}\right\} = \exp\left\{\frac{1}{2}\left(s^2 + 2\rho st + t^2\right)\right\},\end{aligned}$$

as was to be shown.

Next,

$$\begin{aligned}
\mathbb{M}_{X,Y}(s,t) &= \mathbb{E}\left[\exp\left(sX+tY\right)\right] = \mathbb{E}\left[\exp\left(s\left(\mu_1+\sigma_1 U\right)+t\left(\mu_2+\sigma_2 V\right)\right)\right] \\
&= e^{s\mu_1+t\mu_2}\mathbb{M}_{U,V}(\sigma_1 s,\sigma_2 t) \\
&= \exp\left\{\frac{1}{2}\left[\sigma_1^2 s^2+2\rho\sigma_1\sigma_2 st+\sigma_2^2 t^2\right]+\mu_1 s+\mu_2 t\right\}.
\end{aligned}$$

c) From the hint, we use

$$\mathbb{E}\left[XY\right] = \left.\frac{\partial^2\mathbb{M}_{X,Y}(s,t)}{\partial s\partial t}\right|_{s=t=0}.$$

Using Maple for help, it is easily seen that $\mathbb{E}\left[XY\right]=\mu_1\mu_2+\rho\sigma_1\sigma_2$, so that

$$\operatorname{Cov}\left(X,Y\right)=\mathbb{E}\left[XY\right]-\mathbb{E}\left[X\right]\mathbb{E}\left[Y\right]=\rho\sigma_1\sigma_2$$

from which the correlation follows.

d) They are independent if $\mathbb{M}_{X,Z}(s,t)=\mathbb{M}_X(s)\,\mathbb{M}_Z(t)$. We have, using the above expression for $\mathbb{M}_{X,Y}(s,t)$,

$$\begin{aligned}
\mathbb{M}_{X,Z}(s,t) &= \mathbb{E}\left[\exp\left\{sX+t\left(Y-\frac{\rho\sigma_2}{\sigma_1}X\right)\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{tY+\left(s-t\frac{\rho\sigma_2}{\sigma_1}\right)X\right\}\right] \\
&= \mathbb{M}_{X,Y}\left(s-t\frac{\rho\sigma_2}{\sigma_1},t\right) \\
&= \exp\left\{\begin{array}{c}\frac{1}{2}\left[\sigma_1^2\left(s-t\frac{\rho\sigma_2}{\sigma_1}\right)^2+2\rho\sigma_1\sigma_2\left(s-t\frac{\rho\sigma_2}{\sigma_1}\right)t+\sigma_2^2t^2\right] \\ +\mu_1\left(s-t\frac{\rho\sigma_2}{\sigma_1}\right)+\mu_2 t\end{array}\right\}.
\end{aligned}$$

Now recall that the m.g.f. of a $\mathrm{N}\left(\mu,\sigma^2\right)$ r.v. is $\exp\left\{\mu t+\frac{1}{2}\sigma^2t^2\right\}$. As $Z = Y-\rho\sigma_2 X/\sigma_1 \sim \mathrm{N}\left(\left(\mu_2-\rho\frac{\sigma_2}{\sigma_1}\mu_1\right),\left(1-\rho^2\right)\sigma_2^2\right)$, its m.g.f. is

$$\mathbb{M}_Z(t)=\exp\left\{\left(\mu_2-\rho\frac{\sigma_2}{\sigma_1}\mu_1\right)t+\frac{1}{2}\left(1-\rho^2\right)\sigma_2^2t^2\right\}.$$

Factoring the term in the exponent in the above expression for $\mathbb{M}_{X,Z}(s,t)$ yields

$$\begin{aligned}
\mathbb{M}_{X,Z}(s,t) &= \exp\left\{\mu_1 s+\frac{1}{2}\sigma_1^2s^2\right\} \\
&\quad\times\exp\left\{\left(\mu_2-\rho\frac{\sigma_2}{\sigma_1}\mu_1\right)t+\frac{1}{2}\left(1-\rho^2\right)\sigma_2^2t^2\right\},
\end{aligned}$$

so that $\mathbb{M}_{X,Z}(s,t)=\mathbb{M}_X(s)\,\mathbb{M}_Z(t)$, as was to be shown.

Solutions to Chapter 4: Convergence Concepts

Solution to Problem 4.1: Write

$$\frac{\theta-1}{\theta+1} = 1 - \frac{2}{1+\theta}$$

and note that, in general, using the substitution $\zeta = \theta + k$,

$$\begin{aligned}\lim_{\theta\to\infty}\left(1+\frac{\lambda}{k+\theta}\right)^{\theta} &= \lim_{\zeta\to\infty}\left(1+\frac{\lambda}{\zeta}\right)^{(\zeta-k)} = \lim_{\zeta\to\infty}\left(\left(1+\frac{\lambda}{\zeta}\right)^{\zeta}\times\left(1+\frac{\lambda}{\zeta}\right)^{-k}\right)\\ &= \lim_{\zeta\to\infty}\left(1+\frac{\lambda}{\zeta}\right)^{\zeta}\times\lim_{\zeta\to\infty}\left(1+\frac{\lambda}{\zeta}\right)^{-k} = e^{\lambda}\times 1,\end{aligned}$$

it follows that $S = e^{-2}$.

Solution to Problem 4.2: This follows from Hölder's inequality by setting $p = s/r$, $Y \equiv 1$ and substituting $Z = |X|^{1/r}$.

Solution to Problem 4.3: For $t > 0$,

$$\Pr(X \geq c) = \Pr\left(e^{t(X-c)} \geq 1\right) \leq \mathbb{E}\left[e^{t(X-c)}\right]$$

from Markov's inequality. As this holds for any $t > 0$, one would take the infimum of the right hand side with respect to t, which is (13.14).

The bound follows because

$$\begin{aligned}\mathbb{E}\left[e^{t\left(\bar{X}_n-c\right)}\right] &= \mathbb{E}\left[e^{-tc}e^{\frac{t}{n}\sum_{i=1}^{n}X_i}\right]\\ &= \mathbb{M}^n\left(\frac{t}{n}\right)e^{-tc}\\ &= \exp\left(n\log\mathbb{M}\left(\frac{t}{n}\right)-tc\right).\end{aligned}$$

Solution to Problem 4.4: Observe that $\mathbb{E}\left[\bar{X}_n\right] = p$ so $c > p$. Using the fact that $\exp(\cdot)$ is a monotonic function, it follows that

$$\inf_{s>0}\exp\left(n\log\left(1-p+pe^{s/n}\right)\right) = \exp\left(n\inf_{s>0}\left[\log\left(1-p+pe^{s/n}\right)-\frac{cs}{n}\right]\right).$$

Now, define

$$g(u) := \log \mathbb{M}(u) - cu = \log(1 - p + pe^u) - cu,$$

set $\partial g / \partial u = 0$, and then confirm that the value of u which minimizes the function $g(u)$ is given by

$$u^* = \log\left(\frac{c(1-p)}{p(1-c)}\right).$$

Next, show that

$$c > p \Leftrightarrow \frac{c(1-p)}{p(1-c)} > 1,$$

so that $u^* > 0$, which implies that $s > 0$, as required. Now, $\inf_{s>0}$ of $g(u)$ is $g(u^*)$, so simplifying $g(u^*)$ shows that the bound is then $\exp(n \cdot g(u^*))$.

One could examine the difference between the lower (Chernoff) bound of $\Pr\left(\bar{X}_n \geq c\right)$ and the true value,

$$\Pr\left(\bar{X}_n \geq c\right) = 1 - \Pr(X < nc) = \bar{F}_X(nc - 1; n, p)$$

for $n = 20$ and $p = 0.05$, for values $c = \frac{4}{20}, \frac{5}{20}, \ldots, \frac{19}{20}$, where $X \sim \text{Bin}(n, p)$. For $c = \frac{2}{20}$ and $\frac{3}{20}$, the difference is 0.333 and 0.135, respectively. As c increases, the difference is virtually zero, i.e., the Chernoff bound is almost exact. For example, by $c = \frac{9}{20}$, the difference is less than 1×10^{-6}.

Solution to Problem 4.5: $\mathbb{M}_X(t) = \mathbb{E}\left[e^{tX}\right] = e^{\lambda\left(e^t - 1\right)}$, so that, defining

$$g(t) = \log \mathbb{M}_X(t) - ct = \lambda\left(e^t - 1\right) - ct$$

and setting $g'(t)$ to zero shows that $t = \log(c/\lambda)$ gives the minimum. Thus, the bound is given by

$$\begin{aligned}
\Pr\left(\bar{X} \geq c\right) &\leq \exp\left\{ng\left(\log\frac{c}{\lambda}\right)\right\} \\
&= \exp\left\{n\left[\lambda\left(e^{\log\frac{c}{\lambda}} - 1\right) - c\log\left(\frac{c}{\lambda}\right)\right]\right\} \\
&= \exp\left\{n\left[c - \lambda - c\log c + c\log\lambda\right]\right\}
\end{aligned}$$

and, to ensure $t > 0$, the bound is valid for $c > \lambda$.

Solution to Problem 4.6: In the notation of Example 13.12 but letting d_X and d_Y be arbitrary r.v.s, (13.46) and that $C \subset \{A \cup B\}$ implies

$$\Pr(|d_X \pm d_Y| > \epsilon) \leq \Pr\left(|d_X| > \frac{\epsilon}{2}\right) + \Pr\left(|d_Y| > \frac{\epsilon}{2}\right). \qquad \text{(S-4.1)}$$

For (i), letting $d_X = X - X_n$ and $d_Y = Y - X_n$, (S-4.1) implies, in the limit as $n \to \infty$,

$$\Pr(|X - Y| > \epsilon) \leq \Pr\left(|X - X_n| > \frac{\epsilon}{2}\right) + \Pr\left(|Y - X_n| > \frac{\epsilon}{2}\right) \to 0,$$

so that $\Pr(X = Y) = 1$.

For (ii), let $d_X = X_n - X$ and $d_Y = X_m - X$. Then (S-4.1) implies, in the limit as $n, m \to \infty$,

$$\Pr\left(|X_n - X_m| > \epsilon\right) \leq \Pr\left(|X_n - X| > \frac{\epsilon}{2}\right) + \Pr\left(|X_m - X| > \frac{\epsilon}{2}\right) \to 0.$$

Solution to Problem 4.7: Observe that $\sum_{j=1}^{k} Y_j \geq \epsilon$ implies $Y_j \geq \epsilon/k$ for at least one $j \in \{1, \ldots, k\}$. Thus, from (13.25) and the fact that $\Pr(\cdot) \geq 0$,

$$\Pr\left(\sum_{j=1}^{k} Y_j \geq \epsilon\right) \leq \Pr\left(\exists j \text{ such that } Y_j \geq \frac{\epsilon}{k}\right) \leq \sum_{j=1}^{k} \Pr\left(Y_j \geq \frac{\epsilon}{k}\right).$$

Solution to Problem 4.8: We wish to show that $\mathbf{X}_n \xrightarrow{p} \mathbf{X} \Leftrightarrow X_{nj} \xrightarrow{p} X_j$, $j = 1, 2, \ldots, k$. From the hint, we will use the fact that, for any $\mathbf{z} \in \mathbb{R}^m$,

$$|z_i| \leq \|\mathbf{z}\| \leq |z_1| + \cdots + |z_m|, \quad i = 1, \ldots, m. \tag{S-4.2}$$

($\Rightarrow$) Assume $\mathbf{X}_n \xrightarrow{p} \mathbf{X}$. Then, from the first inequality in (S-4.2),

$$|X_{nj} - X_j| \leq \|\mathbf{X}_n - \mathbf{X}\|,$$

so that, for any $\epsilon > 0$,

$$\Pr(|X_{nj} - X_j| \geq \epsilon) \leq \Pr(\|\mathbf{X}_n - \mathbf{X}\| \geq \epsilon),$$

and taking limits,

$$\lim_{n\to\infty} \Pr(|X_{nj} - X_j| \geq \epsilon) \leq \lim_{n\to\infty} \Pr(\|\mathbf{X}_n - \mathbf{X}\| \geq \epsilon) = 0.$$

($\Leftarrow$) Assume $X_{nj} \xrightarrow{p} X_j$, $j = 1, 2, \ldots, k$. From the second inequality in (S-4.2),

$$\|\mathbf{X}_n - \mathbf{X}\| \leq \sum_{j=1}^{k} |X_{nj} - X_j|,$$

so that, for any $\epsilon > 0$,

$$\Pr(\|\mathbf{X}_n - \mathbf{X}\| \geq \epsilon) \leq \Pr\left(\sum_{j=1}^{k} |X_{nj} - X_j| \geq \epsilon\right). \tag{S-4.3}$$

Observe that, for ϵ small enough, it is true that, for a set of nonnegative r.v.s $Y_1, \ldots, Y_k$,

$$\Pr(Y_1 + \cdots + Y_k > \epsilon) < \Pr(Y_1 > \epsilon) + \cdots + \Pr(Y_k > \epsilon),$$

because the l.h.s. approaches one in the limit as $\epsilon \to 0$, while the r.h.s. approaches k. Thus, using a small enough $\epsilon > 0$ gives

$$\Pr\left(\sum_{j=1}^{k} |X_{nj} - X_j| \geq \epsilon\right) \leq \sum_{j=1}^{k} \Pr\left(|X_{nj} - X_j| \geq \epsilon\right).$$

Alternatively, and better (because we do not need to restrict ϵ), from (13.74),

$$\Pr\left(\sum_{j=1}^{k} |X_{nj} - X_j| \geq \epsilon\right) \leq \sum_{j=1}^{k} \Pr\left(|X_{nj} - X_j| \geq \frac{\epsilon}{k}\right). \tag{S-4.4}$$

So, (S-4.4) with (S-4.3), taking limits, and using the assumed fact that $X_{nj} \xrightarrow{p} X_j$, $j = 1, 2, \ldots, k$, gives, for any $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr(\|\mathbf{X}_n - \mathbf{X}\| \geq \epsilon) \leq \sum_{j=1}^{k} \lim_{n\to\infty} \Pr\left(|X_{nj} - X_j| \geq \frac{\epsilon}{k}\right) = 0.$$

Solution to Problem 4.9: From (13.4), $\mathbb{E}\big[|U + V|^r\big] \leq 2^r \left(\mathbb{E}\big[|U|^r\big] + \mathbb{E}\big[|V|^r\big]\right)$ for r.v.s U and V, and with $U = X - X_n$ and $V = X_n - Y$, this reads

$$\mathbb{E}\big[|X - Y|^r\big] \leq 2^r \left(\mathbb{E}\big[|X - X_n|^r\big] + \mathbb{E}\big[|X_n - Y|^r\big]\right).$$

By assumption, the r.h.s. converges to zero in the limit, so that $\mathbb{E}\big[|X - Y|^r\big] = 0$. If X and Y have joint p.d.f. $f_{X,Y}$ and support $\mathcal{S}_{X,Y}$, this implies that

$$0 = \mathbb{E}\big[|X - Y|^r\big] = \iint_{\mathcal{S}_{X,Y}} |x - y|^r f_{X,Y}(x, y)\,dxdy.$$

For $(x, y) \in \mathcal{S}_{X,Y}$ with $x \neq y$, as $|x - y|^r > 0$, it must be the case that (except on a set in $\mathbb{R}^2$ of measure zero), $f_{X,Y}(x, y) = 0$. Thus, $f_{X,Y}(x, y)$ is positive only for $(x, y) \in \mathcal{S}_{X,Y}$ and $x = y$. That is, $\Pr(X \neq Y) = 0$, or $\Pr(X = Y) = 1$.

To show that $X_n + Y_n \xrightarrow{r} X + Y$ if $X_n \xrightarrow{r} X$ and $Y_n \xrightarrow{r} Y$, again from (13.4), with $U = X_n - X$ and $V = Y_n - Y$,

$$\mathbb{E}\big[|(X_n + Y_n) - (X + Y)|^r\big] \leq 2^r \left(\mathbb{E}\big[|X_n - X|^r\big] + \mathbb{E}\big[|Y_n - Y|^r\big]\right),$$

and the r.h.s. converges to zero in the limit.

Solution to Problem 4.10: To show (13.76), i.e.,

$$X_n \xrightarrow{c.c.} X \quad \Rightarrow \quad X_n \xrightarrow{a.s.} X,$$

we use the first Borel-Cantelli lemma (13.38), which states that, for a sequence $\{A_n\}$ of arbitrary events,

$$\sum\nolimits_{n=1}^{\infty} \Pr(A_n) < \infty \quad \Rightarrow \quad \Pr(A_n \text{ i.o.}) = 0,$$

and (13.54), which states that, with $A_n = |X_n - X| > \epsilon$,

$$X_n \overset{a.s.}{\to} X \quad \Leftrightarrow \quad \Pr(A^*) = \lim_{m\to\infty} \Pr\left(\bigcup_{n=m}^{\infty} A_n\right) = 0. \qquad \text{(S-4.5)}$$

The result follows from these and the definition of complete convergence.

To show (13.77), i.e.,

$$X_n \overset{c.c.}{\to} c \quad \Leftrightarrow \quad X_n \overset{a.s.}{\to} c \qquad \text{if the } X_n \text{ are independent,}$$

note that, in light of the previous result, we only need to confirm that

$$X_n \overset{c.c.}{\to} c \quad \Leftarrow \quad X_n \overset{a.s.}{\to} c \qquad \text{if the } X_n \text{ are independent.}$$

From (S-4.5), $X_n \overset{a.s.}{\to} c$ implies that $\Pr(A^*) = 0$. Recall the second Borel-Cantelli lemma (13.39), which states that, for a sequence $\{A_n\}$ of independent events,

$$\sum_{n=1}^{\infty} \Pr(A_n) = \infty \quad \Rightarrow \quad \Pr(A_n \text{ i.o.}) = 1.$$

The contrapositive of this is $\Pr(A_n \text{ i.o.}) \neq 1 \Rightarrow \sum_{n=1}^{\infty} \Pr(A_n) < \infty$, which, with $A_n = |X_n - X| > \epsilon$, is the definition of complete convergence.

Note that, even if the X_n are independent r.v.s, the $X_n - X$ are *not* independent, which is why a constant c is used.

Solution to Problem 4.11: Prove the contrapositive, i.e., $A^c \cap B^c$ implies C^c. In particular, if $|d_X| \leq \epsilon/2$ $(= A^c)$ and $|d_Y| \leq \epsilon/2$ $(= B^c)$, then $|d_X| + |d_Y| \leq \epsilon$ $(= C^c)$.

Solution to Problem 4.12: As in Example 13.26, $X_2/n_2 \overset{p}{\to} 1$. Thus, from Slutsky's theorem (13.65),

$$n_1 F_{n_1,n_2} = n_1 \frac{X_1/n_1}{X_2/n_2} = \frac{X_1}{X_2/n_2} \overset{d}{\to} \chi^2_{n_1}\ .$$

Solution to Problem 4.13:

a) To have length ℓ, $X_{\ell+1}$ has to differ from the common value of the previous $X_1, \ldots, X_\ell$. This, together with the fact that, with $p = 1/2$, the value of X_1 is not important, the length of the first run is geometrically distributed with $\Pr(L_1 = \ell) = 2^{-\ell}$.

b) For $p \neq 1/2$, conditioning on X_1 gives

$$\begin{aligned} \Pr(L_1 = \ell) &= \Pr(L_1 = \ell \mid X_1 = 0)\, p + \Pr(L_1 = \ell \mid X_1 = 1)\,(1-p) \\ &= p^{\ell-1}(1-p)\,p + (1-p)^{\ell-1}\,p\,(1-p) = p^{\ell}(1-p) + (1-p)^{\ell}\,p \end{aligned}$$

which simplifies to $2^{-\ell}$ for $p = 1/2$.

c)

$$\Pr\left(\sum_{i=1}^{m} L_i < n\right) = \Pr\left(\frac{\sum_{i=1}^{m} L_i - m\mathbb{E}[L_1]}{\sqrt{m\mathbb{V}(L_1)}} < \frac{n - m\mathbb{E}[L_1]}{\sqrt{m\mathbb{V}(L_1)}}\right) \approx \Phi(-0.95) = 0.17,$$

where $\mathbb{E}[L_1] = 2$ and $\mathbb{V}(L_1) = (1-p)/p^2 = 2$.

d) $\Pr\left(L_{(50)} \le 6\right) = [F_{L_1}(6)]^{50}$, with the cdf of L_1 given by

$$F_{L_1}(\ell) = \Pr(L_1 \le \ell \mid X_1 = 0)\Pr(X_1 = 0) + \Pr(L_1 \le \ell \mid X_1 = 1)\Pr(X_1 = 1)$$

and

$$\Pr(L_1 \le \ell \mid X_1 = 0) = (1-p)\sum_{i=1}^{\ell} p^{i-1} = (1-p)\frac{1-p^\ell}{1-p} = 1 - p^\ell$$

so that

$$F_{L_1}(\ell) = p\left(1-p^\ell\right) + (1-p)\left(1-(1-p)^\ell\right).$$

For $p = 1/2$, this simplifies to $F_{L_1}(\ell) = 1 - 2^{-\ell}$. Thus,

$$\Pr(M_{100} \le 6 \mid p = 1/2) = \left[1 - 2^{-6}\right]^{50} = 0.455.$$

Solution to Problem 4.14:

a) We wish

$$\Pr\left(\left|\frac{\hat{p} - p}{\sqrt{pq/n}}\right| < \frac{\epsilon_1}{\sqrt{pq/n}}\right) \approx 2\Phi\left(\frac{\epsilon_1}{\sqrt{pq/n}}\right) - 1 \le 1 - \epsilon_2,$$

or,

$$\frac{\epsilon_1}{\sqrt{pq/n}} = \Phi^{-1}(1 - \epsilon_2/2) \quad \text{or} \quad n = pq\left(\frac{\Phi^{-1}(1-\epsilon_2/2)}{\epsilon_1}\right)^2,$$

where $q = 1 - p$.

b) Recall that, if $X \sim \text{Bin}(n, p)$, then

$$\Pr(X \le x) \approx \Pr\left(Y \le x + \frac{1}{2}\right) \quad \text{or} \quad \Pr(X < x) \approx \Pr\left(Y \le x - \frac{1}{2}\right)$$

and

$$\Pr(X \ge x) \approx \Pr\left(Y \ge x - \frac{1}{2}\right) \quad \text{or} \quad \Pr(X > x) \approx \Pr\left(Y > x + \frac{1}{2}\right),$$

where $Y \sim \text{N}(np, npq)$. Thus, to use the continuity correction we must write the expression in terms of X, the number of Heads, where $\hat{p} = X/n$. We get

$$\begin{aligned}
\Pr(|\hat{p} - p| < \epsilon_1) &= \Pr(n(p - \epsilon_1) < X < n(p + \epsilon_1)) \\
&\approx \Pr\left(n(p-\epsilon_1) + \frac{1}{2} < Y < n(p+\epsilon_1) - \frac{1}{2}\right) \\
&= \Phi\left(\frac{n(p+\epsilon_1) - \frac{1}{2} - np}{\sqrt{npq}}\right) - \Phi\left(\frac{n(p-\epsilon_1) + \frac{1}{2} - np}{\sqrt{npq}}\right) \\
&= 2\Phi\left(\frac{n\epsilon_1 - \frac{1}{2}}{\sqrt{npq}}\right) - 1,
\end{aligned}$$

which can be equated to $1-\epsilon_2$ to find n numerically, and then rounded up, or n can be found through trial and error.

c) This entails numerically solving for n such that

$$F\left(n\left(p+\epsilon_1\right)-1;n,p\right)-F\left(n\left(p-\epsilon_1\right);n,p\right)\leq\epsilon_2,$$

where F is the cdf of the binomial.

Solution to Problem 4.15: We have $\mathbb{K}_{S_n}(t)=n\mathbb{K}_Z\bigl(tn^{-1/2}\bigr)$, so that

$$\begin{aligned}\lim_{n\to\infty}\mathbb{K}_{S_n}(t) &= \lim_{n\to\infty}\frac{\mathbb{K}_Z\bigl(tn^{-1/2}\bigr)}{1/n}=\lim_{n\to\infty}\frac{\mathbb{K}_Z'\bigl(tn^{-1/2}\bigr)\,t\,(-1/2)\,n^{-3/2}}{-n^{-2}}\\ &= \frac{t}{2}\lim_{n\to\infty}\frac{\mathbb{K}_Z'\bigl(tn^{-1/2}\bigr)}{1/n^{1/2}}=\frac{t}{2}\lim_{n\to\infty}\frac{\mathbb{K}_Z''\bigl(tn^{-1/2}\bigr)\,t\,(-1/2)\,n^{-3/2}}{(-1/2)n^{-3/2}}\\ &= \frac{t^2}{2}\lim_{n\to\infty}\mathbb{K}_Z''\bigl(tn^{-1/2}\bigr),\end{aligned}$$

having applied L'Hopitals rule twice. Now, assuming exchange of limit and integral,

$$\lim_{n\to\infty}\mathbb{K}_Z''\bigl(tn^{-1/2}\bigr)=\mathbb{K}_Z''\Bigl(\lim_{n\to\infty}tn^{-1/2}\Bigr)=\mathbb{K}_Z''(0)=\mathbb{V}\left(Z\right)=1,$$

so that $\lim_{n\to\infty}\mathbb{K}_{S_n}(t)=t^2/2$.

Solution to Problem 4.16:

a)

i) The exact solution from the binomial is

$$\Pr\left(X>3\right)=1-\sum_{j=0}^{3}\binom{20}{j}0.05^j0.95^{20-j}=0.015901526. \qquad \text{(S-4.6)}$$

ii) The Poisson approximation with $\lambda=np=1$ gives

$$1-\Pr\left(X\leq3\right)=1-\sum_{j=0}^{3}\frac{e^{-1}}{j!}=0.01899$$

with PE

$$100\left(\frac{0.01590-0.01899}{0.01590}\right)=-19.4.$$

iii) The normal approximation without continuity correction yields

$$1-\Pr\left(X\leq3\right)\approx1-\Phi\left(\frac{3-1}{\sqrt{0.95}}\right)=0.020087$$

and PE -26.3.

iv) With continuity correction (13.78), the normal approximation gives

$$1-\Pr\left(X\leq3\right)\approx1-\Phi\left(\frac{3+0.5-1}{\sqrt{0.95}}\right)=0.00516$$

and PE 67.5, which, in this case, is worse than without correction.

b) Trial and error shows that, in this case, ν should be 0.093, yielding

$$1-\Phi\left(\frac{3+\nu-1}{\sqrt{0.95}}\right)\approx 0.01588.$$

Solution to Problem 4.17: With $np=1$, the usual Chebychev gives

$$\Pr\left(|X-1|>3-1\right)\leq\frac{npq}{4}=0.2375$$

and the one-sided Chebychev gives

$$\Pr\left(X>1+2\right)\leq\frac{0.95}{0.95+4}=0.\overline{19}.$$

For Chernoff,

$$\Pr\left(X>3\right)=\Pr\left(\bar{X}_n\geq\frac{4}{20}\right)\leq 0.04975.$$

Finally, the saddlepoint returns $\Pr\left(X>3\right)\approx 0.01597$, which is very close to the true value.

Solution to Problem 4.18:

a) As the sacks are independent,

$$\left(1-\frac{m}{n}\right)^n\to e^{-m}.$$

b) Because we have n independent trials with constant probability m/n of "success", $X\sim\text{Bin}\left(n,m/n\right)$ and $\mathbb{E}\left[X_n\right]=m$.

c) These are given by

$$\begin{aligned}\Pr\left(X_n=m\mid m=1\right) &= \binom{n}{1}\left(\frac{1}{n}\right)^1\left(1-\frac{1}{n}\right)^{n-1}\\ &= (n-1)^{n-1}\,n^{-n+1}\end{aligned}$$

and

$$\begin{aligned}\Pr\left(X_n=m\mid m=2\right) &= \binom{n}{2}\left(\frac{2}{n}\right)^2\left(1-\frac{2}{n}\right)^{n-2}\\ &= 2\left(n-1\right)n^{-n+1}\left(n-2\right)^{n-2}.\end{aligned}$$

d) The limiting probabilities are given by the Poisson distribution with $\lambda=n(m/n)=m$, so that

$$\Pr\left(X_n=m\mid m=1\right)\to\left.\frac{e^{-\lambda}\lambda^x}{x!}\right|_{\substack{\lambda=m=1\\x=m=1}}=e^{-1}$$

and, similarly,

$$\Pr\left(X_n=m\mid m=2\right)\to 2e^{-2}.$$

e) We want

$$\Pr(X_n = m) = \binom{n}{m}\left(\frac{m}{n}\right)^m\left(1-\frac{m}{n}\right)^{n-m} = \frac{n!}{m!\,(n-m)!}\frac{m^m}{n^m}\frac{(n-m)^{n-m}}{n^{n-m}} \qquad \text{(S-4.7)}$$

to be $\simeq 0.05$. Using Stirling's approximation,

$$\begin{aligned}\Pr(X_n = m) &\approx \frac{\sqrt{2\pi}n^{n+0.5}e^{-n}}{\sqrt{2\pi}m^{m+0.5}e^{-m}\sqrt{2\pi}\,(n-m)^{n-m+0.5}\,e^{-(n-m)}}\frac{m^m}{n^m}\frac{(n-m)^{n-m}}{n^{n-m}}\\ &= \sqrt{\frac{n}{2\pi m\,(n-m)}}\,\Bigg|_{m=n/10}\\ &= \sqrt{\frac{50}{9n\pi}},\end{aligned}$$

and solving, $n = 707.4$. From (S-4.7) with $n = 707$ and $m = 70$, $\Pr(X_n = m) = 0.050174$.

f) We have

$$\Pr\left(X_n = \frac{n}{10}\right) \simeq \frac{e^{-\frac{n}{10}}\left(\frac{n}{10}\right)^{\frac{n}{10}}}{\left(\frac{n}{10}\right)!} \simeq \frac{e^{-\frac{n}{10}}\left(\frac{n}{10}\right)^{\frac{n}{10}}}{\sqrt{2\pi}\left(\frac{n}{10}\right)^{\frac{n}{10}+\frac{1}{2}}e^{-\frac{n}{10}}} = \sqrt{\frac{5}{n\pi}},$$

which, for large n, is close to $\sqrt{\frac{50}{9n\pi}}$ from before. Solving yields $n = 636.62$ and, with $n = 637$ and $m = 63$, (S-4.7) gives 0.0529, not quite as good.

Solution to Problem 4.19: The program for using the pdf inversion theorem is modified from those in Chapter 10, and shown in Listings S-4.1 and S-4.2 below. The code used to generate the plot is given below.

```
sim=20000; v=zeros(sim,1);
for i=1:sim, n=40; x=trnd(3,40,1); s=sum(x)/sqrt(3*n); v(i)=s; end
[pdf,grd]=kerngau(v,-2);
x=2.5:0.05:4.5; f40den = invsumtcf(x,40); npdf=normpdf(x);
plot(grd,pdf,'r-',x,f40den,'g--',x,npdf,'b-.')
axis([2.5 4.5 f40den(end) f40den(1)])
set(gca,'fontsize',20), legend('kernel','true','normal')
```

```
function [f,F] = invsumtcf(xvec,n)
bordertol=1e-8; lo=bordertol; hi=1-bordertol; tol=1e-9;
sn=sqrt(1/n);
xl=length(xvec);  F=zeros(xl,1); f=F;
for loop=1:length(xvec)
  x=xvec(loop);
  dopdf=1; f(loop)= quadl('invsumtcf_',lo,hi,tol,0, n,sn,x,dopdf) / pi;
  if nargout>1
   dopdf=0;
   F(loop)=0.5-(1/pi)* quadl('invsumtcf_',lo,hi,tol,0, n,sn,x,dopdf);
  end
end;
```

Program Listing S-4.1: Computes the pdf and, optionally, the cdf, at the points in xvec, of r.v. S in Example 13.27.

```
function I=invsumtcf_(u,n,sn,x,dopdf);
I=zeros(size(u));
t  = (1-u)./u;
cf = (1+t*sn).^n .* exp(-n*t*sn);
z  = exp(-i*t*x) .* cf;
if dopdf==1, g=real(z); else g=imag(z)./t; end
I  = g ./ u.^2;
```

Program Listing S-4.2: Used by the program in Listing S-4.1.

Solutions to Chapter 5: Saddlepoint Approximations

Solution to Problem 5.1: The cgf is $\mathbb{K}_X(s) = s^2/2$, $\mathbb{K}'_X(s) = s$ and $\mathbb{K}''_X(s) = 1$, so that $\hat{s} = x$ and the saddlepoint density

$$\hat{f}_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(0 - \frac{x^2}{2}\right) = f_X(x)$$

is exact. For the cdf, $\hat{w} = \operatorname{sgn}(x)\sqrt{x^2} = x$, and $\hat{u} = x$, so that

$$\hat{F}_X(x) = \Phi(x) + \phi(x)\left\{\frac{1}{x} - \frac{1}{x}\right\} = \Phi(x)$$

is also exact, valid for $x \neq 0 = \mathbb{E}[X]$.

Solution to Problem 5.2: From symmetry, $\mathbb{M}_Y(s) = \mathbb{M}_{-Y}(s)$. In general, as

$$\mathbb{M}_Y(s) = \mathbb{E}\left[e^{sY}\right] = \mathbb{E}\left[e^{(-s)(-Y)}\right] = \mathbb{M}_{-Y}(-s),$$

we have $\mathbb{M}_{-Y}(s) = \mathbb{M}_{-Y}(-s)$ or $\mathbb{M}_Y(s) = \mathbb{M}_Y(-s)$, from which we obtain $\mathbb{K}_Y(s) = \mathbb{K}_Y(-s)$, $\mathbb{K}'_Y(s) = -\mathbb{K}'_Y(-s)$ and $\mathbb{K}''_Y(s) = \mathbb{K}''_Y(-s)$. It then follows that, if $\hat{s}$ is the saddlepoint for y, i.e., the value such that $y = \mathbb{K}'_Y(\hat{s})$, then $-y = \mathbb{K}'_Y(-\hat{s})$, i.e., $-\hat{s}$ is the saddlepoint for $-y$. Thus, (14.3) gives

$$\hat{f}_Y(y) = \frac{\exp\{\mathbb{K}_Y(\hat{s}) - y\hat{s}\}}{\sqrt{2\pi\,\mathbb{K}''_Y(\hat{s})}} = \frac{\exp\{\mathbb{K}_Y(-\hat{s}) - (-y)(-\hat{s})\}}{\sqrt{2\pi\,\mathbb{K}''_Y(-\hat{s})}} = \hat{f}_Y(-y),$$

which shows that the (first order) SPA to f_Y is symmetric if f_Y is symmetric. For $\hat{F}_Y(y)$, the cdf approximation (14.6) at $y \neq 0$, we have

$$\hat{w}(y) = \operatorname{sgn}(\hat{s})\sqrt{2\hat{s}y - 2\mathbb{K}_Y(\hat{s})} \quad \text{and} \quad \hat{u}(y) = \hat{s}\sqrt{\mathbb{K}''_Y(\hat{s})},$$

where the dependence of $\hat{w}$ and $\hat{u}$ on y is emphasized. For $-y \neq 0$,

$$\begin{aligned}
\hat{w}(-y) &= \operatorname{sgn}(-\hat{s})\sqrt{2(-\hat{s})(-y) - 2\mathbb{K}_Y(-\hat{s})} = -\operatorname{sgn}(\hat{s})\sqrt{2\hat{s}y - 2\mathbb{K}_Y(\hat{s})} = -\hat{w}(y) \\
\hat{u}(-y) &= -\hat{s}\sqrt{\mathbb{K}''_Y(-\hat{s})} = -\hat{s}\sqrt{\mathbb{K}''_Y(\hat{s})} = -\hat{u}(y),
\end{aligned}$$

so that, as $\Phi(-x) = 1 - \Phi(x)$ and $\phi(-x) = \phi(x)$,

$$\begin{aligned}
\hat{F}_Y(-y) &= \Phi(-\hat{w}(y)) + \phi(-\hat{w}(y)) \left\{ \frac{1}{-\hat{w}(y)} - \frac{1}{-\hat{u}(y)} \right\} \\
&= 1 - \Phi(\hat{w}(y)) + (-1)\,\phi(\hat{w}(y)) \left\{ \frac{1}{\hat{w}(y)} - \frac{1}{\hat{u}(y)} \right\} \\
& 1 - \hat{F}_Y(y),
\end{aligned}$$

showing that symmetry also holds for the SPA to the cdf.

Solution to Problem 5.3: The density of Y is given by (7.65), i.e.,

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right| = a^{-1} f_X\left(\frac{y-b}{a} \right).$$

The mgf of Y is

$$\mathbb{M}_Y(s) = \mathbb{E}\left[e^{sY}\right] = \mathbb{E}\left[e^{s(aX+b)}\right] = e^{sb}\mathbb{M}_X(as),$$

so that $\mathbb{K}_Y(s) = sb + \mathbb{K}_X(as)$, $\mathbb{K}_Y'(s) = b + a\mathbb{K}_X'(as)$ and $\mathbb{K}_Y''(s) = a^2\mathbb{K}_X''(as)$. Then

$$\begin{aligned}
\hat{f}_Y(y) &= \frac{\exp\{\mathbb{K}_Y(\hat{s}) - y\hat{s}\}}{\sqrt{2\pi\,\mathbb{K}_Y''(\hat{s})}} \\
&= \frac{\exp\{\hat{s}b + \mathbb{K}_X(a\hat{s}) - (ax+b)\,\hat{s}\}}{\sqrt{2\pi\,a^2\mathbb{K}_Y''(a\hat{s})}} \\
&= \frac{\exp\{\mathbb{K}_X(a\hat{s}) - xa\hat{s}\}}{a\sqrt{2\pi\,\mathbb{K}_X''(a\hat{s})}},
\end{aligned}$$

where $y = \mathbb{K}_Y'(\hat{s}) = b + a\mathbb{K}_X'(a\hat{s})$ or $a^{-1}(y-b) = \mathbb{K}_X'(a\hat{s})$, i.e., $x = \mathbb{K}_X'(a\hat{s})$. This implies that $a\hat{s}$ is the saddlepoint for value x when $\hat{s}$ is the saddlepoint for y. But this means that

$$\hat{f}_Y(y) = a^{-1}\hat{f}_X(x) = a^{-1}\hat{f}_X\left(\frac{y-b}{a}\right),$$

as was to be shown.

Now for the cdf. It is clear that

$$F_Y(y) = \Pr(Y \le y) = \Pr(aX + b \le y) = \Pr\left(X \le \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right).$$

For the SPA to $F_Y(y)$ with $y = \mathbb{K}_Y'(\hat{s})$ and the fact that $a > 0$,

$$\begin{aligned}
\hat{w}(y) &= \operatorname{sgn}(\hat{s})\sqrt{2\hat{s}y - 2\mathbb{K}_Y(\hat{s})} = \operatorname{sgn}(\hat{s})\sqrt{2\hat{s}(ax+b) - 2(\hat{s}b + \mathbb{K}_X(a\hat{s}))} \\
&= \operatorname{sgn}(a\hat{s})\sqrt{2(a\hat{s})x - 2\mathbb{K}_X(a\hat{s})} = \hat{w}(x)
\end{aligned}$$

and

$$\hat{u}(y) = \hat{s}\sqrt{\mathbb{K}_Y''(\hat{s})} = \hat{s}\sqrt{a^2\mathbb{K}_X''(a\hat{s})} = a\hat{s}\sqrt{\mathbb{K}_X''(a\hat{s})} = \hat{u}(x),$$

so that

$$\hat{F}_Y(y) = \Phi(\hat{w}(x)) + \phi(\hat{w}(x))\left\{\frac{1}{\hat{w}(x)} - \frac{1}{\hat{u}(x)}\right\} = \hat{F}_X(x) = \hat{F}_X\left(\frac{y-b}{a}\right),$$

as was to be shown.

Solution to Problem 5.4: This is similar to Problem 6.8. From (6.30) and (6.31), the exact pmf and cdf are given by

$$\Pr(X = x) = \sum_{i=0}^{x} \Pr(X_1 = i) \Pr(X_2 = x - i)$$

and

$$\Pr(X \le x) = \sum_{i=0}^{x} \Pr(X_1 = i) \Pr(X_2 \le x - i),$$

which are easily implemented in Matlab; see Listing S-5.1.

```
function pdf = negbin2 (x,r1,p1,r2,p2)
i=0:x;
pdf = sum( negbinpdf(i,r1,p1) .* negbinpdf(x-i,r2,p2) );

function den = negbinpdf(x,r,p)
% den=c(r+x-1,x) .* p.^r .* (1-p).^x;
d=gammaln(r+x) - gammaln(x+1) - gammaln(r) + r*log(p) + x * log(1-p);
den=exp(d);
```

Program Listing S-5.1: Computation of the pmf of X, where x is a scalar, $X = X_1 + X_2$ and $X_i \overset{\text{ind}}{\sim} \text{NBin}(r_i, p_i)$.

For the normal approximation, we require the mean and variance, which are given by

$$\mathbb{E}[X] = \sum_{i=1}^{2} \mathbb{E}[X_i] = \frac{r_1(1-p_1)}{p_1} + \frac{r_2(1-p_2)}{p_2} = 6$$

and

$$\mathbb{V}(X) = \sum_{i=1}^{2} \mathbb{V}(X_i) = \frac{r_1(1-p_1)}{p_1^2} + \frac{r_2(1-p_2)}{p_2^2} = 15,$$

respectively. The following code uses the program to produce the exact mass function, overlaid with the normal approximation.

```
p=zeros(26,1);
for x=0:25, p(x+1)=negbin2(x,2,1/3,4,2/3); end, plot(0:25,p)
hold on, plot(0:25, normpdf(0:25,6,sqrt(15)),'r--'), hold off
set(gca,'fontsize',14), grid
```

For the saddlepoint approximation, first recall from Problem 10.5 that the cumulant generating function of a negative binomially distributed random variable is given by $\mathbb{K}_X(s) = r \ln p - r \ln(1 - qe^s)$, for $s < -\ln q$, where $q = 1 - p$. Thus, with $S = X_1 + X_2$,

$$\mathbb{K}_S(s) = r_1 \ln p_1 - r_1 \ln(1 - q_1 e^s) + r_2 \ln p_2 - r_2 \ln(1 - q_2 e^s), \quad s < \min(-\ln q_i),$$

and we need to solve

$$x = \mathbb{K}_S'(s) = e^s \left(\frac{r_1 q_1}{1 - q_1 e^s} + \frac{r_2 q_2}{1 - q_2 e^s} \right),$$

which is best done numerically for each value of x (those in the interior of the support of S).[25] With

$$\mathbb{K}_S''(s) = e^s \left(r_1 q_1 \left(1 - q_1 e^s\right)^{-2} + r_2 q_2 \left(1 - q_2 e^s\right)^{-2} \right),$$

the density SPA is

$$\hat{f}_X(x) \approx \frac{1}{\sqrt{2\pi\, \mathbb{K}_X''(\hat{s})}} \exp\left\{\mathbb{K}_X(\hat{s}) - x\hat{s}\right\},$$

where $\hat{s}$ is implicitly given via $x = \mathbb{K}_X'(\hat{s})$. This is computed with the program in Listing S-5.2.

```
function f = negbinspa(x,r1,p1,r2,p2)
q1=1-p1; q2=1-p2;
low=-10; % this is just a guess, and might need to
         %   be modified for different parameters
high=min(-log(q1),-log(q2))-0.05; % same thing applies to
                                  % the 0.05 term here
opt=optimset('TolX',1e-6,'Display','none');
s=fzero(@negbinspa_,[low high],opt,x,r1,p1,r2,p2);
E=exp(s);
K=r1*log(p1) - r1*log(1-q1*E) + r2*log(p2) - r2*log(1-q2*E);
Kpp=E * (r1*q1*(1-q1*E)^(-2) + r2*q2*(1-q2*E)^(-2) );
f = (2*pi *Kpp)^(-1/2) * exp(K-x*s);

function d=negbinspa_(s,x,r1,p1,r2,p2)
q1=1-p1; q2=1-p2; E=exp(s);
d = x -  E .* (r1*q1 ./ (1-q1.*E) + r2*q2 ./ (1-q2.*E));
```

Program Listing S-5.2: Saddlepoint mass function approximation.

Then, with values r_1, r_2, p_1, p_2 defined in Matlab, run

```
spa=[]; for x=1:24, spa=[spa, negbinspa(x,r1,p1,r2,p2)]; end
hold on, plot(1:24, spa, 'g--o'), hold off
```

This yields the plot in Figure S-5.1. The SPA is almost exact, while the normal approximation is quite bad.

[25] Maple actually gives an expression for $\hat{s}$, but we don't bother using it here.

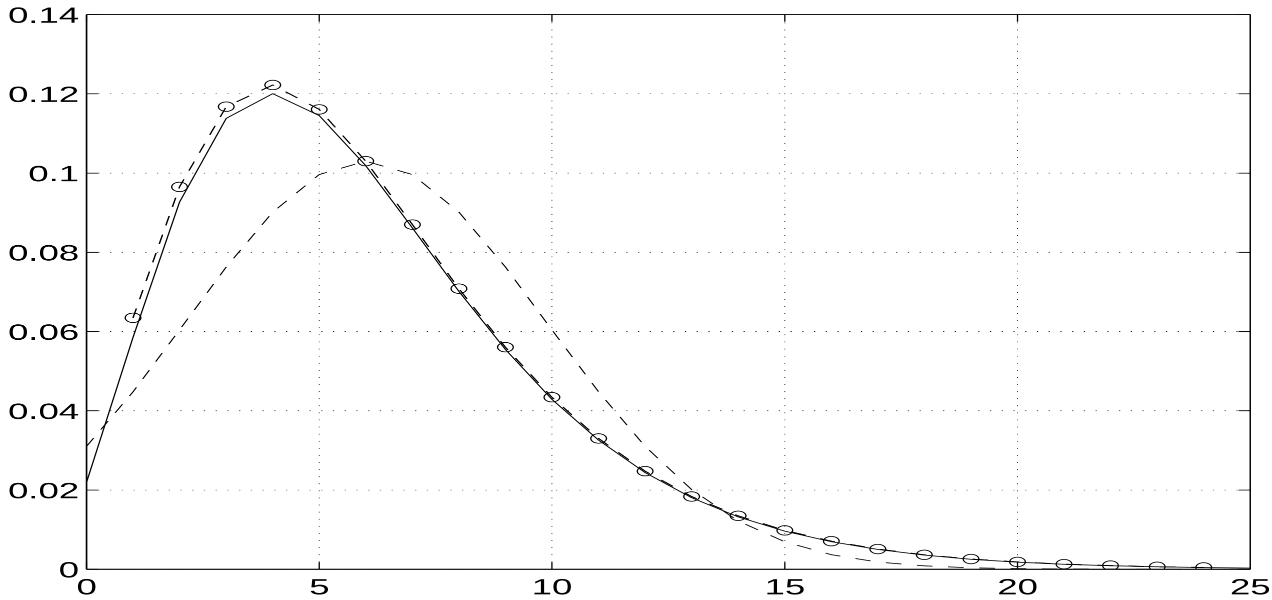

Figure S-5.1: Exact pmf (solid), normal approximation (dashed) and SPA (circles) of $X_1 + X_2$

Solution to Problem 5.5:

a) Simple algebraic rearranging shows that

$$Z_x = (1-x)\,Y_\alpha - xY_\beta.$$

b) The mgf of Z given by

$$\begin{aligned}
\mathbb{M}_{Z_x}(s) &= \mathbb{M}_{\omega_1 Y_\alpha + \omega_2 Y_\beta}(s) = \mathbb{E}\left[e^{-s\left(\omega_1 y_\alpha + \omega_2 y_\beta\right)}\right] = \mathbb{E}\left[e^{-s\omega_1 y_\alpha}\right]\mathbb{E}\left[e^{-s\omega_2 y_\beta}\right] \\
&= \mathbb{M}_{Y_\alpha}(s\omega_1)\,\mathbb{M}_{Y_\beta}(s\omega_2) \\
&= \left(1-2s\,(1-x)\right)^{-\alpha}\left(1+2sx\right)^{-\beta},
\end{aligned}$$

where $\omega_1 = 1-x$, $\omega_2 = -x$. Taking logs,

$$\mathbb{K}_{Z_x}(s) = -\alpha\ln\left(1-2s\,(1-x)\right) - \beta\ln\left(1+2sx\right).$$

c) Differentiating yields

$$\mathbb{K}'_{Z_x}(s) = \frac{2\alpha\,(1-x)}{1-2s\,(1-x)} - \frac{2\beta x}{1+2sx}.$$

Substituting the expression for $\hat{s}$ given in (14.36) and simplifying shows (quite easily using Maple) that

$$\mathbb{K}'_{Z_x}(\hat{s}) = \frac{2\alpha\,(1-x)}{\frac{\alpha}{(\alpha+\beta)x}} - \frac{2\beta x}{\frac{\beta}{(\alpha+\beta)(1-x)}},$$

which is indeed (check with Maple) zero.

d) We require $\Pr(Z_{0.3} < 0)$ with $\hat{s}$ from (14.36) computed as $\hat{s} = -0.873015873$. Then, with $z = 0$, $\mathbb{K}_{Z_x}(\hat{s}) = -4.275390239$, $\hat{w} = -2.924171759$,

$$\mathbb{K}''_{Z_x}(s) = \frac{4\alpha (x-1)^2}{(1-2s(1-x))^2} + \frac{4\beta x^2}{(1+2sx)^2} = 11.907,$$

$\hat{u} = -3.012474066$, so that from (14.6), $\widehat{\Pr}(X < 0.3) = 0.001671$ yielding a percentage error of -0.32, i.e., about one third of one percent error.

e) With $R = (Y_\alpha/n_1) \;/\; (Y_\beta/n_2)$, and from (14.34)

$$\begin{aligned}
\Pr(X < x) &= \Pr\left(\frac{Y_\alpha}{Y_\alpha + Y_\beta} < x\right) = \Pr\left(\frac{Y_\alpha + Y_\beta}{Y_\alpha} > \frac{1}{x}\right) \\
&= \Pr\left(\frac{Y_\beta}{Y_\alpha} > \frac{1}{x} - 1\right) = \Pr\left(\frac{Y_\alpha}{Y_\beta} < \frac{1}{\frac{1}{x} - 1}\right) \\
&= \Pr\left(R < \frac{n_2}{n_1}\frac{x}{1-x}\right)
\end{aligned}$$

and solving

$$r = \frac{n_2}{n_1}\frac{x}{1-x} \quad \text{gives} \quad x = \frac{n_1 r}{n_1 r + n_2}. \tag{S-5.1}$$

f) From (14.35) we have

$$\Pr(R < r) = \Pr\left(X < \frac{n_1 r}{n_1 r + n_2}\right) = \Pr\left(Z_{\frac{n_1 r}{n_1 r + n_2}} < 0\right)$$

with the value of Z_x, $\mathbb{M}_{Z_x}(s)$, $\mathbb{K}_{Z_x}(s)$ and $\mathbb{K}'_{Z_x}(s)$ the same, but with the expression for x in (S-5.1), so that, simplifying (14.36),

$$\hat{s} = \frac{r-1}{2}\frac{n_1 r + n_2}{r(n_1 + n_2)}.$$

This implies

$$\begin{aligned}
\mathbb{K}_{Z_{\frac{n_1 r}{n_1 r + n_2}}}(s) &= -\frac{n_1}{2}\ln\left(\frac{n_1 r + n_2 - 2sn_2}{n_1 r + n_2}\right) - \frac{n_2}{2}\ln\left(\frac{n_1 r + n_2 + 2sn_1 r}{n_1 r + n_2}\right), \\
\mathbb{K}_{Z_{\frac{n_1 r}{n_1 r + n_2}}}(\hat{s}) &= -\frac{n_1}{2}\ln\left(\frac{n_1 r + n_2}{r(n_1 + n_2)}\right) - \frac{n_2}{2}\ln\left(\frac{n_1 r + n_2}{n_1 + n_2}\right), \\
\mathbb{K}'_{Z_{\frac{n_1 r}{n_1 r + n_2}}}(s) &= \frac{n_1 n_2}{n_1 r + n_2 - 2sn_2} - \frac{n_1 n_2 r}{n_1 r + n_2 + 2sn_1 r}, \\
\mathbb{K}''_{Z_{\frac{n_1 r}{n_1 r + n_2}}}(s) &= \frac{2n_1 n_2^2}{(n_1 r + n_2 - 2sn_2)^2} + \frac{2n_1^2 n_2 r^2}{(n_1 r + n_2 + 2sn_1 r)^2}, \\
\mathbb{K}''_{Z_{\frac{n_1 r}{n_1 r + n_2}}}(\hat{s}) &= \frac{2n_1 n_2 r^2 (n_1 + n_2)^3}{(n_1 r + n_2)^4}.
\end{aligned}$$

Substituting these into (14.7) and simplifying yields the expressions (14.37) and (14.38) for $\hat{w}$ and $\hat{u}$.

g) Calculation shows that $\hat{s} = 0.5086032389$, $\hat{w} = 1.317835761$ and $\hat{u} = 1.977259895$, yielding $\widehat{\Pr}(X < 4.75) = 0.9486$ for a RPE of 0.15 percent.

Solution to Problem 5.6:

a) Rearranging gives $\Pr(R < r) = \Pr\left(\frac{n_2}{n_1}C_1 - rC_2 < 0\right)$, so that $Z = \frac{n_2}{n_1}C_1 - rC_2$. From the mgf of a weighted sum of χ^2 random variables (see, for example, (19.13))

$$\begin{aligned}
\mathbb{M}_Z(s) &= \left(1 - 2\frac{n_2}{n_1}s\right)^{-\frac{n_1}{2}} (1+2rs)^{-\frac{n_2}{2}}, \\
\mathbb{K}_Z(s) &= -\frac{n_1}{2}\ln\left(1 - 2\frac{n_2}{n_1}s\right) - \frac{n_2}{2}\ln(1+2rs), \qquad \text{(S-5.2)} \\
\mathbb{K}'_Z(s) &= \frac{n_2}{1 - 2\frac{n_2}{n_1}s} - \frac{n_2 r}{1+2rs}, \\
\mathbb{K}''_Z(s) &= \frac{2n_2^2}{\left(1 - 2\frac{n_2}{n_1}s\right)^2 n_1} + \frac{2n_2 r^2}{(1+2rs)^2}. \qquad \text{(S-5.3)}
\end{aligned}$$

b) Solving $\mathbb{K}'_Z(s) = 0$ yields

$$\hat{s} = \frac{1}{2}\left(1 - r^{-1}\right)\left(1 + \frac{n_2}{n_1}\right)^{-1}. \qquad \text{(S-5.4)}$$

c) From (S-5.4), $\hat{s} = 3.036437247 \times 10^{-2}$, from (S-5.2), $\mathbb{K}_Z(\hat{s}) = -0.8683455462$, and from (S-5.3), $\mathbb{K}''_Z(\hat{s}) = 4240.322567$, so that (14.7) gives $\hat{w} = 1.317835761$ and $\hat{u} = 1.977259896$, which are the same as before, so that $\widehat{\Pr}(X < 4.75)$ is identical to the previous result.

d) From (S-5.2), (S-5.3) and (S-5.4),

$$\begin{aligned}
\mathbb{K}_Z(\hat{s}) &= -\frac{n_1}{2}\ln\left(\frac{n_1 r + n_2}{r(n_1+n_2)}\right) - \frac{n_2}{2}\ln\left(\frac{n_1 r + n_2}{n_1+n_2}\right), \\
\mathbb{K}''_Z(\hat{s}) &= \frac{2n_2 r^2 (n_1+n_2)^3}{n_1 (n_1 r + n_2)^2},
\end{aligned}$$

and when substituted into the saddlepoint expressions for $\hat{w}$ and $\hat{u}$, yield those values in (14.37) and (14.38).

Solution to Problem 5.7:

a) The X_i are uniformly distributed, so that (11.9) gives

$$\begin{aligned}
f_P(p) &= \int_0^1 \frac{1}{x_1}\mathbb{I}_{(0,1)}(x_1)\,\mathbb{I}_{(0,1)}\left(\frac{p}{x_1}\right)dx_1 = \mathbb{I}_{(0,1)}(x_1)\int_p^1 \frac{1}{x_1}dx_1 \\
&= -\ln p\,\mathbb{I}_{(0,1)}(p),
\end{aligned}$$

which follows because $0 < x_1 < 1$ and $0 < \frac{p}{x_1} < 1 \Rightarrow p < x_1 < 1$. The cdf is

$$\Pr(P < p) = -\int_0^p \ln(t)\,dt = p(1 - \ln p)\,\mathbb{I}_{(0,1)}(p) + \mathbb{I}_{[1,\infty)}(p).$$

b) As P is strictly positive,

$$\Pr(P < p) = \Pr(\ln P < \ln p) = \Pr\left(\sum_{i=1}^{n} \ln X_i < \ln p\right) = \Pr(L < \ln p),$$

where $L := \sum_{i=1}^{n} \ln X_i$. It is easily verified that

$$m_{X_i}(s) = \mathbb{M}_{\ln X_i}(s) = \mathbb{E}\left[X_i^s\right] = \frac{\Gamma(p_i + s)\,\Gamma(p_i + q_i)}{\Gamma(p_i)\,\Gamma(p_i + q_i + s)}, \quad s \in (-p_i, \infty)$$

so that

$$m_P(s) = \mathbb{M}_L(s) = \mathbb{E}\left[\left(\prod X_i^s\right)\right] = \prod_{i=1}^{n} \frac{\Gamma(p_i + s)\,\Gamma(p_i + q_i)}{\Gamma(p_i)\,\Gamma(p_i + q_i + s)}$$

convergent on $s \in (-\min(p_i), \infty)$. The cgf of L is given by

$$\mathbb{K}_L(s) = \sum_{i=1}^{n} \{\ln\Gamma(p_i + s) + \ln\Gamma(p_i + q_i) - \ln\Gamma(p_i) - \ln\Gamma(p_i + q_i + s)\}$$

with j^{th} derivative

$$\mathbb{K}_L^{(j)}(s) = \sum_{i=1}^{n} \left(\psi_{(j-1)}(p_i + s) - \psi_{(j-1)}(p_i + q_i + s)\right).$$

Thus, the SPA (14.6) can be readily applied. Applying the transformation via (7.65) shows that

$$f_P(p) = f_{\ln P}(\ln p)\, p^{-1}. \tag{S-5.5}$$

Density $f_{\ln P}(\cdot)$ would then be replaced by the SPA (14.3) to compute $f_P(p)$.

For the special case in Problem 14.7(**a**), the cgf of L reduces to

$$\mathbb{K}_L(s) = 2\left(\ln\Gamma(1 + s) - \ln\Gamma(2 + s)\right),$$

with first and second derivatives

$$\mathbb{K}_L'(s) = 2\left(\psi(1 + s) - \psi(2 + s)\right) = -\frac{2}{s + 1}$$

and

$$\mathbb{K}_L''(s) = 2\left(\psi_{(1)}(1 + s) - \psi_{(1)}(2 + s)\right) = \frac{2}{(s + 1)^2},$$

respectively, using recursion (11.43), i.e.,

$$\psi_{(n)}(z + 1) = \psi_{(n)}(z) + (-1)^n\, n!\, z^{-(n+1)}.$$

Letting $x = \ln p$, the saddlepoint equation $\mathbb{K}_L'(s) = x$ thus admits the closed form solution $\hat{s} = -(1 + 2/x)$. Using the fact that $\Gamma(1 + a) = a\Gamma(a)$,

$$\mathbb{K}_L(\hat{s}) = 2\ln\frac{\Gamma(-2/x)}{\Gamma(1 - 2/x)}, \qquad \mathbb{K}_L''(\hat{s}) = \frac{1}{2}x^2$$

and

$$\exp\left(\mathbb{K}_L(\hat{s})\right) = \left(\frac{\Gamma\left(-2/x\right)}{\Gamma\left(1-2/x\right)}\right)^2 = \frac{1}{4}x^2,$$

so that the density SPA $\hat{f}_{\ln P}\left(x\right)$ is given by

$$\frac{1}{\sqrt{2\pi\mathbb{K}_L''(\hat{s})}}\exp\left(\mathbb{K}_L(\hat{s}) - \hat{s}x\right) = \frac{1}{x\sqrt{\pi}}\left(\frac{1}{4}x^2\right)\exp\left(x+2\right) = \frac{x}{4\sqrt{\pi}}\exp\left(x+2\right)$$

or, from (S-5.5) (and flipping signs)

$$\hat{f}_P\left(p\right) = \frac{1}{p}\frac{\ln p}{4\sqrt{\pi}}\exp\left(\ln p + 2\right) = \ln p \frac{e^2}{4\sqrt{\pi}} = -1.042207 \ln p, \quad 0 < p < 1.$$

Note that the renormalized SPA is exact in this case.

c) Figure S-5.2 shows the RPE for the two approximations. The numbers inscribed next to the asymptotes are the true RPE for the Fan approximation, and show that, in absolute terms as well as in comparison to the SPA, it is a poor approximation and cannot be recommended. It is rather interesting to note that both approximations have trouble in the same places. The SPA and Fan both reach their largest error as x approaches the left tail, and both exhibit a breakdown in accuracy and the same behavior around $x = 0.56$. However, while the Fan approximation continues to break down farther into the left tail, the SPA reaches a maximum RPE of 1.641 around $x = 0.0080$, and then trails off to zero as x approaches zero.

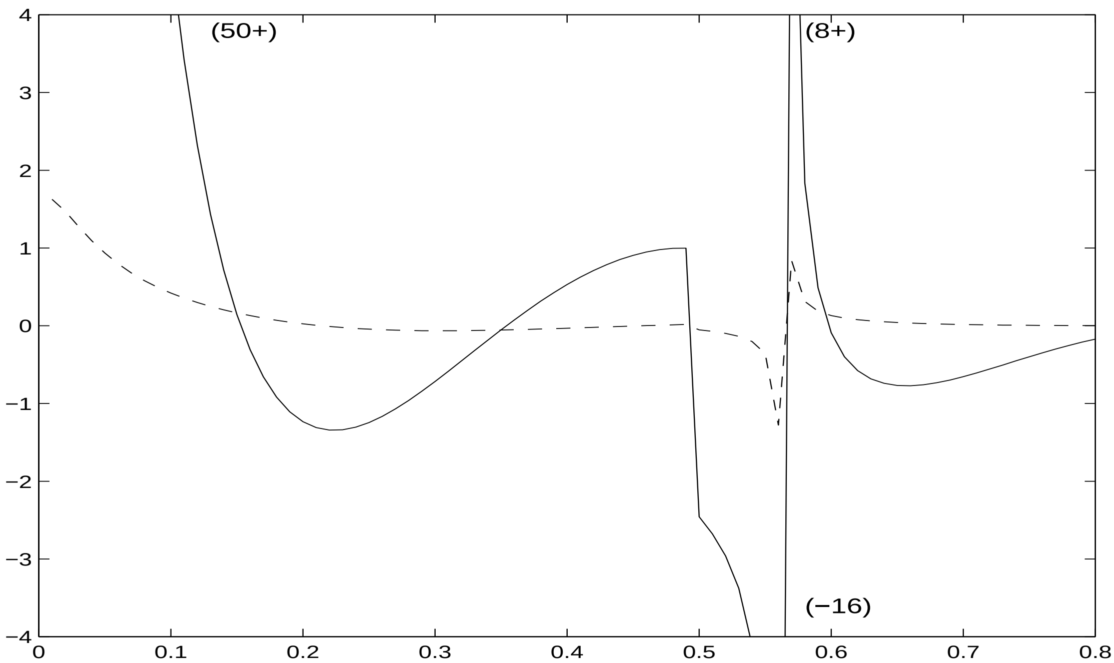

Figure S-5.2: Relative Percentage Error of the SPA (dashed) and Fan (solid, truncated) with exact density (14.42) shown for $x = 0.01, 0.02, \ldots, 0.80$.

d) Using (14.27) and assuming that $p_i, q_i > 0$, the mgf of X_i is seen to be

$$\begin{aligned}\mathbb{M}_{X_i}(t) &= \mathbb{E}\left[e^{tX_i}\right] = B\left(p_i, q_i\right)^{-1} \int_0^1 x^{p_i-1}\left(1-x\right)^{q_i-1} e^{tx} dx \\ &= {}_1F_1\left(p_i, p_i+q_i; t\right),\end{aligned}$$

so that

$$\begin{aligned}\mathbb{M}_S(t) &= \mathbb{E}\left[e^{t\sum_{i=1}^n X_i}\right] = \prod_{i=1}^n {}_1F_1\left(p_i, p_i+q_i; t\right), \\ \mathbb{K}_S(t) &= \sum_{i=1}^n \ln {}_1F_1\left(p_i, p_i+q_i; t\right),\end{aligned}$$

with expressions for $\mathbb{K}_S^{(j)}(t)$ being straightforward. By using (14.27) and (14.30) in conjunction with (14.33), the SPA (14.6) and (14.3) can be easily and quickly calculated. Notice, however, that it involves two approximations (saddlepoint for the density and Laplace for the ${}_1F_1$ function) so that the accuracy may not be startling. The motivated reader is encouraged to investigate the accuracy for $n = 2$ and a variety of p_i, q_i, using the convolution formula and numerical integration to compute the exact pdf.

Solution to Problem 5.8: The first several terms of ${}_1F_1\left(a, b/2; -z/2\right)$ are

$$1 - \frac{a}{b}z + \frac{a\left(a+1\right)}{b\left(b+2\right)}\frac{z^2}{2} - \frac{a\left(a+1\right)\left(a+2\right)}{b\left(b+2\right)\left(b+4\right)}\frac{z^3}{3!} + \cdots$$

which, as $b \to \infty$, can be approximated by

$$1 - \frac{a}{b}z + \frac{a\left(a+1\right)}{b^2}\frac{z^2}{2} - \frac{a\left(a+1\right)\left(a+2\right)}{b^3}\frac{z^3}{3!} + \cdots$$

From (1.12) with $|y/b| < 1$,

$$\begin{aligned}\left(1 - \frac{y}{b}\right)^{-a} &= \sum_{j=0}^{\infty} \frac{\left(a+j-1\right)!}{j!\left(a-1\right)!}\left(\frac{y}{b}\right)^j \\ &= 1 + a\frac{y}{b} + \frac{a\left(a+1\right)}{2}\left(\frac{y}{b}\right)^2 + \frac{a\left(a+1\right)\left(a+2\right)}{3!}\left(\frac{y}{b}\right)^3 + \cdots\end{aligned}$$

so that, with $y = -z$,

$${}_1F_1\left(a, b/2; -z/2\right) \approx \left(1 + \frac{z}{b}\right)^{-a}, \quad \text{as} \quad b \to \infty.$$

Solutions to Chapter 6: Order Statistics

Solution to Problem 6.1:

a) Because the r.v.s are iid, the 3! possible arrangements, i.e., $X_1 < X_2 < X_3$, etc., are equally likely. Among these, three possibilities are such that $X_1 > X_2$, so that $p_0 = 1/2$. Even easier, particularly when n is larger, note that, irrespective of all other $n-2$ r.v.s, events $X_1 < X_2$ and $X_1 > X_2$ are equally likely.

b) This follows directly from the definition of conditional probability,

$$p_1 = \frac{\Pr(X_1 = \max X_i)}{\Pr(X_1 > X_3)} = \frac{1/3}{1/2} = \frac{2}{3}.$$

Alternatively, arrange X_1 and X_3 on a line such that $X_1 > X_3$ so that, with equal probability, X_2 could be in one of the three open places, i.e., either $X_2X_3X_1$ or $X_3X_2X_1$ or $X_3X_1X_2$ is observed. In the first two of these, $X_1 > X_2$, so that $p_1 = 2/3$.

c)

$$p_2 = \frac{\Pr(X_3 > X_1 > X_2)}{\Pr(X_1 < X_3)} = \frac{1/3!}{1/2} = \frac{1}{3}.$$

Solution to Problem 6.2: If all the λ_i are equal, then, because the X_i are iid, the probability is just $1/n$. Otherwise, use the conditional probability formula (8.40), i.e.,

$$\Pr(A) = \int_{-\infty}^{\infty} \Pr(A \mid X = x)\, dF_X(x),$$

the fact that the X_i are independent, and that $F_{X_i}(x) = 1 - e^{-\lambda_i x}$ to get

$$\begin{aligned}
\Pr(X_i = S) &= \int_0^{\infty} \Pr(X_i = S \mid X_i = x)\, \lambda_i e^{-\lambda_i x} dx \\
&= \int_0^{\infty} \Pr(X_j > x,\ j \neq i)\, \lambda_i e^{-\lambda_i x} dx \\
&= \int_0^{\infty} \prod_{j \neq i} \left(1 - F_{X_j}(x)\right) \lambda_i e^{-\lambda_i x} dx \\
&= \lambda_i \int_0^{\infty} e^{-(\lambda_1 + \cdots + \lambda_n)x} dx = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}.
\end{aligned}$$

This clearly reduces to $1/n$ when all the λ_i are equal.

Solution to Problem 6.3: Using $\int udv = uv - \int vdu$ with

$$u = (1-t)^{n-i}, \quad du = -(n-i)(1-t)^{n-i-1}\, dt,$$

$$dv = t^{i-1}dt, \quad v = t^i/i,$$

$F_{Y_i}(y)$ in (15.7) is $\dfrac{n!}{(n-i)!\,(i-1)!}$ times

$$\int_0^{F(y)} t^{i-1}(1-t)^{n-i}\, dt = \int_0^{F(y)} udv =$$

$$\begin{aligned}
&= uv|_0^{F(y)} - \int_0^{F(y)} vdu = (1-t)^{n-i} \left.\frac{t^i}{i}\right|_0^{F(y)} + \frac{n-i}{i}\int_0^{F(y)} t^i (1-t)^{n-i-1}\, dt \\
&= \frac{(1-F(y))^{n-i}F(y)^i}{i} + \frac{n-i}{i}\int_0^{F(y)} t^i(1-t)^{n-i-1}\, dt
\end{aligned}$$

or

$$F_{Y_i}(y) = \binom{n}{i}(1-F(y))^{n-i}F(y)^i + \frac{n!}{(n-i-1)!i!}\int_0^{F(y)} t^i(1-t)^{n-i-1}\, dt.$$

Performing integration by parts on the latter integral with

$$u = (1-t)^{n-i-1}, \quad du = -(n-i-1)(1-t)^{n-i-2}\, dt,$$

$$dv = t^i dt, \quad v = t^{i+1}/(i+1)$$

yields

$$\begin{aligned}
\int_0^{F(y)} t^i(1-t)^{n-i-1}\, dt &= (1-t)^{n-i-1}\left.\frac{t^{i+1}}{i+1}\right|_0^{F(y)} \\
&\quad + \frac{n-i-1}{i+1}\int_0^{F(y)} t^{i+1}(1-t)^{n-i-2}\, dt \\
&= \frac{(1-F(y))^{n-i-1}F(y)^{i+1}}{i+1} \\
&\quad + \frac{n-i-1}{i+1}\int_0^{F(y)} t^{i+1}(1-t)^{n-i-2}\, dt
\end{aligned}$$

or

$$\begin{aligned}
F_{Y_i}(y) &= \binom{n}{i}(1-F(y))^{n-i}F(y)^i + \binom{n}{i+1}(1-F(y))^{n-i-1}F(y)^{i+1} \\
&\quad + \frac{n!}{(n-i-2)!\,(i+1)!}\int_0^{F(y)} t^{i+1}(1-t)^{n-i-2}\, dt.
\end{aligned}$$

The general pattern should now be clear. One more step assuming $n-i=3$ to terminate shows the latter term is, with

$$K = \frac{n!}{(n-i-2)!\,(i+1)!}$$

given by

$$
\begin{aligned}
& K\int_0^{F(y)} t^{i+1}(1-t)^{n-i-2}\,dt \\
= \; & K\frac{(1-F(y))^{n-i-2}F(y)^{i+2}}{i+2} + K\frac{n-i-2}{i+2}\int_0^{F(y)} t^{i+2}dt \\
= \; & \binom{n}{i+2}(1-F(y))^{n-i-2}F(y)^{i+2} + \frac{n!}{(n-i-3)!\,(i+2)!}\frac{F(y)^{i+3}}{i+3} \\
= \; & \binom{n}{i+2}(1-F(y))^{n-i-2}F(y)^{i+2} + \binom{n}{i+3}F(y)^{n},
\end{aligned}
$$

so that

$$F_{Y_i}(y) = \sum_{j=i}^{n}\binom{n}{j}(1-F(y))^{n-j}F(y)^{j}.$$

Solution to Problem 6.4: From (15.13),

$$f_{Y_1,Y_n}(x,y) = n(n-1)[F(y)-F(x)]^{n-2}f(x)f(y)\mathbb{I}_{(x,\infty)}(y), \tag{S-6.1}$$

and with

$$
\begin{aligned}
& R = Y_n - Y_1, \qquad T = (Y_1+Y_n)/2, \\
& r = y - x, \qquad t = (x+y)/2, \qquad x = t - r/2, \quad y = t + r/2,
\end{aligned}
$$

the Jacobian is

$$\mathbf{J} = \begin{bmatrix} \partial x/\partial r & \partial x/\partial t \\ \partial y/\partial r & \partial y/\partial t \end{bmatrix} = \begin{bmatrix} -1/2 & 1 \\ +1/2 & 1 \end{bmatrix}, \quad |\mathbf{J}| = -1,$$

so that a bivariate transformation gives $f_{R,T}(r,t) = f_{Y_1,Y_n}(t-r/2, t+r/2)$, which agrees with (15.18). The indicator function in (S-6.1) is $\mathbb{I}_{(t-r/2,\infty)}(t+r/2)$, which resolves to $t-r/2 < t+r/2$ or $r>0$.

Solution to Problem 6.5:

a) The expected value is $2\theta^{-2}\int_0^\theta x^2dx = 2\theta/3$ and the cdf is

$$F_X(x) = 2\theta^{-2}\int_0^x tdt = x^2\theta^{-2}\mathbb{I}_{(0,\theta)}(x).$$

b) From (15.8) and F_X, f_M simplifies to

$$f_M(m) = 2n\theta^{-2n}m\left(\theta^2-m^2\right)^{n-1}\mathbb{I}_{(0,\theta)}(m).$$

With $u=\theta^2-m^2$, $m=\sqrt{\theta^2-u}$ and $dm = -(1/2)\left(\theta^2-u\right)^{-1/2}du$, f_M integrates to one because

$$2n\theta^{-2n}\int_0^\theta m\left(\theta^2-m^2\right)^{n-1}dm = n\theta^{-2n}\int_0^{\theta^2}u^{n-1}du = 1.$$

For $\mathbb{E}[M]$, with $u = \theta^2 - m^2$, $m = \sqrt{\theta^2 - u}$ and $dm = -(1/2)(\theta^2 - u)^{-1/2} du$,

$$\mathbb{E}[M] = 2n\theta^{-2n} \int_0^\theta m^2 \left(\theta^2 - m^2\right)^{n-1} dm = n\theta^{-2n} \int_0^{\theta^2} \left(\theta^2 - u\right)^{1/2} u^{n-1} du.$$

Next, with $w = (\theta^2 - u)/\theta^2$, $u = (1 - w)\theta^2$, $du = -\theta^2 dw$ and

$$\begin{aligned} n\theta^{-2n} \int_0^{\theta^2} \left(\theta^2 - u\right)^{1/2} u^{n-1} du &= n\theta \int_0^1 w^{1/2} (1 - w)^{n-1} dw = n\theta B\left(\frac{3}{2}, n\right) \\ &= \frac{n\theta}{2}\sqrt{\pi} \frac{\Gamma(n)}{\Gamma(3/2 + n)}. \end{aligned}$$

With $\Gamma(3/2 + 1) = (3/2)\Gamma(3/2)$, for $n = 1$, $\mathbb{E}[M]$ reduces to $(2/3)\theta = \mathbb{E}[X]$. A plot of $\mathbb{E}[M]$ versus n for $\theta = 1$ is shown in Figure S-6.1.

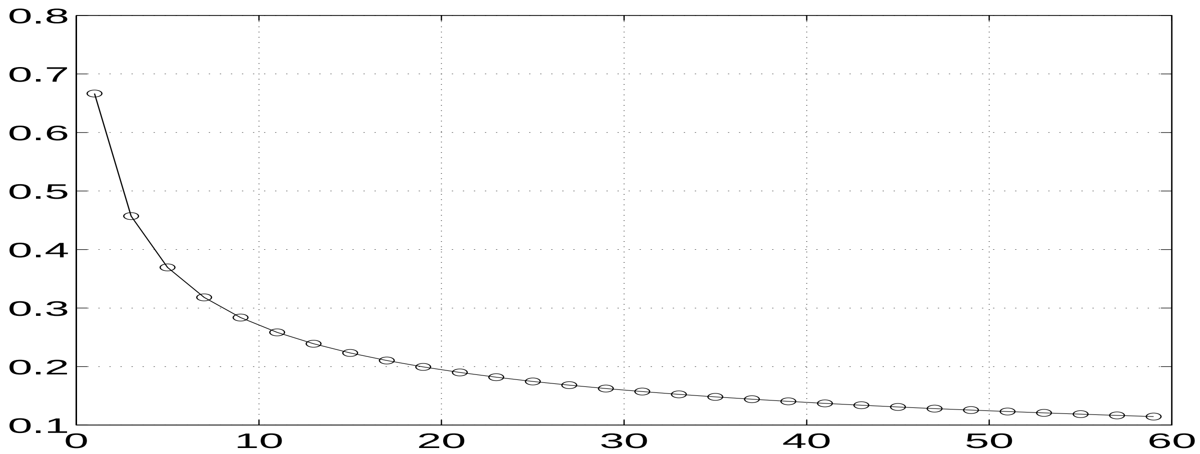

Figure S-6.1: Expected minimum from n iid observations, each with density $f_X(x) = 2\theta^{-2} x \mathbb{I}_{(0,\theta)}(x)$

c) The density of f_M simplifies to $f_M(m) = 2n\theta^{-2n} m^{2n-1} \mathbb{I}_{(0,\theta)}(m)$, which clearly integrates to one. Thus, from (15.8),

$$\mathbb{E}[M] = 2n\theta^{-2n} \int_0^\theta m^{2n} dm = \frac{2n}{2n+1}\theta$$

and, for $n = 1$, $\mathbb{E}[M] = \mathbb{E}[X]$.

Solution to Problem 6.6: Similar to the specific case in Example 15.11, with

$$K = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}$$

and using the binomial theorem,

$$\begin{aligned} f_P(p) &= K \int_p^{\sqrt{p}} x^{i-2} \left(\frac{p}{x} - x\right)^{j-i-1} \left(1 - \frac{p}{x}\right)^{n-j} dx \\ &= K \int_p^1 x^{i-2} \left(\frac{p}{x} - x\right)^{j-i-1} \sum_{k=0}^{n-j} \binom{n-j}{k} (-1)^k \left(\frac{p}{x}\right)^k dx. \end{aligned}$$

Similar substitutions as in the example yield

$$\begin{aligned}
f_P(p) &= \frac{K}{2}\sum_{k=0}^{n-j}\binom{n-j}{k}(-1)^k p^{\frac{1}{2}(-2+j+k)}\int_p^1 v^{-\frac{1}{2}(2-2i+j+k)}(1-v)^{j-i-1}\,dv \\
&= \frac{K}{2}\sum_{k=0}^{n-j}\binom{n-j}{k}(-1)^k p^{\frac{1}{2}(-2+j+k)}(1-p)^{j-i} \\
&\quad \times\int_0^1 u^{j-i-1}(1-u(1-p))^{-\frac{1}{2}(2-2i+j+k)}\,du
\end{aligned}$$

or

$$\begin{aligned}
f_P(p) &= \frac{K}{2}\sum_{k=0}^{n-j}\binom{n-j}{k}(-1)^k p^{\frac{1}{2}(-2+j+k)}(1-p)^{j-i}\;B(j-i,1) \\
&\quad \times {}_2F_1\left(j-i,\frac{1}{2}(2-2i+j+k);j-i+1;1-p\right) \\
&= \frac{1}{2}\frac{n!}{(i-1)!\,(j-i)!\,(n-j)!}(1-p)^{j-i}p^{j/2-1}\sum_{k=0}^{n-j}\binom{n-j}{k}(-1)^k p^{k/2} \\
&\quad \times {}_2F_1\left(j-i,\left(\frac{1}{2}(j+k)-i+1\right);j-i+1;1-p\right).
\end{aligned}$$

The programs are given in Listings S-6.1 and S-6.2. Also, the density and scaled histogram plot was constructed using the following code:

```
v=simorderprod(8,3,6,10000); pvec=0.01:0.01:0.99;
f=orderprodden(8,pvec,3,6); [histcount, histgrd]=hist(v,100);
multiplier = max(histcount)/max(f);
h1=bar(histgrd,histcount/multiplier); ax1=gca;
colormap([0.8 0.8 0.8]); set(get(ax1,'XLabel'),'String','');
set(get(ax1,'YLabel'),'String','');
hold on; h1=plot(pvec,f,'black'); hold off
set(gca,'fontsize',16)
```

Solution to Problem 6.7:

a) Clearly, for $y \leq 0$, $F_Y(y) = 0$, for $y \geq 1$, $F_Y(y) = 1$ and for $0 < y < 1$, $F_Y(y) = \frac{y}{1+y}$. That is, with $u = 1+x$,

$$F_Y(y) = \int_0^y (1+x)^{-2}\,dx = \int_{0+1}^{y+1} u^{-2}du = -u^{-1}\Big|_1^{y+1} = 1 - \frac{1}{y+1} = \frac{y}{1+y}.$$

Thus,

$$F_Y(y) = \begin{cases} 0, & \text{if} \quad y \leq 0, \\ \frac{y}{1+y}, & \text{if} \quad 0 < y < 1, \\ 1, & \text{if} \quad y \geq 1. \end{cases}$$

```
function f=orderprodden(n,pvec,i,j)
% orderprodden(n,pvec=0.01:0.01:0.99,i=1,j=n)
% From n iid uniform(0,1) r.v.s, let P=Y_i * Y_j.
% This computes the density of P over pvec

if nargin<2, pvec=0.01:0.01:0.99; end
if nargin<3, i=1; end
if nargin<4, j=n; end

tol=1e-6; % tolerance for the numeric integration
int=zeros(1,length(pvec));
kon = fact(n) / ( fact(i-1) * fact(j-i-1) * fact(n-j) );
for ploop=1:length(pvec)
  p=pvec(ploop);
  int(ploop) = quadl(@orderprodden_,p,sqrt(p),tol,[],n,i,j,p);
end
f = kon .* int;

function f=fact(t)
if t==0, f=1; else f=t*fact(t-1); end

function f=orderprodden_(x,n,i,j,p)
f = x.^(i-2) .* (p./x - x).^(j-i-1) .* (1-p./x).^(n-j);
```

Program Listing S-6.1: From n iid uniform(0,1) r.v.s, let $P = Y_i * Y_j$, where $Y_1, Y_2, \ldots, Y_n$ are the order statistics. This computes the density of P at values in vector `pvec`.

b) Directly from (4.36), $\mathbb{E}[Y] = \int_0^1 x(1+x)^{-2}\,dx + \int_1^\infty 1(1+x)^{-2}\,dx = \ln(2)$. The former integral follows by taking $u = 1 + x$, so that

$$\int_0^1 \frac{x}{(1+x)^2}dx = \int_1^2 \frac{u-1}{u^2}du = \int_1^2 u^{-1}du - \int_1^2 u^{-2}du = \ln 2 - \frac{1}{2},$$

and the latter, as

$$\int_1^\infty (1+x)^{-2}\,dx = \int_2^\infty u^{-2}du = \frac{1}{2}.$$

c) $\mathbb{E}[Z] = \int_0^1 1(1+x)^{-2}\,dx + \int_1^\infty x(1+x)^{-2}\,dx = +\infty$, i.e., it does not exist.

d) First, $\Pr(G \le g) = \Pr(\ln(1+X) \le g) = \Pr(X \le e^g - 1) = F_X(e^g - 1)$. But as

$$F_X(y) = \int_0^y (1+x)^{-2}\,dx = \frac{y}{1+y},$$

it follows that

$$F_X(e^g - 1) = \frac{e^g - 1}{e^g - 1 + 1} = 1 - e^{-g}$$

```
function v=simorderprod(n,i,j,sim)
v=zeros(sim,1);
for k=1:sim
  h=sort(rand(n,1)); v(k)=h(i)*h(j);
end
```

Program Listing S-6.2: Simulates the density of P

and, hence, $G \sim \text{Exp}(1)$.

Solution to Problem 6.8:

a) Recalling Example 11.13 and noting that the sum of the first two order statistics will be, on average, less than the sum of any two randomly chosen observations, the answer must be less than 1/2. From (15.13),

$$f_{X_{(1)},X_{(2)}}(x_1, x_2) = 3!\,(1 - x_2) \quad 0 < x_1 < x_2 < 1.$$

Note that only for $X_{(2)} > 1/2$ can $X_{(1)} + X_{(2)} \geq 1$. Also, for given $1/2 < X_{(2)} < 1$, in order for $X_{(1)} + X_{(2)} \geq 1$, $X_{(1)}$ has to be at least $1 - X_{(2)}$, but can be at most $X_{(2)}$. Thus

$$\Pr\left(X_{(1)} + X_{(2)} > 1\right) = 6\int_{\frac{1}{2}}^{1}\int_{1-x_2}^{x_2}(1 - x_2)\,dx_1 dx_2 = \frac{1}{4}.$$

b) Similarly, $f_{X_{(2)},X_{(3)}}(x_2, x_3) = 6x_2$, $0 < x_2 < x_3 < 1$, which is a valid bivariate density because $\int_0^1\int_{x_2}^1 x_2 dx_3 dx_2 = 1/6$. As before,

$$\Pr\left(X_{(2)} + X_{(3)} > 1\right) = 6\int_{\frac{1}{2}}^{1}\int_{1-x_3}^{x_3} x_2 dx_2 dx_3 = \frac{3}{4},$$

which might have been guessed, given the symmetry of the problem.

c) With $f_{X_{(1)}}(x) = 3(1 - x)^2\,\mathbb{I}_{(0,1)}(x)$,

$$\mathbb{E}\left[X_{(1)}\right] = 3\int_0^1 x(1 - x)^2\,dx = \frac{1}{4}, \qquad \mathbb{E}\left[X_{(1)}^2\right] = 3\int_0^1 x^2(1 - x)^2\,dx = \frac{1}{10},$$

so that

$$\mathbb{V}\left(X_{(1)}\right) = \frac{1}{10} - \frac{1}{16} = \frac{3}{80}.$$

Solution to Problem 6.9:

a) Consider the value of the joint distribution function of V and W at all real values v and w. We distinguish the cases $v \leq w$ and $v \geq w$. First, let $v \leq w$: we then have $V \leq v$ and $W \leq w$ precisely if $X, Y \leq w$, but not both X and Y have their value in $(v, w]$. This yields

$$\begin{aligned} F_{V,W}(v, w) &= \Pr(X \leq w \wedge Y \leq w) - \Pr(v < X \leq w \wedge v \leq Y \leq w) \\ &= F_X(w)F_Y(w) - (F_X(w) - F_X(v))(F_Y(w) - F_Y(v)). \end{aligned}$$

If X and Y are iid, then this reduces precisely to (15.12).

Now if $v \geq w$, the $W \leq w$ implies $V \leq W \leq w \leq v$, so

$$F_{V,W}(v,w) = \Pr(W \leq w) = \Pr(X, Y \leq w) = F_X(w)F_Y(w).$$

We observe with satisfaction that both formulae coincide if $v = w$.

b) Compute the derivative $\frac{\partial^2}{\partial w \partial v} F_{V,W}(v,w)$ where it is possible. Note that $F_{V,W}(v,w)$ is not necessarily differentiable if $v = w$. If $v > w$, the cdf of (V, W) is independent of v, so

$$\frac{\partial}{\partial v} F_{V,W}(v,w) = 0.$$

Therefore $\frac{\partial^2}{\partial w \partial v} F_{V,W}(v,w) = 0$ as well. If $v < w$,

$$\frac{\partial}{\partial v} F_{V,W}(v,w) = f_X(v)\left(F_Y(w) - F_Y(v)\right) + \left(F_X(w) - F_X(v)\right) f_Y(v).$$

Hence,

$$\frac{\partial^2}{\partial w \partial v} F_{V,W}(v,w) = f_X(v)f_Y(w) + f_X(w)f_Y(v)$$

and the joint density of (V, W) is

$$f_{V,W}(v,w) = \begin{cases} f_X(v)f_Y(w) + f_X(w)f_Y(v), & \text{if } v < w, \\ 0, & \text{if } v > w. \end{cases}$$

If $v = w$, the density could be set arbitrarily as the set where $v = w$ does not matter when integrating the density (it has measure zero).

c) Suppose $v, w \geq 0$. Then

$$F_{V,W}(v,w) = (1 - e^{-\alpha w})(1 - e^{-\beta w})$$

if $v \geq w$ and

$$F_{V,W}(v,w) = (1 - e^{-\alpha w})(1 - e^{-\beta w}) - \left(e^{-\alpha v} - e^{-\alpha w}\right)\left(e^{-\beta v} - e^{-\beta w}\right)$$

if $v \leq w$. If $v < w$ the pdf is

$$f_{V,W}(v,w) = \alpha\beta\left(e^{-\alpha v - \beta w} + e^{-\alpha w - \beta v}\right).$$

If $w > v$ the density is zero.

Solution to Problem 6.10:

a) Let $U_i = \left[X_i - (\alpha - \beta)\right]/2\beta$ so $U_{i:n} = \left[X_{i:n} - (\alpha - \beta)\right]/2\beta$.

b) The mean is given by

$$\mathbb{E}\left[\hat{\alpha}_1\right] = \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \mathbb{E}\left[X\right] = \int_{\alpha-\beta}^{\alpha+\beta} \frac{1}{(\alpha+\beta) - (\alpha-\beta)} x dx = \alpha,$$

so that $\hat{\alpha}_1$ is an "unbiased" estimator of α. For the variance, we recall that $\mathbb{V}\left(U_i\right) = 1/12$, so that $\mathbb{V}\left(X_i\right) = (2\beta)^2/12$ and, hence, $\mathbb{V}\left(\bar{U}_n\right) = 1/\left(12n\right)$ and

$$\mathbb{V}\left(\bar{X}_n\right) = \frac{(2\beta)^2}{12n} = \frac{\beta^2}{3n}.$$

c) Recall that the cdf and pdf of U_i are $F_{U_i}(u) = u$ and $f_{U_i}(u) = 1$, respectively. From (15.6), it follows directly that $U_{(k+1):(2k+1)} \sim \text{Beta}(k+1, k+1)$.

d) Using (7.12) and (7.13),

$$\mathbb{E}[\hat{\alpha}_2] = 2\beta\mathbb{E}\left[U_{(k+1:2k+1)}\right] + (\alpha - \beta) = \alpha$$

and

$$\mathbb{V}\left(U_{(k+1):(2k+1)}\right) = \frac{(k+1)^2}{(2k+3)(2k+2)^2} = \frac{1}{4(n+2)},$$

where $n = 2k+1$, so that

$$\mathbb{V}(\hat{\alpha}_2) = \mathbb{V}\left(2\beta U_{(k+1):(2k+1)}\right) = \frac{\beta^2}{n+2}.$$

e) Their joint distribution is, from (15.13),

$$\begin{aligned} f_{U_{1:n},U_{n:n}}(x,y) &= n(n-1)[F(y) - F(x)]^{n-2} f(x) f(y) \mathbb{I}_{(x,1)}(y) \\ &= n(n-1)(y-x)^{n-2} \mathbb{I}_{(x,1)}(y), \end{aligned}$$

and $U_{1:n} \sim \text{Beta}(1,n)$ and $U_{n:n} \sim \text{Beta}(n,1)$. We know

$$\text{Cov}(U_{1:n}, U_{n:n}) = \mathbb{E}[U_{1:n}U_{n:n}] - \mathbb{E}[U_{1:n}]\mathbb{E}[U_{n:n}],$$

$\mathbb{E}[U_{1:n}] = \frac{1}{1+n}$, $\mathbb{E}[U_{n:n}] = \frac{n}{1+n}$ and

$$\begin{aligned} \mathbb{E}[U_{1:n}U_{n:n}] &= n(n-1)\int_0^1 \int_x^1 xy(y-x)^{n-2}\,dydx \\ &= n(n-1)\int_0^1 \left(\frac{x(1-x)^n}{n} + \frac{x^2(1-x)^{n-1}}{n-1}\right) dx \\ &= \frac{n(n-1)}{(n-2+n^2)\,n} = \frac{1}{n+2}, \end{aligned}$$

so that

$$\text{Cov}(U_{1:n}, U_{n:n}) = \frac{1}{n+2} - \frac{1}{1+n}\frac{n}{1+n} = \frac{1}{(n+2)(n+1)^2}.$$

f) Use $f_X(x) = (2\beta)^{-1}\mathbb{I}_{(\alpha-\beta,\alpha+\beta)}(x)$,

$$F_X(x) = \frac{x-(\alpha-\beta)}{2\beta}\mathbb{I}_{(\alpha-\beta,\alpha+\beta)}(x) + \mathbb{I}_{[\alpha+\beta,\infty)}(x)$$

and (15.18) to get the joint density of the range $R = X_{n:n} - X_{1:n}$ and the midrange $T = \frac{1}{2}(X_{n:n} + X_{1:n})$, i.e., $f_{R,T}(r,t)$ is

$$\begin{aligned} & n(n-1)\left(\frac{r}{2\beta}\right)^{n-2}\left(\frac{1}{2\beta}\right)^2 \mathbb{I}_{(\alpha-\beta,\alpha+\beta)}(t-r/2)\,\mathbb{I}_{(\alpha-\beta,\alpha+\beta)}(t+r/2)\,\mathbb{I}_{(0,\infty)}(r) \\ &= n(n-1)\,r^{n-2}(2\beta)^{-n}\,\mathbb{I}_{(\alpha-\beta+r/2,\alpha+\beta-r/2)}(t)\,\mathbb{I}_{(0,2\beta)}(r). \end{aligned}$$

This follows, as

$$F_X\left(t+r/2\right)-F_X\left(t-r/2\right)=r/\left(2\beta\right)$$

and the fact that, with both β and r positive, the lower and upper bound for t from the two inequalities

$$\mathbb{I}_{(\alpha-\beta,\alpha+\beta)}\left(t-r/2\right)\quad\text{and}\quad\mathbb{I}_{(\alpha-\beta,\alpha+\beta)}\left(t+r/2\right)$$

becomes

$$\alpha-\beta+r/2\leq t\leq\alpha+\beta-r/2. \tag{S-6.2}$$

Also, if $r>2\beta$, say $r=2\beta+\epsilon$, substitution into (S-6.2) gives

$$\alpha+\epsilon/2\leq t\leq\alpha-\epsilon/2,$$

which is invalid for any $\epsilon>0$. Thus, it must be the case that $r\leq 2\beta$. The density of T is given by

$$\begin{aligned}
f_T\left(t\right)&=\int_0^\infty f_{R,T}\left(r,t\right)dr\\
&=\frac{n\left(n-1\right)}{\left(2\beta\right)^n}\left\{\int_0^{2t-2(\alpha-\beta)}r^{n-2}dr\mathbb{I}_{(\alpha-\beta,\alpha)}\left(t\right)+\int_0^{2(\alpha+\beta)-2t}r^{n-2}dr\mathbb{I}_{(\alpha,\alpha+\beta)}\left(t\right)\right\}\\
&=\frac{n}{2\beta}\left\{\left(\frac{t-\alpha}{\beta}+1\right)^{n-1}\mathbb{I}_{(\alpha-\beta,\alpha)}\left(t\right)+\left(1-\frac{t-\alpha}{\beta}\right)^{n-1}\mathbb{I}_{(\alpha,\alpha+\beta)}\left(t\right)\right\}
\end{aligned}$$

where the range of integration follows from a sketch of the support of the joint distribution (see, for example, Figure 5.5.1 in Casella and Berger (1990, p. 236). Some thought also reveals that the two integrals could be combined as

$$f_T\left(t\right)=n\left(n-1\right)\left(2\beta\right)^{-n}\int_0^{\min\{2t-2(\alpha-\beta),2(\alpha+\beta)-2t\}}r^{n-2}dr\mathbb{I}_{(\alpha-\beta,\alpha+\beta)}\left(t\right)$$

(see also Mood *et al.*, 1974, p. 256). Finally, temporarily defining $Z:=\beta^{-1}\left(T-\alpha\right)$, $dt=\beta dz$,

$$\mathbb{I}_{(\alpha-\beta,\alpha)}\left(t\right)\Longleftrightarrow 0<z<1\quad\text{and}\quad\mathbb{I}_{(\alpha,\alpha+\beta)}\left(t\right)\Longleftrightarrow -1<z<0$$

so that $\mathbb{E}\left[T\right]=\beta\mathbb{E}\left[Z\right]+\alpha$ and

$$\begin{aligned}
\mathbb{E}\left[Z\right]&=\frac{n}{2\beta}\int_{-1}^1 f_Z\left(z\right)dz=\frac{n}{2\beta}\left\{\int_{-1}^0 z\left(z+1\right)^{n-1}\beta dz+\int_0^1 z\left(1-z\right)^{n-1}\beta dz\right\}\\
&=\frac{n}{2}\left\{-\int_0^1\left(1-v\right)v^{n-1}+\int_0^1 z\left(1-z\right)^{n-1}dz\right\}\\
&=\frac{n}{2}\left\{-B\left(n,2\right)+B\left(2,n\right)\right\}=0,
\end{aligned}$$

where $v=z+1$, so that $\mathbb{E}\left[T\right]=\alpha$. For the variance, observe

$$\mathbb{V}\left(U_{1:n}\right)=\mathbb{V}\left(U_{n:n}\right)=\frac{n}{\left(n+2\right)\left(n+1\right)^2},$$

so

$$\frac{1}{4}\mathbb{V}\left(U_{1:n}+U_{n:n}\right) = \frac{1}{4}\left\{\mathbb{V}\left(U_{1:n}\right)+\mathbb{V}\left(U_{n:n}\right)+2\operatorname{Cov}\left(U_{1:n},U_{n:n}\right)\right\}$$
$$= \frac{1}{4}\left(\frac{2n+2}{(n+2)\,(n+1)^2}\right) = \frac{1}{2\,(n+1)\,(n+2)}.$$

Thus,

$$\mathbb{V}\left(\hat{\alpha}_3\right) = \mathbb{V}\left(\frac{X_{1:n}+X_{n:n}}{2}\right) = (2\beta)^2\,\mathbb{V}\left(\frac{U_{1:n}+U_{n:n}}{2}\right) = \frac{2\beta^2}{(n+1)\,(n+2)}.$$

g) Summarizing the above results,

$$\mathbb{V}\left(\hat{\alpha}_1\right) = \frac{\beta^2}{3n},\quad \mathbb{V}\left(\hat{\alpha}_2\right) = \frac{\beta^2}{n+2}\quad\text{and}\quad \mathbb{V}\left(\hat{\alpha}_3\right) = \frac{2\beta^2}{(n+1)\,(n+2)}.$$

For $n=1$, the variances are the same, $\beta^2/3$, because the estimators are identical, namely $\hat{\alpha}_i = X_1 = X_{1:1}$, $i=1,2,3$. For $n=2$, $\hat{\alpha}_1=\hat{\alpha}_3$ and their variances are (necessarily) again identical, $\beta^2/6$. Estimator $\hat{\alpha}_2$ is not defined in this case, as we assumed that n is odd. For $n \geq 3$,

$$\mathbb{V}\left(\hat{\alpha}_2\right) > \mathbb{V}\left(\hat{\alpha}_1\right) > \mathbb{V}\left(\hat{\alpha}_3\right)$$

and, as all three estimators are unbiased, i.e., $\mathbb{E}\left[\alpha_i\right]=\alpha$, $i=1,2,3$, it follows that one should prefer $\hat{\alpha}_3$, the sample midrange, as an estimator of α.

Solution to Problem 6.11: From $D_j = Y_{j+1} - Y_j$ and $Z = Y_{j+1}$, we have $y_{j+1} = z$ and $y_j = z - d_j$. From

$$f_{Y_j,Y_{j+1}}\left(y_j,y_{j+1}\right) = K\left(1-e^{-\alpha y_j}\right)^{j-1} e^{-\alpha(n-j-1)y_{j+1}} e^{-\alpha(y_j+y_{j+1})} \mathbb{I}_{(0,\infty)}\left(y_{j+1}\right)\mathbb{I}_{(0,y_{j+1})}\left(y_j\right)$$

with

$$K = \frac{n!}{(j-1)!\,(n-j-1)!}\alpha^2$$

and $|J|=1$,

$$\begin{aligned} f_{D_j,Z}\left(d_j,z\right) &= f_{Y_j,Y_{j+1}}\left(z-d_j,z\right) \\ &= K\left(1-e^{-\alpha(z-d_j)}\right)^{j-1} e^{-\alpha(n-j-1)z} e^{-\alpha(2z-d_j)} \mathbb{I}_{(0,z)}\left(d_j\right). \end{aligned}$$

Thus,

$$f_{D_j}\left(d_j\right) = Ke^{\alpha d_j}\int_{d_j}^{\infty}\left(1-e^{-\alpha(z-d_j)}\right)^{j-1} e^{-\alpha(n-j+1)z}dz.$$

With the substitution

$$u = 1-e^{-\alpha(z-d_j)},\quad z = d_j - \alpha^{-1}\ln\left(1-u\right),\quad dz = \alpha^{-1}\frac{du}{1-u}$$

the limits of integration become

$$z \to \infty \Leftrightarrow u \to 1 \quad \text{and} \quad z = d_j \Leftrightarrow u = 0$$

and

$$e^{-\alpha(n-j+1)z} = e^{-\alpha(n-j+1)\left(d_j - \alpha^{-1}\ln(1-u)\right)} = e^{-\alpha d_j(n-j+1)}\left(1-u\right)^{(n-j+1)}$$

so that

$$\begin{aligned} f_{D_j}\left(d_j\right) &= K e^{\alpha d_j} e^{-\alpha(n-j+1)d_j} \int_0^1 u^{j-1}\left(1-u\right)^{(n-j+1)} \frac{\alpha^{-1}}{1-u} du \\ &= \frac{n!\alpha}{(j-1)!\,(n-j-1)!} e^{-\alpha(n-j)d_j} \int_0^1 u^{j-1}\left(1-u\right)^{(n-j)} du \\ &= \frac{n!\alpha}{(j-1)!\,(n-j-1)!} e^{-\alpha(n-j)d_j} B\left(j, n-j+1\right) \\ &= \frac{n!}{(j-1)!\,(n-j-1)!} \frac{\Gamma(j)\,\Gamma(n-j+1)}{\Gamma(n+1)} \alpha e^{-\alpha(n-j)d_j} \\ &= (n-j)\,\alpha e^{-\alpha(n-j)d_j} \\ &= \kappa e^{-\kappa d_j}, \end{aligned}$$

where $\kappa = \alpha\,(n-j)$, so that $D_j \sim \exp(\kappa)$.

Solution to Problem 6.12: From (11.8) and the joint density (15.34), the density of $-D = Y_1 - Y_2$ is given by

$$\begin{aligned} f_{-D}(d) &= \int_{-\infty}^{\infty} f_{Y_1,Y_2}\left(y_1, y_1 - d\right) dy_1 \\ &= n\,(n-1)\,\alpha^2 \int_0^{\infty} e^{-\alpha(y_1+y_1-d)} \left(e^{-\alpha(y_1-d)}\right)^{n-2} dy_1 \\ &= n\,(n-1)\,\alpha^2 e^{d\alpha(n-1)} \int_0^{\infty} \exp\left(-\alpha n y_1\right) dy_1 \\ &= \alpha\,(n-1)\,e^{d\alpha(n-1)}. \end{aligned}$$

Then, a simple transformation using (7.65) gives

$$f_D(d) = \alpha\,(n-1)\,e^{-d\alpha(n-1)}.$$

Similarly, (15.13) with $i = 1$ and $j = n$ simplifies to

$$f_{Y_1,Y_n}(x,y) = n\,(n-1)\,\left(e^{-\alpha x} - e^{-\alpha y}\right)^{n-2} \alpha^2 e^{-\alpha(x+y)}\,\mathbb{I}_{(x,\infty)}(y),$$

so that, from (11.8) for $-R = Y_1 - Y_n$,

$$\begin{aligned} f_{-R}(r) &= \int_{-\infty}^{\infty} f_{Y_1,Y_n}\left(y_1, y_1 - r\right) dy_1 \\ &= \alpha^2 n\,(n-1) \int_{-\infty}^{\infty} \left(e^{-\alpha y_1} - e^{-\alpha(y_1-r)}\right)^{n-2} e^{-\alpha(y_1+y_1-r)} dy_1 \\ &= \alpha^2 n\,(n-1)\left(1 - e^{\alpha r}\right)^{n-2} e^{r\alpha} \int_0^{\infty} e^{-\alpha n y_1} dy_1 \\ &= \alpha\,(n-1)\left(1 - e^{\alpha r}\right)^{n-2} e^{\alpha r}\,\mathbb{I}_{(0,\infty)}(-r). \end{aligned}$$

Thus, for $R = Y_n - Y_1$,

$$f_R(r) = \alpha(n-1)\left(1-e^{-\alpha r}\right)^{n-2} e^{-\alpha r}\mathbb{I}_{(0,\infty)}(r).$$

Solution to Problem 6.13:

a) Using the cdf $F_X(x) = 6\int_0^x t(1-t)\,dt = x^2(3-2x)$ and (15.8),

$$f_S(s) = 12s(2s+1)(1-s)^3\,\mathbb{I}_{(0,1)}(s)$$

and

$$f_L(l) = 12l^3(3-2l)(1-l)\,\mathbb{I}_{(0,1)}(l).$$

Also,

$$\mu_S = \int_0^1 s f_S(s)\,ds = \frac{13}{35} \approx 0.37143$$

and

$$\sigma_S^2 = \mathbb{E}\left[S^2\right] - \mathbb{E}[S]^2 = \frac{41}{1225} \approx 0.033469,$$

with similar calculations yielding $\mu_L = 22/35 = 1 - \mu_S$ and $\sigma_L^2 = \sigma_S^2$.

b) From the symmetry of $f_X(x)$, the expected values of S and L are equally spaced from the endpoints of the distribution, so that one would expect that their variances coincide.

c) From (15.13), the joint distribution of S and L is

$$f_{S,L}(s,l) = 72s(1-s)\,l(1-l)\,\mathbb{I}_{(0,1)}(s)\,\mathbb{I}_{(s,1)}(l)$$

and using the former equation in (11.10), the density of R is

$$\begin{aligned}
f_R(r) &= \int_{-\infty}^{\infty} \frac{|s|}{r^2} f_{S,L}\left(s, \frac{s}{r}\right) ds \\
&= 72\int_{-\infty}^{\infty} \frac{s}{r^2} s(1-s)\frac{s}{r}\left(1-\frac{s}{r}\right)\mathbb{I}_{(s,1)}\left(\frac{s}{r}\right)\mathbb{I}_{(0,1)}(s)\,ds \\
&= 72\int_0^r \frac{s}{r^2} s(1-s)\frac{s}{r}\left(1-\frac{s}{r}\right) ds = \frac{6}{5} r(3-2r)\,\mathbb{I}_{(0,1)}(r),
\end{aligned}$$

as

$$\mathbb{I}_{(s,1)}\left(\frac{s}{r}\right)\mathbb{I}_{(0,1)}(s) \Rightarrow 0 < s < \frac{s}{r} < 1 \Rightarrow 0 < rs < s < r \Rightarrow 0 < s < r.$$

Direct calculation shows that $\mu_R = 3/5$ and $\sigma_R^2 = 3/50$.

d) By using 12,000 simulated draws, we get a very accurate kernel density estimate of the true density. This is portrayed in Figure S-6.2 as the dashed line, which compares well with the true density, shown as a solid line. For these draws, we obtained a mean of 0.6020, a sample variance of 0.6000 and a sample covariance between s and l of 0.01606, which are very close to their true values.

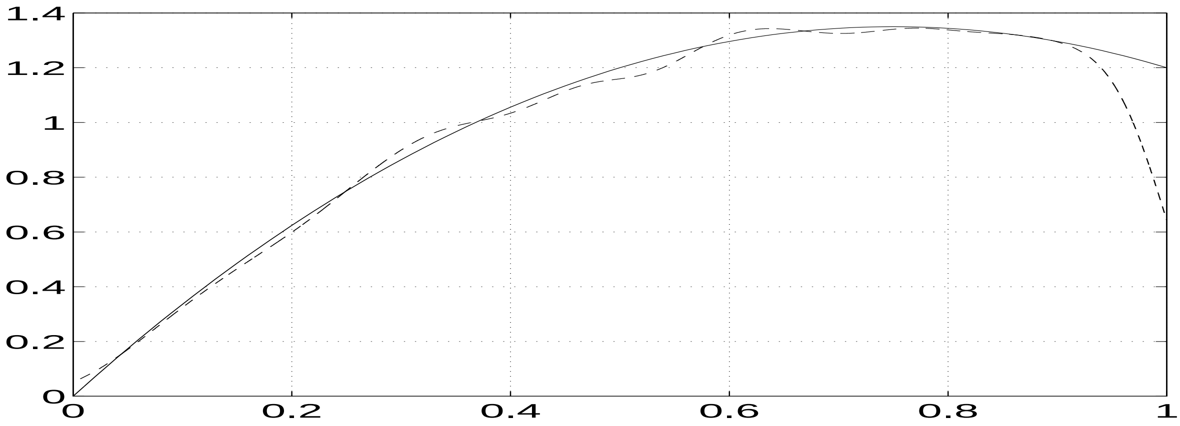

Figure S-6.2: Theoretical (solid) and kernel density from 12,000 simulated Beta(2,2) draws

e)

$$\begin{aligned} c &= \mathbb{E}[SL] - \mathbb{E}[S]\,\mathbb{E}[L] \\ &= 72\int_0^1 \int_s^1 sl \cdot s(1-s)\,l(1-l)\,dlds - \frac{13}{35}\frac{22}{35} \\ &= \frac{81}{4900} \approx 0.01653 \end{aligned}$$

and

$$\operatorname{Corr}(S,L) = \frac{\operatorname{Cov}(S,L)}{\sqrt{\mathbb{V}(S)\,\mathbb{V}(L)}} = \frac{\frac{81}{4900}}{\frac{41}{1225}} \approx 0.4939.$$

f) From (11.32),

$$\begin{aligned} \frac{3}{5} = \mu_R &\approx \frac{\mu_S}{\mu_L} - \mu_L^{-2}\operatorname{Cov}(S,L) + \mu_S\mu_L^{-3}\sigma_L^2 \\ &= \frac{13}{22} - \left(\frac{22}{35}\right)^{-2} c + \frac{13}{35}\left(\frac{22}{35}\right)^{-3}\frac{41}{1225} \end{aligned}$$

which yields $c = 0.0162$, while (11.33) gives

$$\sigma_R^2 = \frac{3}{50} \approx \left(\frac{\mu_S}{\mu_L}\right)^2 \left(\frac{\sigma_S^2}{\mu_S^2} + \frac{\sigma_L^2}{\mu_L^2} - \frac{2\operatorname{Cov}(S,L)}{\mu_S\mu_L}\right)$$

or $c = 0.0182$. Similar to Example 15.14, the approximation using μ_R yields a more accurate result and, in this case, is quite accurate.

Solution to Problem 6.14:

a) The joint density of the order statistics is

$$f_{\mathbf{Y}}(\mathbf{y}) = n!a^n\theta^{-an}\prod_{i=1}^n y_i^{a-1}, \quad 0 < y_1 < y_2 < \cdots < y_n < \theta.$$

For $1 \leq j < n$, $0 < Z_j < 1$ and $0 < Z_n < \theta$. Next, starting with Y_n and backsubstituting, the Y_j can be expressed as functions of the $\{Z_i\}_{i=1}^n$ given by

$$Y_j = \prod_{i=j}^{n} Z_i.$$

It follows that

$$\frac{\partial y_i}{\partial z_j} = \begin{cases} z_{i+1}\, z_{i+2} \cdots z_n, & \text{if} \quad i = j < n, \\ 1, & \text{if} \quad i = j = n, \\ 0, & \text{if} \quad i > j, \end{cases}$$

i.e., the Jacobian is the upper triangular matrix

$$\mathbf{J} = \begin{bmatrix} \prod_{i=2}^{n} z_i & & & & \\ 0 & \prod_{i=3}^{n} z_i & & & \\ 0 & & \ddots & & \\ \vdots & & & z_n & \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

with determinant the product of its diagonal elements

$$|\mathbf{J}| = \prod_{i=2}^{n} z_i \cdot \prod_{i=3}^{n} z_i \cdot \cdots \cdot z_n \cdot 1 = z_2 z_3^2 \ldots z_n^{n-1} = \prod_{i=2}^{n} z_i^{i-1}.$$

Hence,

$$\begin{aligned} f_{\mathbf{Z}}(\mathbf{z}) &= f_{\mathbf{Y}}(\mathbf{y})\,|\mathbf{J}| \\ &= n! a^n \theta^{-an} (z_1 z_2 \ldots z_n)^{a-1} (z_2 \ldots z_n)^{a-1} \cdots z_n^{a-1} \times z_2 z_3^2 \ldots z_n^{n-1} \\ &= n! a^n \theta^{-an} z_1^{a-1}\, z_2^{2(a-1)+1} \cdots\ z_j^{j(a-1)+(j-1)} \cdots\ z_n^{n(a-1)+n-1} \\ &= n! a^n \theta^{-an} \prod\nolimits_{i=1}^{n} z_i^{ia-1}, \end{aligned}$$

with corresponding indicator function

$$\mathbb{I}_{(0,1)}(z_1) \cdot \mathbb{I}_{(0,1)}(z_2) \cdot \cdots \cdot \mathbb{I}_{(0,1)}(z_{n-1}) \cdot \mathbb{I}_{(0,\theta)}(z_n).$$

b) As $f_{\mathbf{Z}}(\mathbf{z}) = \prod_{i=1}^{n} f_{Z_i}(z_i)$, the Z_j are independent and inspection of the indicator function suggests that only $f_{Z_n}(z_n)$ will depend on θ. That is,

$$f_{Z_j}(z_j) = \begin{cases} ja z_j^{ja-1} \mathbb{I}_{(0,1)}(z_j), & \text{if} \quad 1 \leq j < n, \\ na\theta^{-na} z_n^{na-1} \mathbb{I}_{(0,\theta)}(z_n), & \text{if} \quad j = n. \end{cases}$$

For $1 \leq j < n$,

$$ja \int_0^1 z_j^{ja-1} dz_j = ja\,(ja)^{-1}\, z_j^{ja}\Big|_0^1 = 1$$

and, for $j = n$, the density function $f_{Z_n}(z_n)$ is the same as $f_X(x)$ with a replaced by na, so that it is a proper density.

c) Simple integration shows that

$$\mathbb{E}[Z_j] = \begin{cases} \frac{ja}{ja+1}, & \text{if} \quad 1 \le j < n, \\ \theta \frac{na}{na+1}, & \text{if} \quad j = n. \end{cases}$$

d) With $w_i = y_i/y_n$, $i = 1, 2, \ldots, n-1$ and $w_n = y_n$, the inverse transformation is $y_i = w_i w_n$, $i = 1, 2, \ldots, n-1$ and Jacobian

$$J = \left[\frac{\partial y_i}{\partial w_j}\right]_{i,j=1,\ldots,n} = \begin{bmatrix} w_n & 0 & \cdots & 0 & w_1 \\ 0 & w_n & & 0 & w_2 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & & w_n & w_{n-1} \\ 0 & \cdots & & 0 & 1 \end{bmatrix}, \quad |J| = w_n^{n-1},$$

yielding

$$\begin{aligned} f_{\mathbf{W}}(\mathbf{w}) &= |J| \, f_{\mathbf{Y}}(\mathbf{y}) \\ &= w_n^{n-1} n! a^n \theta^{-na} w_n^{n(a-1)} \left(\prod_{i=1}^{n-1} w_i\right)^{a-1} \mathbb{I}(0 < w_1 < \cdots < w_{n-1} < 1 < \theta/w_n) \end{aligned}$$

which is easily expressed as the product

$$(n-1)! a^{n-1} \left(\prod_{i=1}^{n-1} w_i\right)^{a-1} \mathbb{I}(0 < w_1 < \cdots < w_{n-1} < 1) \cdot na\theta^{-na} w_n^{na-1} \mathbb{I}_{(0,\theta)}(w_n)$$

showing independence.

Solution to Problem 6.15: We want

$$\mathbb{E}[\ln X] = \int_0^1 (\ln u) \, u^{a-1} (1-u)^{b-1} \, du.$$

But, from (15.37), with $u = e^{-\alpha y}$ and $y = -(\ln u)/\alpha$, we have

$$\begin{aligned} \sum_{i=0}^{k-1} \frac{1}{n-i} &= \alpha^2 k \binom{n}{k} \int_0^\infty y \left(1 - e^{-\alpha y}\right)^{k-1} e^{-\alpha y(n-k+1)} dy \\ &= -k \binom{n}{k} \int_0^1 (\ln u)(1-u)^{k-1} u^{n-k} du, \end{aligned}$$

or

$$\int_0^1 (\ln u)(1-u)^{k-1} u^{n-k} du = -\frac{1}{k\binom{n}{k}} \sum_{i=0}^{k-1} \frac{1}{n-i}.$$

With $k = b$ and $n = a - 1 + b$, this is

$$\begin{aligned} \int_0^1 (\ln u) \, u^{a-1} (1-u)^{b-1} \, du &= \mathbb{E}[\ln X] \\ &= -\frac{1}{b\binom{a+b-1}{b}} \sum_{i=0}^{b-1} \frac{1}{a+b-1-i}. \end{aligned}$$

Note that, if $a = b = 1$, then $X \sim \text{Unif}(0, 1)$, and the summation formula reduces to

$$-\frac{1}{\binom{1+1-1}{1}} \frac{1}{1+1-1-0} = -1.$$

This agrees with the fact that, if $U \sim \text{Unif}(0, 1)$, then $-\ln U \sim \text{Exp}(1)$ (see Example 7.7) Also, the integral of $\mathbb{E}[\ln X]$ reduces to

$$\mathbb{E}[\ln X] = \int_0^1 \ln u \, du = -1,$$

which follows because

$$\int_0^1 \ln u \, du = (u \ln u - u)|_0^1 = -1 - \lim_{u \to 0} (u \ln u)$$

and, from l'Hôpital's rule and properties of limits,

$$\lim_{u \to 0} \frac{\ln u}{1/u} = -\frac{\lim_{u \to 0} (1/u)}{\lim_{u \to 0} (u^{-2})} = -\lim_{u \to 0} \frac{u^2}{u} = 0.$$

Solution to Problem 6.16:

a) Label the two points X_1 and X_2 and rotate the circle such that X_1 is at 12:00. Then the points are not within a distance of d of each other if X_2 is on the perimeter somewhere between the points situated a distance of d from 12:00 going clockwise and counterclockwise. Thus, the desired probability, say p_2, is just $1 - 2d$.

Now consider working with a straight line of length one, instead of a circle. When measured in terms of the distance from the left origin of the line, the two points are order statistics, say Y_1 and Y_2, with joint density, from (15.17), given by $f_{Y_1,Y_2}(y_1,y_2) = 2!\mathbb{I}(0 < y_1 < y_2 < 1)$. If Y_1 is at least $1 - d$, then Y_2 is certainly within a distance d. Thus, the desired probability is

$$p_2(d) = \Pr(Y_2 > Y_1 + d) = 2! \int_0^{1-d} \int_{y_1+d}^{1} dy_2 dy_1 = (1-d)^2, \quad 0 < d < 1.$$

b) In general, recall from Chapter 3 that there are $(n-1)!$ ways of arranging n distinguishable objects on a circle. So, for $n = 3$, and labeling the first observation X_1 as the first order statistic Y_1, we want (drawing a picture helps to see this)

$$p_3(d) = (3-1)! \int_d^{1-2d} \int_{y_2+d}^{1-d} dy_3 dy_2 = (1-3d)^2.$$

Similarly, for $n = 4$,

$$p_4(d) = (4-1)! \int_d^{1-3d} \int_{y_2+d}^{1-2d} \int_{y_3+d}^{1-d} dy_4 dy_3 dy_2 = (1-4d)^3,$$

and in general, $p_4(d) = (1 - nd)^{n-1}$.

For the straight line: For $n = 3$, with order statistics Y_1, Y_2, Y_3 and joint density $f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = 3!\mathbb{I}(0 < y_1 < y_2 < y_3 < 1)$, note that, if Y_1 is greater than $1 - 2d$, then there is no chance that the three points can be a distance greater than or equal to d from each other. Thus,

$$\begin{aligned} p_3(d) &= \Pr(Y_2 > Y_1 + d, \quad Y_3 > Y_2 + d) \\ &= 3! \int_0^{1-2d} \int_{y_1+d}^{1-d} \int_{y_2+d}^{1} dy_3 dy_2 dy_1 = (1-2d)^3, \quad 0 < d < \frac{1}{2}. \end{aligned}$$

Similar calculation reveals that $p_n(d) = (1 - (n-1)d)^n$, $0 < d < 1/(n-1)$.

Solutions to Chapter 7: Generalized and Mixing

Solution to Problem 7.1: For $Z \sim \text{NLap}(c)$, use of (14.13) gives

$$\begin{aligned}\mathbb{K}_Y''(s) &= \frac{d^2}{ds^2}\left(\frac{1}{2}c^2s^2 - \ln\left(1 - s^2k^2\right)\right) \\ &= \frac{c^2\left(1-sk\right)^2\left(1+sk\right)^2 + 2k^2\left(1+s^2k^2\right)}{\left(1-s^2k^2\right)^2},\end{aligned}$$

where $k = 1 - c$, so that

$$\mathbb{V}(Z) = \mathbb{K}_Z''(0) = c^2 + 2k^2 = 3c^2 - 4c + 2,$$

which reduces to 1 for $c = 1$ (pure normal) and 2 for $c = 0$ (pure Laplace) as it should, recalling the results in Problem 10.2. Next, from

$$\frac{d^4}{ds^4}\left(\frac{1}{2}c^2s^2 - \ln\left(1 - s^2k^2\right)\right) = 12k^4\frac{s^2k^2\left(6+s^2k^2\right)+1}{\left(1-s^2k^2\right)^4},$$

we have

$$\kappa_4 = \mathbb{K}_Z^{(4)}(0) = 12\left(1-c\right)^4,$$

so that

$$\mu_4 = \kappa_4 + 3\mu_2^2 = 12\left(1-c\right)^4 + 3\left(3c^2 - 4c + 2\right)^2$$

and the kurtosis is

$$\frac{\mu_4}{\mu_2^2} = \frac{12\left(1-c\right)^4 + 3\left(3c^2-4c+2\right)^2}{\left(3c^2-4c+2\right)^2},$$

which is the same as the expression in (11.23) and (16.45). For $c = 1$, this reduces to 3, while for $c = 0$, it becomes 6.

For X, the expected value is zero from symmetry, while variance was shown in Problem 7.16 to be $\Gamma\left(3/p\right) / \Gamma\left(1/p\right)$. Similarly, using the same substitution $u = x^p$, we have $x = u^{1/p}$, $dx = p^{-1}u^{1/p-1}du$ and

$$\begin{aligned}E\left[|X|^r\right] &= 2\cdot\frac{p}{2\Gamma\left(p^{-1}\right)}\int_0^\infty x^r \exp\left\{-\left(x^p\right)\right\}dx \\ &= \frac{1}{\Gamma\left(p^{-1}\right)}\int_0^\infty u^{(r+1)/p-1}\exp\left\{-u\right\}du \\ &= \frac{\Gamma\left(p^{-1}\left(r+1\right)\right)}{\Gamma\left(p^{-1}\right)},\end{aligned}$$

so that $\mu_4 = \mathbb{E}[X^4] = \Gamma(5/p) \,/\, \Gamma(1/p)$. Thus,

$$\frac{\mu_4}{\mu_2^2} = \frac{\Gamma(5/p)\,\Gamma(1/p)}{\Gamma^2(3/p)}.$$

For $p = 1$ (Laplace), this correctly reduces to 6, while for $p = 2$ (normal), this correctly reduces to 3.

For Y, from symmetry, $\mathbb{E}[Y] = 0$, so that, with $Z \sim \mathrm{N}(0,1)$ and $L \sim \mathrm{Lap}(0,1)$,

$$\mu_2 = \mathbb{V}(Y) = \mathbb{E}\left[Y^2\right] = w\mathbb{V}(Z) + (1-w)\,\mathbb{V}(L) = w + 2(1-w) = 2 - w.$$

Similarly,

$$\mu_4 = \mathbb{E}\left[Y^4\right] = w\mathbb{E}\left[Z^4\right] + (1-w)\,\mathbb{E}\left[L^4\right] = 3w + 24(1-w) = 24 - 21w$$

so that the kurtosis of Y is

$$\frac{\mu_4}{\mu_2^2} = \frac{24 - 21w}{(2-w)^2}.$$

Solution to Problem 7.2:

a) From (7.65) with $x = \frac{\lambda y}{1+\lambda y - y}$, $dx/dy = \lambda(1 + \lambda y - y)^{-2}$,

$$\begin{aligned} f_Y(y) &= f_X(x)\left|\frac{dx}{dy}\right| \\ &= \left(\frac{\lambda y}{1+\lambda y - y}\right)^{\alpha-1}\left(1 - \frac{\lambda y}{1+\lambda y - y}\right)^{\beta-1}\frac{\lambda B^{-1}(\alpha,\beta)}{(1+\lambda y - y)^2}, \end{aligned}$$

which simplifies to (16.46).

b) Write

$$\begin{aligned} \mathbb{E}[Y^m] &= \int_0^1 \frac{\lambda^\alpha}{B(\alpha,\beta)}\frac{y^{\alpha+m-1}(1-y)^{\beta-1}}{[1-(1-\lambda)\,y]^{\alpha+\beta}} \\ &= \frac{\lambda^\alpha\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\int_0^1 \frac{y^{\alpha+m-1}(1-y)^{\beta-1}}{[1-(1-\lambda)\,y]^{\alpha+\beta}} \\ &= \frac{\Gamma(\alpha+\beta)\,\Gamma(\alpha+m)}{\Gamma(\alpha)\,\Gamma(\alpha+\beta+m)}\lambda^\alpha\frac{\Gamma(\alpha+\beta+m)}{\Gamma(\alpha+m)\,\Gamma(\beta)}\int_0^1 \frac{y^{\alpha+m-1}(1-y)^{\beta-1}}{[1-(1-\lambda)\,y]^{\alpha+\beta}} \\ &= \frac{\Gamma(\alpha+\beta)\,\Gamma(\alpha+m)}{\Gamma(\alpha)\,\Gamma(\alpha+\beta+m)}\lambda^\alpha{}_2F_1(\alpha+m,\alpha+\beta,\alpha+\beta+m,1-\lambda). \end{aligned}$$

It can be further shown that, for $\lambda \le 1$ and large α and β,

$$\mathbb{E}[Y^m] \approx \frac{\Gamma(\alpha+\beta)\,\Gamma(\alpha+m)}{\Gamma(\alpha)\,\Gamma(\alpha+\beta+m)}\left(\frac{\alpha+\beta}{\alpha+\lambda\beta}\right)^m.$$

c)

$$\Pr(Y \le y) = \Pr\left(\frac{X}{\lambda - \lambda X + X} \le y\right) = \Pr\left(X \le \frac{\lambda y}{1+\lambda y - y}\right)$$

so that the cdf of Y can be evaluated by the incomplete beta function.

d) Straightforward computation reveals that

$$\mathbb{E}\left[Z^m\right] = \frac{\Gamma\left(\alpha+\beta\right)\Gamma\left(\alpha+m\right)}{\Gamma\left(\alpha\right)\Gamma\left(\alpha+\beta+m\right)} \frac{{}_2F_1\left(\alpha+m, \kappa, \alpha+\beta+m, 1-\lambda\right)}{{}_2F_1\left(\alpha, \kappa, \alpha+\beta, 1-\lambda\right)}.$$

The cdf of Z must be computed directly with numerical integration.

Solution to Problem 7.3:

a) $F_W\left(w; \alpha, \beta, \lambda\right)$ is given by

$$\begin{aligned}\Pr\left(W \le w\right) &= \Pr\left(\frac{Y}{1-Y} \le w\right) \\ &= \Pr\left(Y \le \frac{w}{w+1}\right) = F_Y\left(\frac{w}{w+1}; \alpha, \beta, \lambda\right).\end{aligned}$$

b) With $y = w\left(w+1\right)^{-1}$ and $dy = \left(w+1\right)^{-2} dw$,

$$\begin{aligned}f_W\left(w; \alpha, \beta, \lambda\right) &= f_Y\left(y\right)\left|\frac{dy}{dw}\right| \\ &= \frac{\lambda^\alpha}{B\left(\alpha, \beta\right)} \frac{\left(\frac{w}{w+1}\right)^{\alpha-1}\left(1-\left(\frac{w}{w+1}\right)\right)^{\beta-1}}{\left[1-\left(1-\lambda\right)\left(\frac{w}{w+1}\right)\right]^{\alpha+\beta}}\left(w+1\right)^{-2} \\ &= \frac{\lambda^\alpha}{B\left(\alpha, \beta\right)} \frac{w^{\alpha-1}}{\left(1+\lambda w\right)^{\alpha+\beta}} \mathbb{I}_{\left(0,\infty\right)}\left(w\right),\end{aligned}$$

as

$$1-\left(1-\lambda\right)\left(\frac{w}{w+1}\right) = \frac{1+w\lambda}{w+1}$$

and $0 < \dfrac{w}{w+1} < 1 \Longleftrightarrow 0 < w < \infty$.

c) Inspection shows that W is a generalization of the usual $F\left(n_1, n_2\right)$ density

$$f_F\left(x\right) = \frac{\left(n_1/n_2\right)^{n_1/2}}{B\left(n_1/2, n_2/2\right)} \frac{x^{\left(n_1-2\right)/2}}{\left(1+\left(n_1/n_2\right)x\right)^{\left(n_1+n_2\right)/2}} \mathbb{I}_{\left(0,\infty\right)}\left(x\right),$$

with equality holding when

$$\alpha' = \frac{n_1}{2}, \quad \beta' = \frac{n_2}{2} \quad \text{and} \quad \lambda' = \frac{\alpha}{\beta} = \frac{n_1}{n_2}. \tag{S-7.1}$$

d) From the usual scale transformation,

$$f_U\left(u; \alpha, \beta, \lambda, c\right) = c^{-1} f_W\left(u/c\right) = \frac{\lambda^\alpha c^{-\alpha}}{B\left(\alpha, \beta\right)} \frac{u^{\alpha-1}}{\left(1+\lambda u/c\right)^{\alpha+\beta}} = f_W\left(u; \alpha, \beta, \lambda/c\right),$$

i.e., $A = \alpha$, $B = \beta$ and $C = \lambda/c$. From the previous question, W is standard F distributed if its additional parameter is α/β. Thus, equating λ/c and α/β shows that cW follows a standard F distribution if $c = \lambda\beta/\alpha$.

e) We wish

$$\mathbb{E}\left[W^r\right] = \frac{\lambda^\alpha}{B(\alpha,\beta)}\int_0^\infty \frac{w^{\alpha-1+r}}{(1+\lambda w)^{\alpha+\beta}}dw.$$

With substitution $t = 1/(1+\lambda w)$, $w = (1-t)/(t\lambda)$, $dw = -\left(t^2\lambda\right)^{-1}dt$, this yields

$$\begin{aligned}
\mathbb{E}\left[W^r\right] &= -\frac{\lambda^\alpha}{B(\alpha,\beta)}\int_1^0 t^{\alpha+\beta}\left(\frac{1-t}{t\lambda}\right)^{\alpha-1+r}\left(t^2\lambda\right)^{-1}dt \\
&= \frac{\lambda^{-r}}{B(\alpha,\beta)}\int_0^1 t^{\beta-1-r}(1-t)^{\alpha-1+r}dt = \frac{\lambda^{-r}}{B(\alpha,\beta)}B(\beta-r,\alpha+r) \\
&= \lambda^{-r}\frac{\Gamma(\beta-r)\Gamma(\alpha+r)}{\Gamma(\alpha)\Gamma(\beta)}, \quad -\alpha < r < \beta.
\end{aligned}$$

For positive integer $r < \beta$, this reduces to

$$\mathbb{E}\left[W^r\right] = \lambda^{-r}\frac{\alpha(\alpha+1)\cdots(\alpha+r-1)}{(\beta-1)\cdots(\beta-r)}.$$

f) With $r = 1$, the first two raw moments of $W \sim \mathrm{G3F}(\alpha,\beta,\lambda)$ are

$$\mathbb{E}[W] = \lambda^{-1}\frac{\alpha}{\beta-1} \quad \text{and} \quad \mathbb{E}\left[W^2\right] = \lambda^{-2}\frac{\alpha(\alpha+1)}{(\beta-1)(\beta-2)}$$

so that

$$\begin{aligned}
\mathbb{V}(W) &= \lambda^{-2}\frac{\alpha(\alpha+1)}{(\beta-1)(\beta-2)} - \left(\lambda^{-1}\frac{\alpha}{\beta-1}\right)^2 \\
&= \frac{\alpha(\alpha+\beta-1)}{\lambda^2(\beta-1)^2(\beta-2)}.
\end{aligned}$$

Using values $\alpha' = n_1/2$, $\beta' = n_2/2$ and $\lambda' = n_1/n_2$ from (S-7.1), the previous formulae for $W \sim \mathrm{G3F}(\alpha',\beta',\lambda')$ reduce to

$$\mathbb{E}[W] = \frac{n_2}{n_2-2}, \quad \mathbb{E}\left[W^2\right] = n_2^2\frac{n_1+2}{n_1(n_2-2)(n_2-4)}$$

and

$$\mathbb{V}(W) = \frac{2n_2^2(n_1+n_2-2)}{n_1(n_2-2)^2(n_2-4)},$$

which agree with those of the F distribution.

Solution to Problem 7.4:

a) Write

$$\begin{aligned}
f_R(r) &= \frac{d}{dr}\Pr\left(G_1 < \frac{r}{1-r}G_2\right) \\
&= K\int_0^\infty e^{-g_2 s_2}g_2^{n_2-1}\frac{d}{dr}\int_0^{\frac{r}{1-r}g_2} e^{-g_1 s_1}g_1^{n_1-1}dg_1 dg_2
\end{aligned}$$

with $K = s_1^{n_1} s_2^{n_2} / \Gamma(n_1) \Gamma(n_2)$. From Leibniz' rule (assuming $n_1 > 1$),

$$
\begin{aligned}
& \frac{d}{dr} \int_0^{g_2 r/(1-r)} \exp(-g_1 s_1) g_1^{n_1-1} dg_1 \\
= & \exp\left(-\frac{r}{1-r} g_2 s_1\right) \left(\frac{r}{1-r} g_2\right)^{n_1-1} g_2 (1-r)^{-2} \\
= & \exp\left(-\frac{r g_2 s_1}{1-r}\right) g_2^{n_1} r^{n_1-1} (1-r)^{-n_1-1},
\end{aligned}
$$

so that

$$
f_R(r) = K r^{n_1-1} (1-r)^{-n_1-1} \int_0^\infty \exp\left\{-g_2 \left(\frac{s_2 - s_2 r + r s_1}{1-r}\right)\right\} g_2^{n_1+n_2-1} dg_2.
$$

The latter integral is the gamma density and so integrates to

$$
\Gamma(n_1+n_2) \left[(s_2 - s_2 r + r s_1) / (1-r)\right]^{-(n_1+n_2)},
$$

giving

$$
\begin{aligned}
f_R(r) & = \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} s_1^{n_1} s_2^{n_2} \frac{r^{n_1-1}(1-r)^{n_2-1}}{(s_2 - s_2 r + r s_1)^{n_1+n_2}} \mathbb{I}_{(0,1)}(r) \\
& = \frac{s_1^{n_1} s_2^{n_2}}{B(n_1,n_2)} \frac{r^{n_1-1}(1-r)^{n_2-1}}{(s_1 r + s_2(1-r))^{n_1+n_2}} \mathbb{I}_{(0,1)}(r) \\
& = \frac{\lambda^{n_1}}{B(n_1,n_2)} \frac{r^{n_1-1}(1-r)^{n_2-1}}{(1-(1-\lambda)r)^{n_1+n_2}} \mathbb{I}_{(0,1)}(r),
\end{aligned}
$$

where $\lambda = s_1/s_2$.

b) Recalling the integral expression for ${}_2F_1$, the raw moments are easily seen to be

$$
\mathbb{E}\left[R^j\right] = \frac{\lambda^{n_1} B(n_1+j, n_2)}{B(n_1,n_2)} {}_2F_1(n_1+j, n_1+n_2, n_1+n_2+j, 1-\lambda).
$$

for $j < n_1$.

c) With $y = 1/r$, the usual transformation technique simplifies to

$$
f_Y(y) = f_R\left(\frac{1}{y}\right) \frac{dr}{dy} = \frac{1}{B(n_1,n_2)} y^{-(n_1+n_2)} (y-1)^{n_2-1} \mathbb{I}_{(1,\infty)}(r)
$$

so that

$$
\mathbb{E}\left[Y^j\right] = \frac{1}{B(n_1,n_2)} \int_1^\infty y^{j-n_1-n_2} (y-1)^{n_2-1} dy
$$

or, with $z = (y-1)/y$, $y = (1-z)^{-1}$, $dy = (1-z)^{-2} dz$,

$$
\mathbb{E}\left[Y^j\right] = \frac{1}{B(n_1,n_2)} \int_0^1 (1-z)^{n_1-j-1} z^{n_2-1} dz = \frac{B(n_1-j, n_2)}{B(n_1,n_2)}, \qquad \text{(S-7.2)}
$$

which exists if $n_1 > j$. For $j = 1$ (and $n_1 > 1$),

$$
\mathbb{E}[Y] = (n_1+n_2-1)/(n_1-1). \qquad \text{(S-7.3)}
$$

d) Similarly, with $y = 1/r$,

$$f_Y(y) = f_R\left(\frac{1}{y}\right)\frac{dr}{dy} = \frac{s_1^{n_1} s_2^{n_2}}{B(n_1, n_2)} \frac{(y-1)^{n_2-1}}{(s_1 + s_2(y-1))^{n_1+n_2}} \mathbb{I}_{(1,\infty)}(y) \qquad \text{(S-7.4)}$$

and, as $dv = dy$,

$$\begin{aligned} f_V(v) &= f_Y(v+1) = \frac{s_1^{n_1} s_2^{n_2}}{B(n_1, n_2)} \frac{v^{n_2-1}}{(s_1 + s_2 v)^{n_1+n_2}} \mathbb{I}_{(0,\infty)}(v) \\ &= \frac{h^{n_2}}{B(n_1, n_2)} \frac{v^{n_2-1}}{(1 + hv)^{n_1+n_2}} \mathbb{I}_{(0,\infty)}(v), \quad h = \frac{s_2}{s_1}. \end{aligned}$$

Because of symmetry,

$$\begin{aligned} f_W(w) &= \frac{s_1^{n_1} s_2^{n_2}}{B(n_1, n_2)} \frac{w^{n_1-1}}{(s_2 + s_1 w)^{n_1+n_2}} \mathbb{I}_{(0,\infty)}(w) \\ &= \frac{\lambda^{n_1}}{B(n_1, n_2)} \frac{w^{n_1-1}}{(1 + \lambda v)^{n_1+n_2}} \mathbb{I}_{(0,\infty)}(w) \end{aligned}$$

where $\lambda = s_1/s_2$.

e) Substituting

$$u = \frac{s_1 w}{s_2 + s_1 w}, \quad w = \frac{s_2 u}{s_1(1-u)}, \quad dw = \frac{s_2}{s_1(1-u)^2} du$$

and simplifying gives

$$\frac{B(n_1, n_2)}{s_1^{n_1} s_2^{n_2}} \mathbb{E}\left[W^j\right] = \int_0^\infty \frac{w^{n_1-1+j} dw}{(s_2 + s_1 w)^{n_1+n_2}} = \frac{s_2^{j-n_2}}{s_1^{n_1+j}} \int_0^1 (1-u)^{n_2-1-j} u^{n_1-1+j} du$$

or

$$\mathbb{E}\left[V^j\right] = \left(\frac{s_2}{s_1}\right)^j \frac{B(n_1 + j, n_2 - j)}{B(n_1, n_2)}$$

for all $j \in \mathbb{R}$ such that $n_2 - j > 0$.

f) For the mean,

$$\mathbb{E}[Y] = 1 + \mathbb{E}[V] = 1 + \frac{s_1 n_2}{s_2(n_1 - 1)},$$

which simplifies to (S-7.3) for $s_1 = s_2$. Higher order *integer* moments can be obtained directly from the binomial expansion as

$$\mathbb{E}\left[Y^j\right] = \mathbb{E}\left[(V+1)^j\right] = \sum_{i=0}^{j} \binom{j}{i} \mathbb{E}\left[V^i\right].$$

In particular,

$$\begin{aligned} \mathbb{E}\left[Y^2\right] &= 1 + 2\mathbb{E}[V] + \mathbb{E}\left[V^2\right] \\ &= 1 + 2\frac{s_1 n_2}{s_2(n_1-1)} + \frac{s_1^2}{s_2^2} \frac{(n_2+1) n_2}{(n_1-1)(n_1-2)} \\ &= \frac{s_2^2(n_1-1)(n_1-2) + 2 s_1 s_2 n_2 (n_1-2) + s_1^2 n_2 (n_2+1)}{s_2^2 (n_1-1)(n_1-2)} \end{aligned}$$

so that, for $n_1 > 2$,

$$\mathbb{V}(Y) = \mathbb{E}\left[Y^2\right] - \mathbb{E}\left[Y\right]^2 = \left(\frac{s_1}{s_2}\right)^2 \frac{n_2(n_2+n_1-1)}{(n_1-1)^2(n_1-2)}.$$

g) One way is as follows. From (S-7.4),

$$\mathbb{E}\left[Y^j\right] = \frac{s_1^{n_1} s_2^{n_2}}{B(n_1,n_2)} \int_1^\infty \frac{(y-1)^{n_2-1}}{(s_1+s_2(y-1))^{n_1+n_2}} dy,$$

and with $z = (y-1)/y$, $y = (1-z)^{-1}$, $dy = (1-z)^{-2}\,dz$, we get, after some simplification,

$$\begin{aligned}\mathbb{E}\left[Y^j\right] &= \frac{s_1^{n_1} s_2^{n_2}}{B(n_1,n_2)} \int_0^1 \frac{z^{n_2-1}(1-z)^{n_1-j-1}}{((1-z)s_1+zs_2)^{n_1+n_2}} dz \\ &= \frac{s_1^{-n_2} s_2^{n_2}}{B(n_1,n_2)} \int_0^1 z^{n_2-1}(1-z)^{n_1-j-1}\left(1-z\left(1-\frac{s_2}{s_1}\right)\right)^{n_1+n_2} dz,\end{aligned}$$

and comparing to the ${}_2F_1$ function, we see that

$$\begin{aligned}\mathbb{E}\left[Y^j\right] &= \frac{s_1^{-n_2} s_2^{n_2}}{B(n_1,n_2)} B(n_2, n_1-j) \times \\ &\quad {}_2F_1\left(n_2, -(n_1+n_2), n_1+n_2-j, 1-\frac{s_2}{s_1}\right)\end{aligned}$$

valid when $1 - \dfrac{s_2}{s_1} < 1$, $n_2 > 0$ and $n_1 - j > 0$. For $s = s_1 = s_2$, ${}_2F_1 = 1$, so that

$$\mathbb{E}\left[Y^j\right] = \frac{B(n_2, n_1-j)}{B(n_1,n_2)}$$

agrees with (S-7.2).

h) If $h = k$, $R = 1$. Otherwise, for $h < k$, as $\sum_{i=1}^h G_i \sim \text{Gam}\left(\sum_{i=1}^h n_i, s\right)$ independent of

$$\sum_{i=h+1}^{k} G_i \sim \text{Gam}\left(\sum_{i=h+1}^{k} n_i,\ s\right), \quad R \sim \text{Beta}\left(\sum_{i=1}^{h} n_i,\ \sum_{i=h+1}^{k} n_i\right).$$

Solution to Problem 7.5:

a) Part (i) follows by differentiating both sides. For (ii), with $v = 1 - u$,

$$\begin{aligned}\int_0^1 \ln \frac{u\,du}{1-u} &= \int_0^1 \ln u\,du - \int_0^1 \ln(1-u)\,du \\ &= \int_0^1 \ln u\,du - \int_1^0 (-1)\ln v dv \\ &= \int_0^1 \ln u\,du - \int_0^1 \ln v dv = 0,\end{aligned}$$

if $\int_0^1 \ln u \, du$ exists. But,

$$\int_0^1 \ln u du = \lim_{a \to 0^+} \int_a^1 \ln u du = \lim_{a \to 0^+} \left(u \ln u - u \right)\Big|_a^1 = \lim_{a \to 0^+} \left(u \ln u \right)\Big|_a^1 - 1$$
$$= 0 - \lim_{a \to 0^+} (a \ln a) - 1.$$

From l'Hôpital's rule,

$$\lim_{a \to 0^+} (a \ln a) = \lim_{a \to 0^+} \left(\frac{\ln a}{1/a} \right) = - \lim_{a \to 0^+} \frac{1/a}{1/a^2} = - \lim_{a \to 0^+} a = 0,$$

so that $\int_0^1 \ln u \, du = -1$ exists.

(iii) Using the proposed transformation,

$$x = -\beta \ln \frac{u}{1-u} + \alpha, \quad f_X(x) = \frac{u(1-u)}{\beta}$$

and

$$dx = -\beta e^{\frac{x-\alpha}{\beta}} \left(1 + e^{-\frac{x-\alpha}{\beta}} \right)^2 du = -\frac{\beta}{u(1-u)}.$$

As $x = -\infty \Leftrightarrow u = 1$ and $x = +\infty \Leftrightarrow u = 0$,

$$\begin{aligned} \mathbb{E}[X] &= \int_{-\infty}^{+\infty} x f_X(x) \, dx \\ &= \int_1^0 \left\{ -\beta \ln \frac{u}{1-u} + \alpha \right\} \left\{ \beta^{-1} u (1-u) \right\} \left\{ \frac{-\beta}{u(1-u)} du \right\} \\ &= -\beta \int_0^1 \ln \frac{u \, du}{1-u} + \int_0^1 \alpha \, du = \alpha. \end{aligned}$$

b) With $u = (e^y + 1)^{-1}$, $y = \ln \dfrac{1-u}{u}$ and $dy = -(1-u)^{-1} u^{-1} du$,

$$\begin{aligned} \mathbb{M}_Y(t) &= \int_{-\infty}^{\infty} e^{-(1-t)y} \left(1 + e^{-y} \right)^{-2} dy \\ &= -\int_1^0 \left(\frac{1-u}{u} \right)^{-(1-t)} \left(1 + \frac{u}{1-u} \right)^{-2} \frac{du}{u(1-u)} \\ &= \int_0^1 u^{-t} (1-u)^t \, du = B(1-t, 1+t) = \Gamma(1-t)\Gamma(1+t). \end{aligned}$$

In terms of the digamma function $\psi(s) = \dfrac{d}{ds} \ln \Gamma(s) = \dfrac{\Gamma'(s)}{\Gamma(s)}$,

$$\begin{aligned} \frac{d}{dt}\mathbb{M}_Y(t) &= \Gamma(1-t)\Gamma'(1+t) - \Gamma'(1-t)\Gamma(1+t) \\ &= \Gamma(1-t)\Gamma(1+t)\psi(1+t) - \Gamma(1+t)\Gamma(1-t)\psi(1-t), \end{aligned}$$

so that $\mathbb{M}_Y'(0) = \psi(1) - \psi(1) = 0$, because $\Gamma(1) = 1$. Next,

$$\begin{aligned} \frac{d^2}{dt^2}\mathbb{M}_Y(t) &= \Gamma(1-t)\Gamma''(1+t) - \Gamma'(1-t)\Gamma'(1+t) \\ &\quad -\Gamma'(1-t)\Gamma'(1+t) + \Gamma''(1-t)\Gamma(1+t) \end{aligned}$$

or

$$\mathbb{M}_Y''(0) = 2\Gamma''(1) - 2\Gamma'(1)\Gamma'(1).$$

With

$$\begin{aligned}
\frac{d^2}{ds^2}\Gamma(s) &= \frac{d}{ds}\psi(s)\Gamma(s) \\
&= \psi(s)\Gamma'(s) + \Gamma(s)\psi'(s) \\
&= \psi^2(s)\Gamma(s) + \psi'(s)\Gamma(s),
\end{aligned}$$

where $\psi'(s) = \psi(1, s) = \frac{d^2}{ds^2}\ln\Gamma(s)$ is the trigamma function,

$$\begin{aligned}
2^{-1}\cdot\mathbb{M}_Y''(0) &= \Gamma''(1) - \Gamma'(1)\Gamma'(1) \\
&= \psi^2(1) + \psi'(1) - \psi^2(1) = \psi'(1) = \frac{\pi^2}{6},
\end{aligned}$$

or $\mathbb{M}_Y''(0) = \pi^2/3$. Thus,

$$\mathbb{V}(Y) = \mathbb{M}_Y''(0) - (\mathbb{E}[Y])^2 = \pi^2/3,$$

$\mathbb{E}[X] = \alpha$ and $\mathbb{V}(X) = \beta^2\pi^2/3$.

Solution to Problem 7.6:

a) One useful substitution is $u = (e^x + 1)^{-1}$, so that $x = \ln\frac{1-u}{u}$ and

$$\frac{e^{-qy}}{(1+e^{-y})^{p+q}} = \frac{\left(\frac{u}{1-u}\right)^q}{\left(\frac{1}{1-u}\right)^{p+q}} = u^q(1-u)^p.$$

As $y = -\infty \Longleftrightarrow u = 1$, $y = \infty \Longleftrightarrow u = 0$, $dx = -(1-u)^{-1}u^{-1}du$,

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{e^{-qy}}{(1+e^{-y})^{p+q}}dy &= -\int_1^0 u^q(1-u)^p(1-u)^{-1}u^{-1}du \\
&= \int_0^1 u^{q-1}(1-u)^{p-1}du \\
&= B(p,q).
\end{aligned}$$

The result also follows from the cdf result given next.

b) Another useful substitution is

$$y = e^x(1+e^x)^{-1}, \quad x = \ln\left(\frac{y}{1-y}\right), \quad dx = \frac{dy}{y(1-y)},$$

yielding

$$\begin{aligned}
F_X(t) &= \frac{1}{B(p,q)}\int_{-\infty}^t \frac{e^{-qx}}{(1+e^{-x})^{p+q}}dx \\
&= \frac{1}{B(p,q)}\int_0^{e^t\left(1+e^t\right)^{-1}} y^{p-1}(1-y)^{q-1}dy \\
&= \bar{B}_z(p,q), \quad z = e^t\left(1+e^t\right)^{-1},
\end{aligned}$$

after simplifying.

As an aside, note that the integral associated with the expected shortfall can be expressed as

$$\frac{1}{B(p,q)}\int_0^z \ln\left(\frac{y}{1-y}\right) y^{p-1}(1-y)^{q-1}\,dy.$$

This form is more conducive to numerical integration than the direct form $\int_{-\infty}^{t} x f_X(x)\,dx$.

c) With the substitution $u=(e^x+1)^{-1}$,

$$\begin{aligned}
B(p,q)\,\mathbb{M}_X(t) &= \int_{-\infty}^{\infty}\frac{e^{-x(q-t)}}{(1+e^{-x})^{p+q}}\\
&= -\int_1^0\left(\frac{1-u}{u}\right)^{-(q-t)}\left(1+\frac{u}{1-u}\right)^{-(p+q)}\frac{du}{u(1-u)}\\
&= \int_0^1 u^{q-t-1}(1-u)^{t-q+p+q-1}\,du\\
&= B(q-t,p+t)=\frac{\Gamma(q-t)\Gamma(p+t)}{\Gamma(p+q)}
\end{aligned}$$

so that

$$\mathbb{M}_X(t)=\frac{\Gamma(p+t)\Gamma(q-t)}{\Gamma(p)\Gamma(q)}.$$

d)

$$\frac{d}{dt}\mathbb{M}_X(t)=\frac{\Gamma'(p+t)\Gamma(q-t)-\Gamma(p+t)\Gamma'(q-t)}{\Gamma(p)\Gamma(q)}$$

so that, with $\psi(s)=\Gamma'(s)/\Gamma(s)$ and $\Gamma(1)=1$,

$$\mathbb{E}[X]=\mathbb{M}_X'(0)=\frac{\Gamma'(p)}{\Gamma(p)}-\frac{\Gamma'(q)}{\Gamma(q)}=\psi(p)-\psi(q).$$

Next,

$$\begin{aligned}
\Gamma(p)\Gamma(q)\frac{d^2}{dt^2}\mathbb{M}_Y(t) &= -\Gamma'(p+t)\Gamma'(q-t)+\Gamma(q-t)\Gamma''(p+t)\\
&\quad+\Gamma(p+t)\Gamma''(q-t)-\Gamma'(q-t)\Gamma'(p+t)
\end{aligned}$$

or

$$\begin{aligned}
\mathbb{M}_X''(0) &= \frac{-\Gamma'(p)\Gamma'(q)+\Gamma(q)\Gamma''(p)+\Gamma(p)\Gamma''(q)-\Gamma'(q)\Gamma'(p)}{\Gamma(p)\Gamma(q)}\\
&= -\frac{\Gamma'(p)\Gamma'(q)}{\Gamma(p)\Gamma(q)}+\frac{\Gamma(q)\Gamma''(p)}{\Gamma(p)\Gamma(q)}+\frac{\Gamma(p)\Gamma''(q)}{\Gamma(p)\Gamma(q)}-\frac{\Gamma'(q)\Gamma'(p)}{\Gamma(p)\Gamma(q)}\\
&= \frac{\Gamma''(p)}{\Gamma(p)}+\frac{\Gamma''(q)}{\Gamma(q)}-2\psi(p)\psi(q).
\end{aligned}$$

As $\mathbb{V}(X) = \mu_2' - \mu^2$,

$$\begin{aligned}\mathbb{V}(X) &= \frac{\Gamma''(p)}{\Gamma(p)} + \frac{\Gamma''(q)}{\Gamma(q)} - \left[\psi^2(p) - 2\psi(p)\psi(q) + \psi^2(q)\right] \\ &= \frac{\Gamma''(p)}{\Gamma(p)} + \frac{\Gamma''(q)}{\Gamma(q)} - \psi^2(p) - \psi^2(q)\end{aligned}$$

From the hint, it follows directly that

$$\mathbb{V}(X) = \psi'(p) + \psi'(q).$$

Solution to Problem 7.7:

a) We need to calculate $\int_{-\infty}^{\infty} z^r f(z; d, \nu, \theta)\, dz$

$$\begin{aligned}&= \int_{-\infty}^{0} z^r \left(1 + \frac{(-z \cdot \theta)^d}{\nu}\right)^{-\left(\nu + \frac{1}{d}\right)} dz + \int_{0}^{\infty} z^r \left(1 + \frac{(z/\theta)^d}{\nu}\right)^{-\left(\nu + \frac{1}{d}\right)} dz \\ &= I_1 + I_2\end{aligned}$$

for values of r such that I_1 and I_2 are defined. First consider I_1, and use

$$\begin{aligned}u = 1 + (-z\theta)^d \nu^{-1}, \quad z = -\theta^{-1}\nu^{1/d}(u-1)^{1/d}, \\ dz = -\theta^{-1}\nu^{1/d}d^{-1}(u-1)^{1/d-1}\, du, \quad z^r = (-1)^r \theta^{-r}\nu^{r/d}(u-1)^{r/d}.\end{aligned}$$

Substituting and simplifying gives

$$I_1 = -\frac{(-1)^r}{\theta^{r+1}d}\nu^{\frac{r+1}{d}} \int_{\infty}^{1} (u-1)^{\frac{r+1}{d}-1} u^{-\left(\nu+\frac{1}{d}\right)} du \tag{S-7.5}$$

which almost looks like a beta integral. With the transformation

$$x = \frac{u-1}{u}, \quad u = \frac{1}{1-x}, \quad u - 1 = \frac{x}{1-x}, \quad du = \frac{dx}{(1-x)^2},$$

we get an expression for I_1 as

$$-\frac{(-1)^r}{\theta^{r+1}d}\nu^{\frac{r+1}{d}} \int_{1}^{0} x^{\frac{r+1}{d}-1}(1-x)^{\nu-\frac{r}{d}-1}\, dx = \frac{(-1)^r}{\theta^{r+1}d}\nu^{\frac{r+1}{d}} B\left(\frac{r+1}{d}, \nu - \frac{r}{d}\right).$$

For I_2, using the same sequence of substitutions,

$$\begin{aligned}u = 1 + (z/\theta)^d \nu^{-1}, \quad z = \theta\nu^{1/d}(u-1)^{1/d}, \\ dz = \theta\nu^{1/d}d^{-1}(u-1)^{1/d-1}\, du, \quad z^r = \theta^r\nu^{r/d}(u-1)^{r/d},\end{aligned}$$

it follows that

$$I_2 = \frac{\theta^{r+1}}{d}\nu^{\frac{r+1}{d}} \int_{1}^{\infty} (u-1)^{\frac{r+1}{d}-1} u^{-\left(\nu+\frac{1}{d}\right)} du,$$

and noticing that the integral is just the minus of that in (S-7.5),

$$I_2 = \frac{\theta^{r+1}}{d} \nu^{\frac{r+1}{d}} B\left(\frac{r+1}{d}, \nu - \frac{r}{d}\right).$$

Combining yields

$$I_1 + I_2 = \left[(-1)^r \theta^{-(r+1)} + \theta^{r+1}\right] d^{-1} \nu^{\frac{r+1}{d}} B\left(\frac{r+1}{d}, \nu - \frac{r}{d}\right).$$

For $r = 0$, we get the integrating constant, or

$$K^{-1} = \left(\theta^{-1} + \theta\right) d^{-1} \nu^{1/d} B\left(\frac{1}{d}, \nu\right),$$

so that, simplifying,

$$\mathbb{E}\left[Z^r\right] = \frac{I_1 + I_2}{K^{-1}} = \frac{(-1)^r \theta^{-(r+1)} + \theta^{r+1}}{\theta^{-1} + \theta} \frac{B\left(\frac{r+1}{d}, \nu - \frac{r}{d}\right)}{B\left(\frac{1}{d}, \nu\right)} \nu^{\frac{r}{d}},$$

which is valid for integer r such that $0 \le r < \nu d$.

b) We need to calculate $\int_{-\infty}^{\infty} \left(|z| - \gamma z\right)^r f\left(z; d, \nu, \theta\right) dz$, which can be expressed as

$$\begin{aligned} &= \int_{-\infty}^{0} \left(-z\left(1+\gamma\right)\right)^r u_1^{-\left(\nu+\frac{1}{d}\right)} dz + \int_{0}^{\infty} \left(z\left(1-\gamma\right)\right)^r u_2^{-\left(\nu+\frac{1}{d}\right)} dz \\ &= J_1 + J_2, \end{aligned}$$

where $u_1 = 1 + \left(-z \cdot \theta\right)^d \nu^{-1}$ and $u_2 = 1 + \left(z/\theta\right)^d \nu^{-1}$. Using the above results, it easily follows that

$$J_1 = (-1)^r \left(1+\gamma\right)^r I_1 = \frac{\left(1+\gamma\right)^r}{\theta^{r+1} d} \nu^{\frac{r+1}{d}} B\left(\frac{r+1}{d}, \nu - \frac{r}{d}\right),$$

$$J_2 = \left(1-\gamma\right)^r I_2 = \frac{\theta^{r+1}\left(1-\gamma\right)^r}{d} \nu^{\frac{r+1}{d}} B\left(\frac{r+1}{d}, \nu - \frac{r}{d}\right),$$

from which we get

$$\begin{aligned} \mathbb{E}\left[\left(|Z| - \gamma Z\right)^r\right] &= \frac{J_1 + J_2}{K^{-1}} \\ &= \frac{\left(1+\gamma\right)^r \theta^{-(r+1)} + \theta^{r+1}\left(1-\gamma\right)^r}{\theta^{-1} + \theta} \frac{B\left(\frac{r+1}{d}, \nu - \frac{r}{d}\right)}{B\left(\frac{1}{d}, \nu\right)} \nu^{\frac{r}{d}}. \end{aligned}$$

c) For $z \le 0$, similar to the previous calculations, let $u = 1 + \left(-z\theta\right)^d \nu^{-1}$ and then $x = \left(u-1\right)/u$ to get

$$\begin{aligned} F_Z\left(z\right) &= K \int_{-\infty}^{z} \left(1 + \frac{\left(-z \cdot \theta\right)^d}{\nu}\right)^{-\left(\nu+\frac{1}{d}\right)} dz \\ &= \frac{K}{\theta d} \nu^{1/d} \int_{1+\left(-z\theta\right)^d/\nu}^{\infty} u^{-\left(\nu+\frac{1}{d}\right)} \left(u-1\right)^{1/d-1} du \\ &= \frac{K}{\theta d} \nu^{1/d} \int_{L'}^{1} x^{1/d-1} \left(1-x\right)^{\nu-1} dx, \qquad L' = \frac{\left(-z\theta\right)^d}{\nu + \left(-z\theta\right)^d}, \end{aligned}$$

or

$$F_Z(z) = \frac{1-\bar{B}_{L'}(1/d,\nu)}{(1+\theta^2)} = \frac{\bar{B}_L(\nu,1/d)}{(1+\theta^2)}, \qquad L = 1-L' = \frac{\nu}{\nu+(-z\theta)^d},$$

as

$$\int_L^1 x^{1/d-1}(1-x)^{\nu-1}\,dx = \int_0^{1-L} y^{\nu-1}(1-y)^{1/d-1}\,dy$$

and $B(a,b) = B(b,a)$. Notice that, with $\theta = 1$ and $d = 2$,

$$F_Z(z;\nu) = \frac{\bar{B}_L(\nu,1/2)}{2}, \quad L = \frac{\nu}{\nu+z^2}.$$

This is almost the cdf of $T \sim t(\nu)$. Take $u = 2\nu$ to get

$$F_Z(z;\nu) = \frac{\bar{B}_L(u/2,1/2)}{2}, \quad L = \frac{u}{u+\left(z\sqrt{2}\right)^2},$$

so that $F_Z(z;\nu) = F_T\left(z\sqrt{2};2\nu\right)$. Thus, in order for T to be a special case of Z, a scale–family has to be used.

For $z > 0$, we add $F_Z(0) = (1+\theta^2)^{-1}$ to the following integral:

$$\begin{aligned}
I &= K\int_0^z \left(1+\frac{(z/\theta)^d}{\nu}\right)^{-\left(\nu+\frac{1}{d}\right)} dz \\
&= \frac{(\theta^{-2}+1)^{-1}}{B(1/d,\nu)} \int_1^{1+(z/\theta)^d/\nu} u^{-\left(\nu+\frac{1}{d}\right)}(u-1)^{1/d-1}\,du \\
&= \frac{(\theta^{-2}+1)^{-1}}{B(1/d,\nu)} \int_0^U x^{1/d-1}(1-x)^{\nu-1}\,dx \\
&= \frac{\bar{B}_U(1/d,\nu)}{(1+\theta^{-2})}, \qquad U = \frac{(z/\theta)^d}{\nu+(z/\theta)^d}.
\end{aligned}$$

The following Matlab code will compute the cdf for a given vector of ordinates `xvec`:

```
cdf = zeros(size(xvec));
k = find(xvec<0);
if any(k)
  y=xvec(k); L = v./(v+(-y*theta).^d);
  cdf(k) = betainc(L,v,1/d)/(1+theta^2);
end
k = find(xvec==0);
if any(k)
  y=xvec(k); cdf(k) = 1/(1+theta^2);
end
k = find(xvec>0);
```

```
if any(k)
  y=xvec(k); top=(y/theta).^d; U=top./(v+top);
  cdf(k) = 1/(1+theta^2) + betainc(U,1/d,v)/(1+theta^(-2));
end
```

d) This is given by

$$S_r(c) = \frac{K}{F_Z(c)} \int_{-\infty}^{c} z^r \left(1 + \frac{(-z\cdot\theta)^d}{\nu}\right)^{-\left(\nu+\frac{1}{d}\right)} dz,$$

for integer r. Similar to the previous calculations, let $u = 1 + (-z\theta)^d \nu^{-1}$ and then $x = (u-1)/u$, so that $S_r(c) K^{-1} F_Z(c)$ is equal to

$$\begin{aligned}
& -\frac{(-1)^r}{\theta^{r+1} d} \nu^{\frac{r+1}{d}} \int_{\infty}^{1+(-c\theta)^d \nu^{-1}} (u-1)^{\frac{r+1}{d}-1} u^{-\left(\nu+\frac{1}{d}\right)} du \\
= & \frac{(-1)^r}{\theta^{r+1} d} \nu^{\frac{r+1}{d}} \int_{L'}^{1} x^{\frac{r+1}{d}-1} (1-x)^{\nu-\frac{r}{d}-1} dx, \qquad L' = \frac{(-c\theta)^d}{\nu + (-c\theta)^d} \\
= & \frac{(-1)^r}{\theta^{r+1} d} \nu^{\frac{r+1}{d}} \int_{0}^{L} y^{\nu-r/d-1} (1-y)^{(r+1)/d-1} dy, \qquad L = 1 - L' \\
= & \frac{(-1)^r}{\theta^{r+1} d} \nu^{(r+1)/d} B_L(\nu - r/d, (r+1)/d)
\end{aligned}$$

or

$$S_r(c) = (-1)^r \nu^{r/d} \frac{(1+\theta^2)}{(\theta^r + \theta^{r+2})} \frac{B_L(\nu - r/d, (r+1)/d)}{B_L(\nu, 1/d),}, \qquad L = \frac{\nu}{\nu + (-c\theta)^d}.$$

The more direct substitution is

$$\begin{aligned}
y &= \frac{1}{1 + (-z\theta)^d / v}, \quad \frac{(-z\theta)^d}{v} = \frac{1-y}{y}, \quad z = -\frac{v^{1/d}}{\theta} \frac{(1-y)^{1/d}}{y^{1/d}} \\
dz &= -\frac{v^{1/d}}{\theta} \frac{y^{1/d} d^{-1} (1-y)^{1/d-1} (-1) - (1-y)^{1/d} d^{-1} y^{1/d-1}}{y^{2/d}} dy = \frac{v^{1/d}}{\theta d} y^{-1/d-1} (1-y)^{1/d-1} dy,
\end{aligned}$$

so that

$$\begin{aligned}
I &= \int_{-\infty}^{c} z^r \left(1 + \frac{(-z\cdot\theta)^d}{v}\right)^{-\left(v+\frac{1}{d}\right)} dz \\
&= \int_{0}^{1/\left(1+(-c\theta)^d/v\right)} \left(-\frac{v^{1/d}}{\theta} \frac{(1-y)^{1/d}}{y^{1/d}}\right)^r y^{v+1/d} \frac{v^{1/d}}{\theta d} y^{-1/d-1} (1-y)^{1/d-1} dy \\
&= (-1)^r \frac{v^{(r+1)/d}}{d\theta^{r+1}} \int_{0}^{1/\left(1+(-c\theta)^d/v\right)} y^{v-r/d-1} (1-y)^{(r+1)/d-1} dy \\
&= (-1)^r \frac{v^{(r+1)/d}}{d\theta^{r+1}} B_L\left(v - \frac{r}{d}, \frac{r+1}{d}\right), \quad L = \frac{1}{1+(-c\theta)^d/v} = \frac{v}{v+(-c\theta)^d}
\end{aligned}$$

and

$$S_r(c) = \frac{IK}{F_Z(c)} = (-1)^r v^{r/d} \frac{1+\theta^2}{\theta^r + \theta^{r+2}} \frac{B_L(v - r/d, (r+1)/d)}{B_L(\nu, 1/d)}.$$

Solution to Problem 7.8:

a) We need only examine the kernel of the density. With $a = b$ and $v = 2a$,

$$\begin{aligned} f_T(t) &\propto \left(\left(1+\frac{t}{(2a+t^2)^{1/2}}\right)\left(1-\frac{t}{(2a+t^2)^{1/2}}\right)\right)^{a+1/2} \\ &= \left(\left(1-\frac{t^2}{2a+t^2}\right)\right)^{(2a+1)/2} \\ &\propto \left(v+t^2\right)^{-(v+1)/2}. \end{aligned}$$

b) Squaring both sides and simple manipulations show that this relation is equivalent to

$$0 = 4\left(S^2+k\right)B^2 - 4\left(S^2+k\right)B + k,$$

where $k = a + b$, so that

$$B_{\pm} = \frac{4\left(S^2+k\right) \pm \sqrt{16\left(S^2+k\right)^2 - 16k\left(S^2+k\right)}}{8\left(S^2+k\right)} = \frac{1}{2} \pm \frac{1}{2}\frac{|S|}{\left(S^2+k\right)^{1/2}}.$$

Observe from (16.52) that $\{B < 1/2\} \Leftrightarrow \{S < 0\}$ and $\{B > 1/2\} \Leftrightarrow \{S > 0\}$, i.e., for $S < 0$, $B = B_-$ and for $S > 0$, $B = B_+$. That is,

$$B = \frac{1}{2} + \frac{1}{2}\frac{S}{\left(S^2+k\right)^{1/2}}. \qquad \text{(S-7.6)}$$

Next,

$$\frac{d}{ds}\left(\frac{1}{2}\frac{s}{\left(s^2+k\right)^{1/2}}\right) = \frac{1}{2}\frac{\left(s^2+k\right)^{1/2} - s^2\left(s^2+k\right)^{-1/2}}{\left(s^2+k\right)} = \frac{k}{2}\left(s^2+k\right)^{-3/2},$$

so that, with $f_B(x) = x^{a-1}(1-x)^{b-1}/B(a,b)$, $f_S(s) = f_B(x)\,|dx/ds|$ is given by (with $Q = \left(s^2+k\right)^{1/2}$ for convenience)

$$\begin{aligned} &\frac{k}{2}\frac{\left(s^2+k\right)^{-3/2}}{B(a,b)}\left(\frac{1}{2}+\frac{1}{2}\frac{s}{Q}\right)^{a-1}\left(\frac{1}{2}-\frac{1}{2}\frac{s}{Q}\right)^{b-1} \\ = \;& \frac{k\left(s^2+k\right)^{-3/2}}{B(a,b)}\frac{1}{2^{a+b-1}}\left(1+\frac{s}{Q}\right)^{a-1}\left(1-\frac{s}{Q}\right)^{b-1} \\ = \;& \frac{1}{k^{1/2}B(a,b)}\frac{1}{2^{a+b-1}}\left(1+\frac{s^2}{k}\right)^{-3/2}\left(1+\frac{s}{Q}\right)^{a-1}\left(1-\frac{s}{Q}\right)^{b-1} \\ = \;& \frac{1}{k^{1/2}B(a,b)}\frac{1}{2^{a+b-1}}\left(1+\frac{s}{Q}\right)^{a+1/2}\left(1-\frac{s}{Q}\right)^{b+1/2} \end{aligned}$$

as

$$\begin{aligned} &\left(1+\frac{s^2}{k}\right)^{-3/2}\left(1+\frac{s}{Q}\right)^{-3/2}\left(1-\frac{s}{Q}\right)^{-3/2} \\ = \;& \left(1+\frac{s^2}{k}\right)^{-3/2}\left(1-\frac{s^2}{s^2+k}\right)^{-3/2} \\ = \;& \left(\frac{k+s^2}{k}\right)^{-3/2}\left(\frac{k}{s^2+k}\right)^{-3/2} = 1. \end{aligned}$$

Finally, from (S-7.6),

$$\Pr(S \le t) = \Pr\left(\frac{\sqrt{k}\,(2B-1)}{2\sqrt{B(1-B)}} \le t\right) = \Pr\left(B \le \frac{1}{2} + \frac{1}{2}\frac{t}{(t^2+k)^{1/2}}\right),$$

from which the cdf expression immediately follows.

c) From Example 9.11 and the fact that χ^2 is a special case of gamma (whereby both U and V have the same scale parameter, $1/2$), it follows that $U/(U+V) \sim \text{Beta}(a,b)$. Thus, from (16.52),

$$\frac{\sqrt{k}\,(2B-1)}{2\sqrt{B(1-B)}} = \frac{\sqrt{k}\left(2\left(\frac{U}{U+V}\right)-1\right)}{2\sqrt{\left(\frac{U}{U+V}\right)\left(1-\left(\frac{U}{U+V}\right)\right)}} = R,$$

showing the result. Next, recalling how F r.v.s arise (see Example 9.8), we can let

$$F = \frac{U/(2a)}{V/(2b)} = \frac{bU}{aV},$$

yielding

$$Z = \frac{\sqrt{k}}{2}\left(\left(\frac{U}{V}\right)^{1/2} - \left(\frac{U}{V}\right)^{-1/2}\right) = \frac{\sqrt{k}}{2}\frac{U-V}{\sqrt{(UV)}}.$$

Using (16.53), the result follows.

d) From (16.52),

$$\begin{aligned}\mathbb{E}[S^r] &= 2^{-r}k^{r/2}\mathbb{E}\left[\frac{(2B-1)^r}{B^{r/2}(1-B)^{r/2}}\right] \\ &= \frac{2^{-r}k^{r/2}}{B(a,b)}\int_0^1 (2x-1)^r x^{a-1-r/2}(1-x)^{b-1-r/2}\,dx\end{aligned}$$

and, from the binomial theorem,

$$(2x-1)^r = \sum_{j=0}^{r}\binom{r}{j}(-1)^j(2x)^{r-j},$$

so that

$$\begin{aligned}\mathbb{E}[S^r] &= \frac{2^{-r}k^{r/2}}{B(a,b)}\sum_{j=0}^{r}\binom{r}{j}(-1)^j 2^{r-j}\int_0^1 x^{a-1-r/2+r-j}(1-x)^{b-1-r/2}\,dx \\ &= \frac{k^{r/2}}{B(a,b)}\sum_{j=0}^{r}\binom{r}{j}(-1)^j 2^{-j}B(a+r/2-j,b-r/2).\end{aligned}$$

In particular, for $r=1$, $\mathbb{E}[S]$ is given by

$$\begin{aligned}
& \frac{(a+b)^{1/2}}{B(a,b)}\left(B(a+1/2,b-1/2)-2^{-1}B(a-1/2,b-1/2)\right) \\
= \; & (a+b)^{1/2}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\left(\frac{\Gamma(a+1/2)\Gamma(b-1/2)}{\Gamma(a+b)}-\frac{1}{2}\frac{\Gamma(a-1/2)\Gamma(b-1/2)}{\Gamma(a+b-1)}\right) \\
= \; & \frac{(a+b)^{1/2}}{\Gamma(a)\Gamma(b)} \\
& \times\left(\frac{(a-1/2)\Gamma(a-1/2)\Gamma(b-1/2)}{1}-\frac{1}{2}\frac{(a+b-1)\Gamma(a-1/2)\Gamma(b-1/2)}{1}\right) \\
= \; & (a-b)\frac{(a+b)^{1/2}}{2}\frac{\Gamma(a-1/2)\Gamma(b-1/2)}{\Gamma(a)\Gamma(b)}.
\end{aligned}$$

e) Use the substitution given in (16.52), i.e.,

$$S=\left(\sqrt{k}\,(2B-1)\right)/\left(2\sqrt{B(1-B)}\right),$$

and first observe that

$$\left(1+\frac{S}{(k+S^2)^{1/2}}\right)^{a+1/2}\left(1-\frac{S}{(k+S^2)^{1/2}}\right)^{b+1/2}=2^{k+1}B^{a+1/2}(1-B)^{b+1/2}.$$

Let $y=\left(1+c\left(c^2+k\right)^{-1/2}\right)/2$ and note that

$$\frac{dB}{dS}=k\left(S^2+k\right)^{-3/2}/2\Leftrightarrow\frac{dS}{dB}=\frac{2}{k}\left(S^2+k\right)^{3/2}=\frac{\sqrt{k}}{4}\left(B(1-B)\right)^{-3/2},$$

yielding

$$\begin{aligned}
\int_{-\infty}^{c} sf_S(s)\,ds &= \frac{\sqrt{k}/2}{B(a,b)}\int_0^y (2B-1)\,B^{a-3/2}(1-B)^{b-3/2}\,dB \\
&= \frac{\sqrt{k}}{B(a,b)}\left(B_y(a+1/2,b-1/2)-\frac{1}{2}B_y(a-1/2,b-1/2)\right)
\end{aligned}$$

or

$$\mathbb{E}[S\mid S<c]=\frac{\sqrt{k}}{B_y(a,b)}\left(B_y(a+1/2,b-1/2)-\frac{1}{2}B_y(a-1/2,b-1/2)\right).$$

Solution to Problem 7.9:

a) The density of X is given by

$$\begin{aligned}
f_X(x; n, a, b) &= \int_{-\infty}^{\infty} f_{X,P}(x, p)\, dp = \int_{-\infty}^{\infty} f_{X|P}(x \mid p)\, f_P(p)\, dp \\
&= \int_0^1 \binom{n}{x} \frac{1}{B(a,b)} p^{x+a-1} (1-p)^{n-x+b-1}\, dp \\
&= \frac{n!}{(n-x)!x!} \frac{B(x+a, n-x+b)}{B(a,b)} \\
&= \frac{1}{n+1} \frac{(n+1)!}{x!\,(n-x)!} \frac{B(x+a, n-x+b)}{B(a,b)} \\
&= \frac{1}{n+1} \frac{\Gamma(2+n)}{\Gamma(x+1)\,\Gamma(n-x+1)} \frac{B(x+a, n-x+b)}{B(a,b)} \\
&= \frac{1}{n+1} \frac{1}{B(x+1, n-x+1)} \frac{B(x+a, n-x+b)}{B(a,b)},
\end{aligned}$$

for $x = 0, 1, \ldots, n$.

b) From the moments of the beta distribution, $\mathbb{E}_P[P] = a/(a+b)$,

$$\begin{aligned}
\mathbb{E}_P[P(1-P)] &= \mathbb{E}_P[P] - \mathbb{E}_P\left[P^2\right] \\
&= \frac{a}{a+b} - \frac{a(a+1)}{(a+b)(a+b+1)} = \frac{ab}{(a+b)(a+b+1)}
\end{aligned}$$

and

$$\mathbb{V}(P) = \frac{ab}{(a+b)^2 (a+b+1)},$$

so that, from (8.31),

$$\mathbb{E}[X] = \mathbb{E}_P[\mathbb{E}[X \mid P]] = \mathbb{E}[nP] = \frac{na}{a+b}$$

and, from (8.37),

$$\begin{aligned}
\mathbb{V}(X) &= \mathbb{E}_P[\mathbb{V}(X \mid P)] + \mathbb{V}_P(\mathbb{E}[X \mid P]) \\
&= \mathbb{E}_P[nP(1-P)] + \mathbb{V}_P(nP) \\
&= n \frac{ab}{(a+b)(a+b+1)} + n^2 \frac{ab}{(a+b)^2 (a+b+1)} \\
&= \frac{nab(a+b+n)}{(a+b)^2 (a+b+1)}.
\end{aligned}$$

c) Using algebraic software, we get $a = 10/3$ and $b = 5$, with mass function plotted in Figure S-7.1.

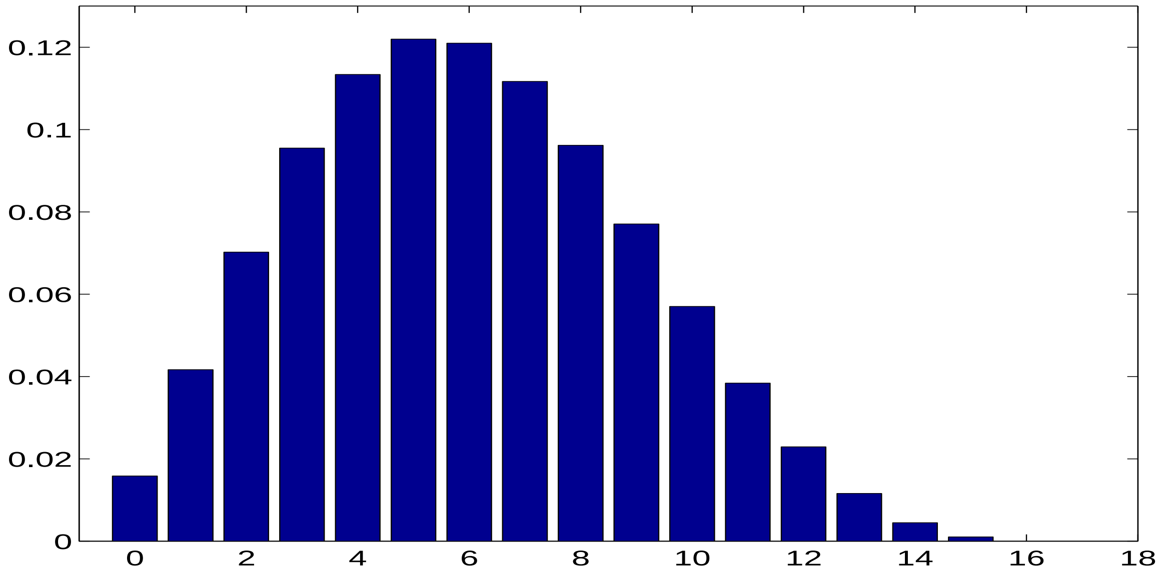

Figure S-7.1: The beta-binomial mass function for $n = 15$, $a = 10/3$ and $b = 5$

Solutions to Chapter 8: The Stable Paretian Distribution

Solutions to Chapter 9: Generalized Inverse Gaussian and Generalized Hyperbolic Distributions

Solution to Problem 9.1: We use Leibniz' rule, and let $\nu \in \mathbb{R}$ and $x > 0$. Then

$$\begin{aligned}
\frac{d}{dx}K_\nu(x) &= \frac{d}{dx}\frac{1}{2}\int_0^\infty t^{\nu-1}e^{-\frac{1}{2}x(t+t^{-1})}dt \\
&= \frac{1}{2}\int_0^\infty t^{\nu-1}\frac{d}{d\,x}e^{-\frac{1}{2}x(t+t^{-1})}dt \\
&= \frac{1}{2}\int_0^\infty t^{\nu-1}\left(-\frac{1}{2}(t+t^{-1})\right)e^{-\frac{1}{2}x(t+t^{-1})}dt \\
&= -\frac{1}{2}\frac{1}{2}\int_0^\infty t^{\nu-1}(t+t^{-1})e^{-\frac{1}{2}x(t+t^{-1})}dt \\
&= -\frac{1}{2}(K_{\nu-1}(x)+K_{\nu+1}(x)),
\end{aligned}$$

so finally

$$-2K'_\nu(x) = K_{\nu-1}(x) + K_{\nu+1}(x). \tag{S-9.1}$$

Solution to Problem 9.2: Let $x > 0$. We are given the equality

$$K_{1/2}(x) = K_{-1/2}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}.$$

So using (18.4), we can calculate $K_{3/2}(x)$ by letting $\nu = 1/2$:

$$K_{3/2}(x) = \frac{1}{x}K_{1/2}(x) + K_{-1/2}(x) = (1/x+1)\sqrt{\frac{\pi}{2x}}e^{-x}.$$

Similarly, $K_{5/2}(x)$ can be calculated from $K_{3/2}(x)$ and $K_{1/2}(x)$ by (18.4), $K_{7/2}(x)$ can be calculated from $K_{5/2}(x)$ and $K_{3/2}(x)$ and so on. In general, one can determine $K_{n+1/2}(x)$ for every integer n by a recursive computation. The first results are

$$K_{5/2}(x) = (3x^{-2} + 3x^{-1} + 1)K_{1/2}(x)$$

and

$$K_{7/2}(x) = (15x^{-3} + 15x^{-2} + 6x^{-1} + 1)K_{1/2}(x).$$

Solution to Problem 9.3:

a) The variance equation: Note that for every $z > 0$ we have

$$\int_{-\infty}^{\infty} x^2 f_{\mathrm{N}}(x;\mu+\beta z, z)dx = z + (\mu+\beta z)^2,$$

because this is just the second raw moment of a $\mathrm{N}(\mu+\beta z, z)$ distributed random variable. So

$$\begin{aligned}
\mathbb{E}[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_{-\infty}^{\infty} x^2 \int_0^{\infty} f_{\mathrm{N}}(x;\mu+\beta z, z) f_Z(z)dzdx \\
&= \int_0^{\infty}\int_{-\infty}^{+\infty} x^2 f_{\mathrm{N}}(x;\mu+\beta z, z)dx f_Z(z)dz \\
&= \int_0^{\infty} \left[z + (\mu+\beta z)^2\right] f_Z(z)dz = \mathbb{E}[Z] + \mu^2 + 2\mu\beta\mathbb{E}[Z] + \beta^2\mathbb{E}[Z^2].
\end{aligned}$$

Thus, the variance of X is

$$\mathbb{V}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathbb{E}[Z] + \beta^2\mathbb{V}(Z).$$

b) The third moment: We use the expression for the mean of X we have already calculated to compute the third central moment of X. Note that, for every $z \in \mathbb{Z}$ (where N denotes a $\mathrm{N}(\mu+\beta z, z)$ distributed random variable),

$$\begin{aligned}
&\int_{-\infty}^{\infty} (x-\mu-\beta\mathbb{E}[Z])^3 f_{\mathrm{N}}(x;\mu+\beta z, z)dx \\
= &\int_{-\infty}^{\infty} (x-\mu-\beta z+\beta z-\beta\mathbb{E}[Z])^3 f_{\mathrm{N}}(x;\mu+\beta z, z)dx \\
= &\int_{-\infty}^{\infty} \big[(x-\mu-\beta z)^3 + 3(x-\mu-\beta z)^2(\beta z-\beta\mathbb{E}[Z]) \\
&\quad + 3(x-\mu-\beta z)(\beta z-\beta\mathbb{E}[Z])^2 + (\beta z-\beta\mathbb{E}[Z])^3\big] f_{\mathrm{N}}(x;\mu+\beta z, z)dx \\
= &\ \mu_3(N) + 3\beta\mathbb{V}(N)(z-\mathbb{E}[Z]) \\
&+3(\mathbb{E}[N]-\mu-\beta z)\beta^2(z-\mathbb{E}[Z])^2 + \beta^3(z-\mathbb{E}[Z])^3 \\
= &\ 0 + 3\beta z(z-\mathbb{E}[Z]) + 0 + \beta^3(z-\mathbb{E}[Z])^3.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mu_3(X) &= \int_{-\infty}^{\infty} (x-\mathbb{E}[X])^3 f_X(x)dx \\
&= \int_{-\infty}^{\infty} (x-\mathbb{E}[X])^3 \int_0^{\infty} f_{\mathrm{N}}(x;\mu+\beta z, z) f_Z(z)dzdx \\
&= \int_0^{\infty}\int_{-\infty}^{\infty} (x-\mu-\beta\mathbb{E}[Z])^3 f_{\mathrm{N}}(x;\mu+\beta z, z)dx f_Z(z)dz \\
&= \int_0^{\infty} 3\beta z(z-\mathbb{E}[Z]) + \beta^3(z-\mathbb{E}[Z])^3 f_Z(z)dz \\
&= 3\beta\mathbb{V}(Z) + \beta^3\mu_3(Z).
\end{aligned}$$

Solution to Problem 9.4: By the transformation formula (7.65), X^{-1} has density

$$f_{X^{-1}}(x) = |x|^{-2} f_X(x^{-1}) = \frac{1}{\mathrm{k}_\lambda(\chi,\,\psi)}\, x^{-\lambda-1} e^{-\frac{1}{2}(\chi x + \psi x^{-1})}, \qquad x > 0.$$

But this is the density of $\mathrm{GIG}(-\lambda, \psi, \chi)$.

Solution to Problem 9.5: Again via (7.65), random variable rX has density

$$\begin{aligned} f_{rX}(x) &= r^{-1} f_X(x/r) = \frac{1}{\mathrm{k}_\lambda(\chi,\,\psi)}\, r^{-1}(x/r)^{\lambda-1} e^{-\frac{1}{2}(\chi(x/r)^{-1} + \psi x/r)} \\ &= \frac{1}{\mathrm{k}_\lambda(\chi,\,\psi)}\, r^{-\lambda} x^{\lambda-1} e^{-\frac{1}{2}((r\chi)x^{-1} + (r^{-1}\psi)x)}, \qquad x > 0. \end{aligned}$$

This is the density of $\mathrm{GIG}(\lambda, r\chi, r^{-1}\psi)$.

Solution to Problem 9.6: For **boundary case I**, we have

$$\mathbb{E}[X^r] \overset{(18.18)}{=} \frac{\mathrm{k}_{\lambda+r}(0,\,\psi)}{\mathrm{k}_\lambda(0,\,\psi)} \overset{(18.9)}{=} \frac{(\psi/2)^{-\lambda-r}\Gamma(\lambda+r)}{(\psi/2)^{-\lambda}\Gamma(\lambda)} = \left(\frac{\psi}{2}\right)^{-r} \frac{\Gamma(\lambda+r)}{\Gamma(\lambda)}.$$

This makes sense whenever $r > -\lambda$. Moreover,

$$\mathbb{M}_X(t) \overset{(18.18)}{=} \frac{\mathrm{k}_\lambda(0,\,\psi-2t)}{\mathrm{k}_\lambda(0,\,\psi)} \overset{(18.9)}{=} \frac{((\psi-2t)/2)^{-\lambda}}{(\psi/2)^{-\lambda}} = \left(1 - \frac{2t}{\psi}\right)^{-\lambda}.$$

And as above, t has to be smaller than $\psi/2$ for this to make sense.

There are two ways to determine the raw moments in **boundary case II**. The first would be a similar calculation as in boundary case I. But we have shown above that, if $X \sim \mathrm{GIG}(\lambda, \chi, \psi)$, then $X^{-1} \sim \mathrm{GIG}(-\lambda, \psi, \chi)$. So, as the r^{th} raw moment of X is the $-r^{\text{th}}$ raw moment of X^{-1}, we can use the formula for the raw moments in boundary case I to derive

$$\mathbb{E}[X^r] = \left(\frac{\chi}{2}\right)^{r} \frac{\Gamma(-\lambda-r)}{\Gamma(-\lambda)}, \quad \text{for} \quad r < \lambda.$$

The mgf is calculated as in boundary case I,

$$\mathbb{M}_X(t) \overset{(18.18)}{=} \frac{\mathrm{k}_\lambda(\chi,\,-2t)}{\mathrm{k}_\lambda(\chi,\,0)} \overset{(18.8),(18.10)}{=} \frac{2K_\lambda(\sqrt{-2t\chi})\sqrt{-\chi/(2t)}^{\lambda}}{(\chi/2)^\lambda \Gamma(-\lambda)} = \frac{2K_\lambda\left(\sqrt{-2\chi t}\right)}{\Gamma(-\lambda)\left(\frac{-\chi t}{2}\right)^{\lambda/2}}, \quad t \le 0.$$

Solution to Problem 9.7: Suppose $\nu > 0$. We show that

$$\mathrm{k}_\nu(\chi,\,\psi) \to \mathrm{k}_\nu(0,\,\psi)$$

if $\chi \to 0$. If $\chi > 0$ then

$$\mathrm{k}_\nu(\chi,\,\psi) = 2(\chi/\psi)^{\nu/2} K_\nu(\sqrt{\chi\psi}) = \frac{2\chi^{\nu/2} K_\nu(\sqrt{\chi\psi})}{\psi^{\nu/2}}.$$

If $\chi \to 0$, then $\omega = \sqrt{\chi\psi} \to 0$. So we can use relation (18.6), i.e.,

$$K_\nu(\sqrt{\chi\psi}) \simeq \Gamma(\nu)2^{\nu-1}(\chi\psi)^{-\nu/2}.$$

This means that

$$\frac{K_\nu(\sqrt{\chi\psi})}{\Gamma(\nu)2^{\nu-1}(\chi\psi)^{-\nu/2}} \to 1$$

as $\chi \to 0$, and it follows that

$$K_\nu(\sqrt{\chi\psi})\chi^{\nu/2} \to \Gamma(\nu)2^{\nu-1}\psi^{-\nu/2}.$$

Hence,

$$\mathrm{k}_\nu\left(\chi,\, \psi\right) \to \frac{2\Gamma(\nu)2^{\nu-1}\psi^{-\nu/2}}{\psi^{\nu/2}} = \frac{\Gamma(\nu)2^\nu}{\psi^\nu} = \left(\frac{\psi}{2}\right)^{-\nu}\Gamma(\nu) = \mathrm{k}_\nu\left(0,\, \psi\right).$$

Now we have shown this, we can conclude that

$$\frac{\mathrm{k}_{\lambda+r}\left(\chi,\, \psi\right)}{\mathrm{k}_\lambda\left(\chi,\, \psi\right)} \to \frac{\mathrm{k}_{\lambda+r}\left(0,\, \psi\right)}{\mathrm{k}_\lambda\left(0,\, \psi\right)}.$$

The lhs is the r^{th} raw moment of $\mathrm{GIG}(\lambda, \chi, \psi)$, whereas the rhs is the r^{th} raw moment of $\mathrm{GIG}(\lambda, 0, \psi)$.

Solution to Problem 9.8:

a) Note that $\mathbb{M}_Z(t) = \mathbb{E}\left(e^{t\ln X}\right) = \mathbb{E}\left(X^t\right)$. So Equation (18.18) gives

$$\mathbb{M}_Z(t) = \eta^t \frac{K_{\lambda+t}\left(\omega\right)}{K_\lambda\left(\omega\right)}$$

with $\eta = \sqrt{\chi/\psi}$ and $\omega = \sqrt{\chi\psi}$.

b) The c.g.f. of Z is

$$\mathbb{K}_Z(t) = \ln\left(\eta^t \frac{K_{\lambda+t}\left(\omega\right)}{K_\lambda\left(\omega\right)}\right) = t\ln\eta + \ln\left(K_{\lambda+t}\left(\omega\right)\right) - \ln\left(K_\lambda\left(\omega\right)\right),$$

so

$$\frac{d}{dt}\mathbb{K}_Z(t) = \ln\eta + \frac{d}{dt}\ln\left(K_{\lambda+t}\left(\omega\right)\right).$$

We evaluate this at $t = 0$ to get

$$\mathbb{E}\left(\ln X\right) = \mathbb{E}\left(Z\right) = \ln\eta + \frac{\frac{\partial}{\partial\lambda}K_\lambda(\omega)}{K_\lambda(\omega)}.$$

c) Assume $X \sim \mathrm{GIG}(\lambda, 0, \psi)$ with $\lambda, \psi > 0$. This means that $X \sim \mathrm{Gam}\left(\lambda, \psi/2\right)$. Define $Z := \ln X$. The m.g.f. of Z is

$$\mathbb{M}_Z(t) = \mathbb{E}\left(X^t\right) \overset{(18.26)}{=} \frac{\Gamma(\lambda+t)}{\Gamma(\lambda)}\left(\frac{\psi}{2}\right)^{-t},$$

for $t \in (-\lambda, \infty)$, so

$$\mathbb{K}_Z(t) = \ln(\Gamma(\lambda + t)) - \ln(\Gamma(\lambda)) - t\ln(\psi/2).$$

Differentiating with respect to t yields

$$\frac{d}{dt}\mathbb{K}_Z(t) = \frac{d}{dt}\ln(\Gamma(\lambda + t)) - \ln\frac{\psi}{2},$$

and hence,

$$\mathbb{E}(\ln X) = \mathbb{E}(Z) = \frac{\Gamma'(\lambda)}{\Gamma(\lambda)} - \ln\frac{\psi}{2}.$$

After setting $\lambda = \nu/2$ and $\psi = 1$, i.e., in the case $X \sim \mathrm{Gam}(\nu/2, 1/2) = \chi^2_\nu$, this simplifies to $\frac{\Gamma'(\nu/2)}{\Gamma(\nu/2)} + \ln 2$, the result we have already met in Example 10.8.

d) Assume $X \sim \mathrm{GIG}(\lambda, \chi, 0)$ with $\lambda < 0$ and $\chi > 0$. This means that $X \sim \mathrm{IGam}(-\lambda, \chi/2)$. Define $Z := \ln X$. The m.g.f. of Z is

$$\mathbb{M}_Z(t) = \mathbb{E}\left(X^t\right) \overset{(18.29)}{=} \frac{\Gamma(-\lambda - t)}{\Gamma(-\lambda)}\left(\frac{\chi}{2}\right)^t,$$

for $t \in (-\infty, -\lambda)$, so

$$\mathbb{K}_Z(t) = \ln(\Gamma(-\lambda - t)) - \ln(\Gamma(-\lambda)) + t\ln(\chi/2).$$

Differentiating with respect to t yields

$$\frac{d}{dt}\mathbb{K}_Z(t) = \frac{d}{dt}\ln(\Gamma(-\lambda - t)) + \ln\frac{\chi}{2},$$

and hence,

$$\mathbb{E}(\ln X) = \mathbb{E}(Z) = \frac{-\Gamma'(-\lambda)}{\Gamma(-\lambda)} + \ln\frac{\chi}{2}.$$

This result can also be obtained by using the fact that $X^{-1} \sim \mathrm{GIG}(-\lambda, 0, \chi) = \mathrm{Gam}(-\lambda, \chi/2)$, which was shown in Problem 18.4. Now

$$\mathbb{E}(\ln X) = -\mathbb{E}\left(\ln X^{-1}\right) = -\left(\frac{\Gamma'(-\lambda)}{\Gamma(-\lambda)} - \ln\frac{\chi}{2}\right)$$

from the preceding subproblem.

Solution to Problem 9.9: We use the mean equation (18.13) and the expression (18.27) for the mean of the GIG distribution in boundary case I. Suppose $Z \sim \mathrm{GIG}(\lambda, 0, \psi)$ and $X \sim \mathrm{GHyp}(\lambda, \alpha, \beta, 0, \mu)$, where $\psi = \alpha^2 - \beta^2$. Then

$$\mathbb{E}[X] = \mu + \beta\mathbb{E}[Z] = \mu + \beta\frac{2\lambda}{\psi}.$$

The variance can be calculated in the same fashion:

$$\mathbb{V}(X) = \mathbb{E}[Z] + \beta^2\mathbb{V}(Z) = \frac{\lambda}{\psi/2} + \beta^2\frac{\lambda}{(\psi/2)^2} = \frac{2\lambda\psi + 4\beta^2\lambda}{\psi^2} = 2\lambda\frac{\alpha^2 + \beta^2}{\psi^2},$$

where we used $\psi = \alpha^2 - \beta^2$. The mgf is

$$\begin{aligned}\mathbb{M}_X(t) &= e^{\mu t}\mathbb{M}_Z\big(\beta t + t^2/2\big) = e^{\mu t}\left(1 - \frac{\beta t + t^2/2}{\psi/2}\right)^{-\lambda} \\ &= e^{\mu t}\left(\frac{\alpha^2 - \beta^2 - 2\beta t + t^2}{\psi}\right)^{-\lambda} = e^{\mu t}\left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + t)^2}\right)^{\lambda}\end{aligned}$$

Solution to Problem 9.10: Suppose $Z \sim \mathrm{GIG}(\lambda, \chi, 0)$ and $X \sim \mathrm{GHyp}(\lambda, |\beta|, \beta, \delta, \mu)$, where $\chi = \delta^2$. Then from (18.13) and (18.30),

$$\mathbb{E}[X] = \mu + \beta\mathbb{E}[Z] = \mu - \beta\frac{\chi}{2}\frac{1}{\lambda + 1} \quad (\text{if } \lambda < -1).$$

The variance is (if $\lambda < -2$):

$$\mathbb{V}(X) = \mathbb{E}[Z] + \beta^2\mathbb{V}(Z) = \frac{\chi}{2}\frac{-1}{\lambda + 1} + \beta^2\left(\frac{\chi}{2}\right)^2\frac{-1}{(\lambda + 1)^2(\lambda + 2)}.$$

If $\beta \neq 0$ then the mgf is

$$\mathbb{M}_X(t) = e^{\mu t}\mathbb{M}_Z\big(\beta t + t^2/2\big) = e^{\mu t}\frac{2K_\lambda\left(\sqrt{-2\chi\left(\beta t + t^2/2\right)}\right)}{\Gamma(-\lambda)\left(\frac{-\chi(\beta t + t^2/2)}{2}\right)^{\lambda/2}},$$

where $-2\beta \leq t \leq 0$. If $\beta = 0$, i.e., if X is symmetric, then the formulae for mean and variance still hold, but the mgf is only defined for $t = 0$, in other words, there is no mgf.

Solution to Problem 9.11: The pdf of Hyp is

$$\begin{aligned}f_{\mathrm{Hyp}_3}(x;\ \alpha, \beta, \delta, \mu) &= f_{\mathrm{GHyp}}(x;\ 1, \alpha, \beta, \delta, \mu) \\ &\overset{(18.41)}{=} \frac{\sqrt{\alpha^2 - \beta^2}y^{1/2}}{\sqrt{2\pi}\alpha^{1/2}\delta K_1(\delta\sqrt{\alpha^2 - \beta^2})}K_{\frac{1}{2}}\left(\alpha y\right)e^{\beta(x-\mu)} \\ &\overset{(18.3)}{=} \frac{\sqrt{\alpha^2 - \beta^2}y^{1/2}}{\sqrt{2\pi}\alpha^{1/2}\delta K_1(\delta\sqrt{\alpha^2 - \beta^2})}\sqrt{\frac{\pi}{2\alpha y}}e^{-\alpha y}e^{\beta(x-\mu)} \\ &= \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1(\delta\sqrt{\alpha^2 - \beta^2})}\exp\left[-\alpha y + \beta(x - \mu)\right],\end{aligned}$$

using the abbreviation $y = \sqrt{\delta^2 + (x - \mu)^2}$. Similarly, the pdf of NIG is calculated:

$$\begin{aligned}f_{\mathrm{NIG}}(x;\ \alpha, \beta, \delta, \mu) &= f_{\mathrm{GHyp}}(x;\ -1/2, \alpha, \beta, \delta, \mu) \\ &\overset{(18.41)}{=} \frac{(\sqrt{\alpha^2 - \beta^2})^{-1/2}y^{-1}}{\sqrt{2\pi}\alpha^{-1}\delta^{-1/2}K_{-1/2}(\delta\sqrt{\alpha^2 - \beta^2})}K_{-1}\left(\alpha y\right)e^{\beta(x-\mu)} \\ &\overset{(18.3)}{=} e^{\delta\sqrt{\alpha^2-\beta^2}}\frac{(\sqrt{\alpha^2 - \beta^2})^{-1/2}y^{-1}}{\sqrt{2\pi}\alpha^{-1}\delta^{-1/2}\sqrt{\frac{\pi}{2\delta\sqrt{\alpha^2-\beta^2}}}}K_{-1}\left(\alpha y\right)e^{\beta(x-\mu)} \\ &= e^{\delta\sqrt{\alpha^2-\beta^2}}\frac{\alpha\delta}{\pi y}K_1\left(\alpha y\right)e^{\beta(x-\mu)}.\end{aligned}$$

Solution to Problem 9.12:

a) Suppose $X \sim \mathrm{IG}_1(\chi, \psi)$, where $\chi > 0$ and $\psi > 0$. By (18.24) and (18.3),

$$\mathbb{E}[X] = \eta \frac{K_{-1/2+1}(\omega)}{K_{-1/2}(\omega)} = \eta,$$

and the second raw moment can be computed as follows using (18.22):

$$\mathbb{E}[X^2] = \eta^2 \frac{K_{3/2}(\omega)}{K_{1/2}(\omega)} = \eta^2(1/\omega + 1).$$

So the variance is

$$\mathbb{V}(X) = \frac{\eta^2}{\omega}.$$

The third raw moment is

$$\mathbb{E}[X^3] = \eta^3 \frac{K_{5/2}(\omega)}{K_{1/2}(\omega)} = \eta^3(3\omega^{-2} + 3\omega^{-1} + 1).$$

From this, we can compute the third central moment μ_3 using (4.49):

$$\mu_3(X) = \mathbb{E}[X^3] - 3\mathbb{V}(X)\mathbb{E}[X] - \mathbb{E}[X]^3 = \eta^3(3\omega^{-2} + 3\omega^{-1} + 1 - 3\omega^{-1} - 1) = 3\eta^3\omega^{-2}$$

Alternatively, the raw moments of X can be computed using the mgf (18.36) of X.

b) Suppose $Z \sim \mathrm{IG}_1(\chi, \psi)$ and $X \sim \mathrm{NIG}(\alpha, \beta, \delta, \mu)$, where $\chi = \delta^2 > 0$ and $\psi = \alpha^2 - \beta^2 > 0$. We set $\eta := \sqrt{\chi/\psi}$ and $\omega := \sqrt{\chi\psi}$. Then

$$\mathbb{E}[X] = \mu + \beta\mathbb{E}[Z] = \mu + \beta\eta,$$

$$\mathbb{V}(X) = \mathbb{E}[Z] + \beta^2\mathbb{V}[Z] = \eta + \beta^2\frac{\eta^2}{\omega},$$

and

$$\mu_3(X) = 3\beta\mathbb{V}[Z] + \beta^3\mu_3(Z) = 3\beta\frac{\eta^2}{\omega} + 3\beta^3\frac{\eta^3}{\omega^2}.$$

As far as the mgf is concerned, we derive

$$\begin{aligned} \mathbb{M}_X(t) &= e^{\mu t}\mathbb{M}_Z\big(\beta t + t^2/2\big) = e^{\mu t}e^{\sqrt{\chi}\left(\sqrt{\psi} - \sqrt{\psi - 2(\beta t + t^2/2)}\right)} \\ &= e^{\mu t}e^{\delta\left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2 - (\beta+t)^2}\right)}, \end{aligned}$$

where $-\alpha - \beta \le t \le \alpha - \beta$.

Solution to Problem 9.13:

a) This follows directly by applying (18.55).

b) Suppose that π is infinitely divisible and $n \in \mathbb{N}$. Let ρ be a positive n^{th} convolution root of π. Then, by the first part of this problem,

$$\mathrm{Mix}_{\rho}(\mu/n, \beta)^{\star n} = \mathrm{Mix}_{\rho^{\star n}}(n\mu/n, \beta) = \mathrm{Mix}_{\pi}(\mu, \beta),$$

so $\mathrm{Mix}_{\rho}(\mu/n, \beta)$ is an n^{th} convolution root of $\mathrm{Mix}_{\pi}(\mu, \beta)$. Hence $\mathrm{Mix}_{\pi}(\mu, \beta)$ is infinitely divisible.

Solution to Problem 9.14:

a) Let $\psi = \alpha^2 - \beta^2 > 0$, $\chi_1 = \delta_1^2 > 0$ and $\chi_2 = \delta_2^2 > 0$. Then

$$\begin{aligned} \mathrm{NIG}(\alpha,\beta,\delta_1,\mu_1) * \mathrm{NIG}(\alpha,\beta,\delta_2,\mu_2) &= \mathrm{Mix}_{\mathrm{IG}_1(\chi_1,\psi)}(\mu_1,\beta) \star \mathrm{Mix}_{\mathrm{IG}_1(\chi_2,\psi)}(\mu_2,\beta) \\ &= \mathrm{Mix}_{\mathrm{IG}_1(\chi_1,\psi)\star \mathrm{IG}_1(\chi_2,\psi)}(\mu_1+\mu_2,\beta) \\ &= \mathrm{Mix}_{\mathrm{IG}_1\left(\left(\sqrt{\chi_1}+\sqrt{\chi_2}\right)^2,\psi\right)}(\mu_1+\mu_2,\beta) \\ &= \mathrm{NIG}\left(\alpha,\beta,\sqrt{\chi_1}+\sqrt{\chi_2},\mu_1+\mu_2\right) \\ &= \mathrm{NIG}\left(\alpha,\beta,\delta_1+\delta_2,\mu_1+\mu_2\right). \end{aligned}$$

b) We have

$$\begin{aligned} & \mathbb{M}_{\mathrm{NIG}(\alpha,\beta,\delta_1,\mu_1)*\mathrm{NIG}(\alpha,\beta,\delta_2,\mu_2)}(t) \\ =\; & \mathbb{M}_{\mathrm{NIG}(\alpha,\beta,\delta_1,\mu_1)}(t)\, \mathbb{M}_{\mathrm{NIG}(\alpha,\beta,\delta_2,\mu_2)}(t) \\ =\; & e^{\mu_1 t} e^{\delta_1\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+t)^2}\right)} e^{\mu_2 t} e^{\delta_2\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+t)^2}\right)} \\ =\; & e^{(\mu_1+\mu_2)t} e^{(\delta_1+\delta_2)\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+t)^2}\right)} \\ =\; & \mathbb{M}_{\mathrm{NIG}(\alpha,\beta,\delta_1+\delta_2,\mu_1+\mu_2)}(t). \end{aligned}$$

This is valid for all $t \in [-\alpha-\beta, \alpha-\beta]$. Therefore, the distributions to which these mgfs belong must be equal.

c) There are two possible ways to prove the convolution formula for the variance–gamma distribution: We have calculated the mgf of the variance–gamma distribution in problem 18.9. We use this and the fact that the mgf of a convolution is the product of the mgfs of the factors to calculate the mgf of the convolution of to variance–gamma distribution:

$$\begin{aligned} & \mathbb{M}_{\mathrm{VG}(\lambda_1,\alpha,\beta,\mu_1)\star \mathrm{VG}(\lambda_2,\alpha,\beta,\mu_2)}(t) \\ =\; & \mathbb{M}_{\mathrm{VG}(\lambda_1,\alpha,\beta,\mu_1)}(t)\, \mathbb{M}_{\mathrm{VG}(\lambda_2,\alpha,\beta,\mu_2)}(t) \\ =\; & \mathbb{M}_{\mathrm{GHyp}(\lambda_1,\alpha,\beta,0,\mu_1)}(t)\, \mathbb{M}_{\mathrm{GHyp}(\lambda_2,\alpha,\beta,0,\mu_2)}(t) \\ =\; & e^{\mu_1 t}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+t)^2}\right)^{\lambda_1} e^{\mu_2 t}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+t)^2}\right)^{\lambda_2} \\ =\; & e^{(\mu_1+\mu_2)t}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+t)^2}\right)^{\lambda_1+\lambda_2} = \mathbb{M}_{\mathrm{VG}(\lambda_1+\lambda_2,\alpha,\beta,\mu_1+\mu_2)}(t). \end{aligned}$$

This is true for all t between $-\alpha-\beta$ and $\alpha-\beta$. From the uniqueness property of the mgf we can deduce that

$$\mathrm{VG}(\lambda_1,\alpha,\beta,\mu_1) \star \mathrm{VG}(\lambda_2,\alpha,\beta,\mu_2) = \mathrm{VG}(\lambda_1+\lambda_2,\alpha,\beta,\mu_1+\mu_2).$$

An alternative approach to proving this formula uses the fact that the gamma distribution is closed under convolution (i.e. the sum of two independent gamma distributed random variables with appropriate parameters is again gamma distributed):

$$\mathrm{Gam}(\lambda_1, \psi/2) \star \mathrm{Gam}(\lambda_2, \psi/2) = \mathrm{Gam}(\lambda_1 + \lambda_2, \psi/2)$$

for all $\lambda, \psi > 0$. Note that $\mathrm{Gam}(\lambda, \psi/2) = \mathrm{GIG}(\lambda, 0, \psi)$. So using our knowledge of mixtures and convolution we get

$$\begin{aligned}
& \mathrm{VG}(\lambda_1, \alpha, \beta, \mu_1) \star \mathrm{VG}(\lambda_2, \alpha, \beta, \mu_2) \\
= \; & \mathrm{GHyp}(\lambda_1, \alpha, \beta, 0, \mu_1) \star \mathrm{GHyp}(\lambda_2, \alpha, \beta, 0, \mu_2) \\
= \; & \mathrm{Mix}_{\mathrm{GIG}(\lambda_1, 0, \alpha^2-\beta^2)}(\mu_1, \beta) \star \mathrm{Mix}_{\mathrm{GIG}(\lambda_2, 0, \alpha^2-\beta^2)}(\mu_2, \beta) \\
= \; & \mathrm{Mix}_{\mathrm{Gam}(\lambda_1, (\alpha^2-\beta^2)/2)}(\mu_1, \beta) \star \mathrm{Mix}_{\mathrm{Gam}(\lambda_2, (\alpha^2-\beta^2)/2)}(\mu_2, \beta) \\
= \; & \mathrm{Mix}_{\mathrm{Gam}(\lambda_1, (\alpha^2-\beta^2)/2) \star \mathrm{Gam}(\lambda_2, (\alpha^2-\beta^2)/2)}(\mu_1 + \mu_2, \beta) \\
= \; & \mathrm{Mix}_{\mathrm{Gam}(\lambda_1+\lambda_2, (\alpha^2-\beta^2)/2)}(\mu_1 + \mu_2, \beta) \\
= \; & \mathrm{Mix}_{\mathrm{GIG}(\lambda_1+\lambda_2, 0, \alpha^2-\beta^2)}(\mu_1 + \mu_2, \beta) \\
= \; & \mathrm{GHyp}(\lambda_1 + \lambda_2, \alpha, \beta, 0, \mu_1 + \mu_2) = \mathrm{VG}(\lambda_1 + \lambda_2, \alpha, \beta, \mu_1 + \mu_2).
\end{aligned}$$

Solutions to Chapter 10: Noncentral Distributions

Solution to Problem 10.1: Let $G \sim \text{Gam}(a, 1)$ with density

$$f_G(x; a, b) = \Gamma^{-1}(a)\; x^{a-1} e^{-bx} \mathbb{I}_{(0,\infty)}(x)$$

and cdf $F_G(x; a, 1) = \Gamma_x(a) \,/\, \Gamma(a)$, where $\Gamma_x(a) = \int_0^x t^{a-1} e^{-t} dt$ is the incomplete gamma function. If $C \sim \chi^2(k)$, then $C = 2G \sim \text{Gam}(a, 1/2)$ for $a = k/2$, and

$$\Gamma_x(a) = \Gamma(a) \Pr(G \leq x) = \Gamma(k/2) \Pr(C \leq 2x).$$

For example, in Matlab, running
`x=1; a=2.5; k=2*a; myimhof (2*x,1,k), gammainc(x,a)`
returns (essentially) the same results. The latter method is far faster and accurate to machine precision.

Solution to Problem 10.2: From the multivariate normal distribution we immediately have the identity

$$|\boldsymbol{\Sigma}|^{1/2} = \underbrace{\int \cdots \int}_{n \text{ times}} (2\pi)^{-n/2} \exp\left\{-\tfrac{1}{2}\mathbf{x}'\boldsymbol{\Sigma}^{-1}\mathbf{x}\right\} dx_1 \cdots dx_n. \qquad \text{(S-10.1)}$$

Write $S = \mathbf{X}'\boldsymbol{\Lambda}\mathbf{X}$, where $\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and $\mathbf{X} \sim \text{N}(\mathbf{0}, \mathbf{I})$. Then, as the X_i are independent,

$$\begin{aligned}
\mathbb{M}_S(s) &= \mathbb{E}\left[e^{s\mathbf{X}'\boldsymbol{\Lambda}\mathbf{X}}\right] = \mathbb{E}\left[e^{s\left(\lambda_1 X_1^2 + \cdots \lambda_n X_n^2\right)}\right] \\
&= \mathbb{E}\left[e^{\lambda_1 X_1^2}\right] \cdot \cdots \cdot \mathbb{E}\left[e^{\lambda_n X_n^2}\right] \\
&= \int e^{s\lambda_1 x_1^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_1^2} \cdot \cdots \cdot \text{similar} \\
&= \int \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x_1^2(-2s\lambda_1 + 1)\right\} \cdot \cdots \cdot \text{similar} \\
&= \int \cdots \int (2\pi)^{-n/2} \exp\left\{-\tfrac{1}{2}\mathbf{x}'(\mathbf{I} - 2s\boldsymbol{\Lambda})\mathbf{x}\right\} dx_1 \cdots dx_n \\
&\overset{\text{S-10.1}}{=} |\mathbf{I} - 2s\boldsymbol{\Lambda}|^{-1/2} = \prod_{i=1}^{n} (1 - 2\lambda_i s)^{-1/2},
\end{aligned}$$

where in the second-to-last equality, $\mathbf{\Sigma}$ in (S-10.1) was replaced by $(\mathbf{I}-2s\mathbf{\Lambda})^{-1}$.

Identity (S-10.1) holds only when $\mathbf{\Sigma}$ is positive definite. But, as $\mathbf{\Lambda}$ is diagonal, that corresponds to requiring that $1-2s\lambda_i>0$, $\forall i$.

Solution to Problem 10.3: Let $\mathbf{Y}=\mathbf{X}/\sigma\sim\mathrm{N}(\mathbf{0},\mathbf{I})$ so that $S=\mathbf{Y}'\mathbf{A}\mathbf{Y}$. Similar to the previous problem,

$$\mathbb{M}_S(s)=\int\cdots\int(2\pi)^{-n/2}\exp\left\{-\tfrac{1}{2}\mathbf{y}'(\mathbf{I}-2s\mathbf{A})\,\mathbf{y}\right\}dy_1\cdots dy_n=|\mathbf{I}-2s\mathbf{A}|^{-1/2}.$$

Let $\mathbf{O}$ be an orthogonal matrix and $\mathbf{D}=\operatorname{diag}(\lambda_1,\ldots,\lambda_n)$ such that $\mathbf{A}=\mathbf{O}\mathbf{D}\mathbf{O}'$. Then

$$\mathbf{O}'(\mathbf{I}-2s\mathbf{A})\,\mathbf{O}=\mathbf{O}'\mathbf{O}-2s\mathbf{O}'\mathbf{A}\mathbf{O}=\mathbf{I}-2s\mathbf{D}=\operatorname{diag}(1-2s\lambda_i)$$

and, using basic properties of determinants,

$$|\mathbf{I}-2s\mathbf{A}|=|\mathbf{O}'(\mathbf{I}-2s\mathbf{A})\,\mathbf{O}|=|\mathbf{I}-2s\mathbf{D}|=\prod_{i=1}^{n}(1-2s\lambda_i),$$

so that

$$\mathbb{M}_S(s)=\prod_{i=1}^{n}(1-2s\lambda_i)^{-1/2},$$

which is valid for s in a neighborhood of zero. The precise range of s is given below (19.13).

Solution to Problem 10.4: We know the first row of $\mathbf{B}$ has to be given by $\boldsymbol{\mu}\theta^{-1/2}$, where $\theta=\mu_1^2+\mu_2^2+\mu_3^2$ and that the constraints $\mathbf{B}\boldsymbol{\mu}=\left(\theta^{1/2},0,0\right)'$ and $\mathbf{B}'\mathbf{B}=\mathbf{B}\mathbf{B}'=\mathbf{I}_3$ have to be satisfied. Let

$$\mathbf{B}=\theta^{-1/2}\begin{pmatrix}\mu_1&\mu_2&\mu_3\\a&b&c\\d&q&f\end{pmatrix}.$$

Then the constraints are

$$\theta^{-1/2}\begin{pmatrix}\mu_1&\mu_2&\mu_3\\a&b&c\\d&q&f\end{pmatrix}\begin{pmatrix}\mu_1\\\mu_2\\\mu_3\end{pmatrix}=\begin{pmatrix}\theta^{1/2}\\\frac{a\mu_1+b\mu_2+c\mu_3}{\sqrt{\theta}}\\\frac{d\mu_1+q\mu_2+f\mu_3}{\sqrt{\theta}}\end{pmatrix}=\begin{pmatrix}\theta^{1/2}\\0\\0\end{pmatrix},\qquad\text{(S-10.2)}$$

$$\mathbf{I}_3=\mathbf{B}\mathbf{B}'=\theta^{-1}\begin{pmatrix}\mu_1^2+\mu_2^2+\mu_3^2&a\mu_1+b\mu_2+c\mu_3&d\mu_1+q\mu_2+f\mu_3\\a\mu_1+b\mu_2+c\mu_3&a^2+b^2+c^2&ad+bq+cf\\d\mu_1+q\mu_2+f\mu_3&ad+bq+cf&d^2+q^2+f^2\end{pmatrix}$$

and

$$\mathbf{I}_3=\mathbf{B}'\mathbf{B}=\theta^{-1}\begin{pmatrix}\mu_1^2+a^2+d^2&\mu_1\mu_2+ab+dq&\mu_1\mu_3+ac+df\\\mu_1\mu_2+ab+dq&\mu_2^2+b^2+q^2&\mu_2\mu_3+bc+qf\\\mu_1\mu_3+ac+df&\mu_2\mu_3+bc+qf&\mu_3^2+c^2+f^2\end{pmatrix}.$$

We picked the two constraints from (S-10.2) and "some" (at least 4) of the others, say

$$\begin{aligned} a\mu_1 + b\mu_2 + c\mu_3 &= 0 \\ d\mu_1 + q\mu_2 + f\mu_3 &= 0 \\ a^2 + b^2 + c^2 &= \mu_1^2 + \mu_2^2 + \mu_3^2 \\ d^2 + q^2 + f^2 &= \mu_1^2 + \mu_2^2 + \mu_3^2 \\ \mu_1^2 + a^2 + d^2 &= \mu_1^2 + \mu_2^2 + \mu_3^2 \\ \mu_2^2 + b^2 + q^2 &= \mu_1^2 + \mu_2^2 + \mu_3^2 \\ \mu_1\mu_2 + ab + dq &= 0, \end{aligned}$$

and invoked Maple's solve routine. Four solutions are returned, the first of which is

$$a = \frac{-\mu_1\mu_2}{U},\ b = U,\ c = \frac{-\mu_2\mu_3}{U},\ d = \rho,\ q = 0,\ f = -\rho\frac{\mu_1}{\mu_3},$$

where $U = \sqrt{\mu_1^2 + \mu_3^2}$ and ρ is a root of

$$U^2Z^2 + 2Z\mu_1 q\mu_2 + q^2\mu_2^2 + q^2\mu_3^2 - \mu_2^2\mu_3^2 - \mu_3^4 - \mu_1^2\mu_3^2.$$

One of the two roots of the latter equation is (using the fact that $q = 0$),

$$\rho = \frac{1}{U^2}\mu_3\sqrt{2\mu_1^2\mu_3^2 + \mu_2^2\mu_3^2 + \mu_3^4 + \mu_1^4 + \mu_1^2\mu_2^2} = \frac{\mu_3\theta^{1/2}}{U},$$

i.e.,

$$a = \frac{-\mu_1\mu_2}{U},\ b = U,\ c = \frac{-\mu_2\mu_3}{U},\ d = \frac{\mu_3\theta^{1/2}}{U},\ q = 0,\ f = -\frac{\mu_1\theta^{1/2}}{U}.$$

Using these values for **B**, we check (with Maple) that the constraints are indeed satisfied.

Solution to Problem 10.5: First note that the integral expression for ${}_1F_1$ is not valid, as the second argument minus the first yields $-s = -1$. Instead, from the definition, with $\psi = \theta/2$ and $m = n/2$, $\mathbb{E}[X]$ is given by

$$\begin{aligned} &\frac{2}{e^\psi}\frac{\Gamma(m+1)}{\Gamma(m)}\,{}_1F_1(m+1, m; \psi) = \sum_{j=0}^{\infty}\frac{(m+1)^{[j]}}{(mb)^{[j]}}\frac{\psi^j}{j!} \\ =\ & \frac{2m}{e^\psi}\left[1 + \frac{m+1}{m}\psi + \frac{(m+1)(m+2)}{m(m+1)}\frac{\psi^2}{2!} + \frac{(m+1)(m+2)(m+3)}{m(m+1)(m+2)}\frac{\psi^3}{3!} + \cdots\right] \\ =\ & \frac{2m}{e^\psi}\left[1 + \frac{m+1}{m}\psi + \frac{(m+2)}{m}\frac{\psi^2}{2!} + \frac{(m+3)}{m}\frac{\psi^3}{3!} + \cdots\right] \\ =\ & \frac{2m}{e^\psi}\left[\frac{1}{m}\sum_{j=0}^{\infty}(m+j)\frac{\psi^j}{j!} = 1 + \sum_{j=1}^{\infty}\frac{\psi^j}{j!} + \frac{1}{m}\sum_{j=1}^{\infty}\frac{\psi^j}{(j-1)!}\right] \\ =\ & \frac{2m}{e^\psi}\left[e^\psi + \frac{\psi}{m}\sum_{i=0}^{\infty}\frac{\psi^i}{i!}\right] = \frac{2m}{e^\psi}\left[e^\psi + \frac{\psi}{m}e^\psi\right] = n\left[1 + \frac{\theta}{n}\right] \\ =\ & n + \theta. \end{aligned}$$

Solution to Problem 10.6: Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$. The mgf of X is

$$\begin{aligned}
\mathbb{M}_X(t) &= \mathbb{E}\left[e^{t(\mathbf{X}'\mathbf{X})}\right] \\
&= (2\pi)^{-n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{t\left(\mathbf{x}'\mathbf{x}\right) - \frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu}\right)'\left(\mathbf{x}-\boldsymbol{\mu}\right)\right\} d\mathbf{x} \\
&= (2\pi)^{-n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left[(1-2t)\left(\mathbf{x}'\mathbf{x}\right) - 2\mathbf{x}'\boldsymbol{\mu} + \boldsymbol{\mu}'\boldsymbol{\mu}\right]\right\} d\mathbf{x},
\end{aligned}$$

which can be solved by using (19.78) with

$$\mathbf{A} = \mathbf{0}, \quad \mathbf{a} = \mathbf{0}, \quad a = (2\pi)^{-n/2},$$

and

$$\mathbf{B} = (1-2t)/2\,\mathbf{I}_n, \quad \mathbf{b} = -\boldsymbol{\mu}, \quad b = \boldsymbol{\mu}'\boldsymbol{\mu}/2,$$

and $\mathbf{B}$ is positive definite for $t < 1/2$. Thus, substituting in (19.78) and simplifying yields

$$\begin{aligned}
\mathbb{M}_X(t) &= 2^{-n/2}\left|(1-2t)/2\,\mathbf{I}_n\right|^{-1/2} \exp\left\{\frac{1}{4}\left(\boldsymbol{\mu}'\mathbf{B}^{-1}\boldsymbol{\mu}\right) - \boldsymbol{\mu}'\boldsymbol{\mu}/2\right\} \\
&= (1-2t)^{-n/2} \exp\left\{\frac{t\theta}{1-2t}\right\}
\end{aligned}$$

for $\theta = \boldsymbol{\mu}'\boldsymbol{\mu}$ and $t < 1/2$.

For r.v. Y with density (19.3) and taking $u = y(1-2t)/2$,

$$\begin{aligned}
\mathbb{M}_Y(t) &= \mathbb{E}\left[e^{tY}\right] = \int_0^{\infty} e^{ty}\, e^{-y/2} \sum_{i=0}^{\infty} \frac{e^{-\theta/2}(\theta/2)^i}{i!} \frac{y^{n/2+i-1}}{2^{n/2+i}\Gamma(n/2+i)} dy \\
&= \sum_{i=0}^{\infty} \frac{e^{-\theta/2}(\theta/2)^i}{i!} \frac{1}{2^{n/2+i}\Gamma(n/2+i)} \int_0^{\infty} y^{n/2+i-1} \exp\left\{-y(1-2t)/2\right\} dy \\
&= \sum_{i=0}^{\infty} \frac{e^{-\theta/2}(\theta/2)^i}{i!} (1-2t)^{-(n/2+i)}.
\end{aligned}$$

Multiplying and dividing by $\exp\left((\theta/2)/(1-2t)\right)$, using the fact from the Poisson distribution that $1 = \sum_{i=0}^{\infty} e^{-\lambda}\lambda^i/i!$ for $\lambda > 0$, and the fact that

$$-\frac{\theta}{2} + \frac{\theta/2}{1-2t} = \frac{t\theta}{1-2t},$$

we have

$$\begin{aligned}
\mathbb{M}_Y(t) &= (1-2t)^{-n/2} e^{-\theta/2} e^{\frac{\theta/2}{1-2t}} \sum_{i=0}^{\infty} \frac{e^{-\frac{\theta/2}{1-2t}}\left(\frac{\theta/2}{1-2t}\right)^i}{i!} \\
&= (1-2t)^{-n/2} e^{-\theta/2} e^{\frac{\theta/2}{1-2t}} \\
&= (1-2t)^{-n/2} \exp\left\{\frac{t\theta}{1-2t}\right\},
\end{aligned}$$

which equals $\mathbb{M}_X(t)$.

Solution to Problem 10.7: Let

$$S=\frac{\sigma_2^2}{\sigma_1^2}R=\left(\frac{X_1/\sigma_1}{X_2/\sigma_2}\right)^2,$$

so that $S\sim \mathrm{F}\left(1,1,\theta_1,\theta_2\right)$, $\theta_i=\mu_i/\sigma_i$, with density

$$f_S\left(x\right)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-\theta_1/2}\left(\theta_1/2\right)^i}{i!}\frac{e^{-\theta_2/2}\left(\theta_2/2\right)^j}{j!}\frac{x^{i-1/2}}{\left(1+x\right)^{1+i+j}}\frac{\Gamma\left(i+j+1\right)}{\Gamma\left(i+1/2\right)\Gamma\left(j+1/2\right)},$$

from which that for R, being just a scale transform, can be easily given.

With $\sigma_1^2=\sigma_2^2=1$ and $\mu_2=0$,

$$\begin{aligned}f_R\left(x\right)&=\frac{1}{\sqrt{\pi}}\sum_{i=0}^{\infty}\frac{e^{-\theta_1/2}\left(\theta_1/2\right)^i}{i!}\frac{x^{i-1/2}}{\left(1+x\right)^{1+i}}\frac{\Gamma\left(i+1\right)}{\Gamma\left(i+1/2\right)}\\&=\frac{e^{-\mu_1/2}}{\sqrt{\pi}}\frac{x^{-1/2}}{1+x}\sum_{i=0}^{\infty}\left(\frac{\mu_1}{2}\frac{x}{1+x}\right)^i\frac{1}{\Gamma\left(i+1/2\right)}.\end{aligned}$$

Using the second identity in (19.2),

$$\begin{aligned}f_R\left(x\right)&=\frac{e^{-\mu_1/2}}{\pi}\frac{x^{-1/2}}{1+x}\sum_{i=0}^{\infty}\left(\frac{\mu_1}{2}\frac{x}{1+x}\right)^i\frac{2^i}{1\cdot3\cdot5\cdot\cdots\cdot\left(2i-1\right)}\\&=\frac{e^{-\mu_1/2}}{\pi}\frac{x^{-1/2}}{1+x}\sum_{i=0}^{\infty}\frac{2^i}{1\cdot3\cdot5\cdot\cdots\cdot\left(2i-1\right)}\left(\frac{\mu_1}{2}\frac{x}{1+x}\right)^i\\&=\frac{e^{-\mu_1/2}}{\pi}\frac{x^{-1/2}}{1+x}\sum_{i=0}^{\infty}\frac{i!}{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\cdots\left(\frac{1}{2}+i-1\right)}\frac{1}{i!}\left(\frac{\mu_1}{2}\frac{x}{1+x}\right)^i\\&=\frac{e^{-\mu_1/2}}{\pi}\frac{1}{\sqrt{x}\left(1+x\right)}\,{}_1F_1\left(1,\frac{1}{2},\frac{\mu_1}{2}\frac{x}{1+x}\right).\end{aligned}$$

Solution to Problem 10.8: Let $G=X_1/X_2$. Using the rhs of (11.10) and substituting $u=y\left(r+1\right)/2$,

$$f_G\left(r\right)=\int_{-\infty}^{\infty}\left|y\right|f_{X_1,X_2}\left(ry,y\right)dy$$

is given by

$$\begin{aligned}&\int_0^{\infty}y\sum_{i=0}^{\infty}\omega_{i,\theta_1}g_{n_1+2i}\left(ry\right)\sum_{j=0}^{\infty}\omega_{j,\theta_2}g_{n_2+2j}\left(y\right)dy\\&=\int_0^{\infty}y\sum_{i=0}^{\infty}\frac{e^{-\theta_1/2}\left(\theta_1/2\right)^i}{i!}\frac{e^{-ry/2}\left(ry\right)^{n_1/2+i-1}}{2^{n_1/2+i}\Gamma\left(i+n_1/2\right)}\sum_{j=0}^{\infty}\frac{e^{-\theta_2/2}\left(\theta_2/2\right)^j}{j!}\frac{e^{-y/2}y^{n_2/2+j-1}}{2^{n_2/2+j}\Gamma\left(j+n_2/2\right)}dy\\&=\sum_{i=0}^{\infty}\frac{e^{-\theta_1/2}\left(\theta_1/2\right)^i}{i!}\sum_{j=0}^{\infty}\frac{e^{-\theta_2/2}\left(\theta_2/2\right)^j}{j!}r^{n_1/2+i-1}\\&\quad\times\frac{2^{-\left(n_1+n_2\right)/2-i-j}}{\Gamma\left(i+n_1/2\right)\Gamma\left(j+n_2/2\right)}\int_0^{\infty}e^{-y\left(r+1\right)/2}y^{\left(n_1+n_2\right)/2+i+j-1}dy\end{aligned}$$

or

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-\theta_1/2}\left(\theta_1/2\right)^i}{i!}\frac{e^{-\theta_2/2}\left(\theta_2/2\right)^j}{j!}\frac{r^{n_1/2+i-1}}{B\left(i+n_1/2,j+n_2/2\right)}\frac{1}{\left(r+1\right)^{(n_1+n_2)/2+i+j}}.$$

Then, with $F=\left(X_1/n_1\right)/\left(X_2/n_2\right)=\left(n_2/n_1\right)G$, $f_F\left(x\right)=f_G\left(r\right)dr/dx$ is

$$\begin{aligned}
&\frac{n_1}{n_2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-\theta_1/2}\left(\theta_1/2\right)^i}{i!}\frac{e^{-\theta_2/2}\left(\theta_2/2\right)^j}{j!}\frac{\left(x\frac{n_1}{n_2}\right)^{n_1/2+i-1}}{B\left(i+n_1/2,j+n_2/2\right)}\frac{1}{\left(x\frac{n_1}{n_2}+1\right)^{(n_1+n_2)/2+i+j}}\\
=\;&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-\theta_1/2}\left(\theta_1/2\right)^i}{i!}\frac{e^{-\theta_2/2}\left(\theta_2/2\right)^j}{j!}\frac{n_1^{n_1/2+i}}{n_2^{-n_2/2-j}}\frac{x^{n_1/2+i-1}\left(xn_1+n_2\right)^{-(n_1+n_2)/2-i-j}}{B\left(i+n_1/2,j+n_2/2\right)}.
\end{aligned}$$

For the cdf, with substitution

$$z=\frac{t\frac{n_1}{n_2}}{t\frac{n_1}{n_2}+1}=\frac{n_1t}{n_1t+n_2},\quad t\frac{n_1}{n_2}=\frac{z}{1-z},\quad dt=\frac{n_2}{n_1}\left(1-z\right)^{-2}dz,$$

it follows that

$$\begin{aligned}
F_F\left(x\right)&=\frac{n_1}{n_2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\omega_{i,\theta_1}\omega_{j,\theta_2}}{B\left(i+n_1/2,j+n_2/2\right)}\int_0^x\frac{\left(t\frac{n_1}{n_2}\right)^{n_1/2+i-1}}{\left(\left(t\frac{n_1}{n_2}\right)+1\right)^{(n_1+n_2)/2+i+j}}dt\\
&=\frac{n_1}{n_2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\omega_{i,\theta_1}\omega_{j,\theta_2}}{B\left(i+n_1/2,j+n_2/2\right)}\int_0^x z^{i+n_1/2}\left(1-z\right)^{j+n_2/2}dz\\
&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{i,\theta_1}\omega_{j,\theta_2}\frac{B_y\left(i+n_1/2,\ j+n_2/2\right)}{B\left(i+n_1/2,\ j+n_2/2\right)},\qquad y=\frac{n_1x}{n_1x+n_2},
\end{aligned}$$

where $\omega_{i,\theta}=e^{-\theta/2}\left(\theta/2\right)^i\,/\,i!$.

Solution to Problem 10.9: Let $W=\sqrt{Y}$, i.e., in terms of $Z=\sqrt{Y/k}$, $W=k^{1/2}Z$, with density

$$f_W\left(w\right)=f_Z\left(z\right)\left|dz/dw\right|=\frac{2^{-k/2+1}}{\Gamma\left(k/2\right)}w^{k-1}\exp\left(-\frac{1}{2}w^2\right)\mathbb{I}_{(0,\infty)}\left(w\right),$$

so that, from (8.41),

$$\begin{aligned}
F_T\left(t\right)&=\Pr\left(k^{1/2}X\,/\,W\leq t\right)\\
&=\Pr\left(X\leq tk^{-1/2}W\right)=\int_0^{\infty}F_X\left(tk^{-1/2}z\right)f_W\left(w\right)dw\\
&=\frac{2^{-k/2+1}}{\Gamma\left(k/2\right)}\int_0^{\infty}w^{k-1}e^{-w^2/2}\,\Phi\left(tk^{-1/2}z;\mu,1\right)dw.
\end{aligned}$$

As

$$\begin{aligned}\Phi(q;\mu,1) &= (2\pi)^{-1/2}\int_{-\infty}^{q}\exp\left\{-\frac{1}{2}(x-\mu)^2\right\}dx\\ &= (2\pi)^{-1/2}\int_{-\infty}^{q-\mu}\exp\left\{-\frac{1}{2}p^2\right\}dp,\end{aligned}$$

we have

$$F_T(t)=\frac{2^{-k/2+1}}{\Gamma(k/2)}\int_0^{\infty}w^{k-1}e^{-w^2/2}\,\Phi\left(tk^{-1/2}z-\mu\right)dw,$$

where $\Phi(q)=\Phi(q;0,1)$ is the standard normal cdf.

Solution to Problem 10.10: As $z>0$, let $u=(t^2+k)\,z^2/2$ (and $z=\left(2u/\left(t^2+k\right)\right)^{1/2}$ and $dz=2^{-1/2}u^{-1/2}\left(t^2+k\right)^{-1/2}du$) so that

$$\begin{aligned}&K\int_0^{\infty}z^{k+i}\exp\left\{-\frac{1}{2}\left[\left(t^2+k\right)z^2\right]\right\}dz\\ &= 2^{\frac{k+i-1}{2}}\left(\frac{1}{t^2+k}\right)^{\frac{k+i+1}{2}}K\int_0^{\infty}u^{\frac{k+i-1}{2}}\exp\{-u\}\,du\\ &= \frac{2^{\frac{i+1}{2}}k^{k/2}}{\Gamma(k/2)\sqrt{2\pi}}\left(\frac{1}{t^2+k}\right)^{\frac{k+i+1}{2}}\Gamma\left(\frac{k+i+1}{2}\right).\end{aligned}$$

This yields

$$f_T(t)=e^{-\mu^2/2}\sum_{i=0}^{\infty}\frac{(t\mu)^i}{i!}\frac{2^{\frac{i+1}{2}}k^{k/2}}{\Gamma(k/2)\sqrt{2\pi}}\left(\frac{1}{\sqrt{t^2+k}}\right)^{k+i+1}\Gamma\left(\frac{k+i+1}{2}\right),$$

which, in turn, is equivalent to (19.58).

Solution to Problem 10.11: As x is positive, take

$$u=\frac{x^2}{k+x^2},\quad x=(1-u)^{-1/2}u^{1/2}k^{1/2},\quad dx=\frac{k^{1/2}}{2}(1-u)^{-3/2}u^{-1/2}du$$

to get

$$\begin{aligned}&\int_0^t x^i\left(k+x^2\right)^{-\frac{k+1+i}{2}}dx\\ &= \int_0^{\frac{t^2}{k+t^2}}\left((1-u)^{-1/2}u^{1/2}k^{1/2}\right)^i\left(\frac{k}{1-u}\right)^{-\frac{k+1+i}{2}}\frac{k^{1/2}}{2}(1-u)^{-3/2}u^{-1/2}du\\ &= \frac{1}{2}k^{-\frac{k+2+2i}{2}}\int_0^{\frac{t^2}{k+t^2}}u^{(i-1)/2}(1-u)^{\frac{k-2}{2}}du=\frac{1}{2}k^{-k/2}B_m\left(\frac{1}{2}(1+i),\frac{k}{2}\right),\end{aligned}$$

i.e., with $m=t^2/\left(k+t^2\right)$,

$$\Pr(0\le T\le t)=\frac{1}{2}e^{-\mu^2/2}\sum_{i=0}^{\infty}\frac{\Gamma((k+i+1)/2)}{\sqrt{\pi}\,\Gamma(k/2)\,i!}2^{i/2}\mu^i\,B_m\left(\frac{1}{2}(1+i),\frac{k}{2}\right).$$

This is easily seen to be equivalent to (19.60).

For i even, the equality of (19.60) and (19.61) follows from directly from (19.2), i.e.,

$$\frac{2^{i/2}\Gamma\left(\frac{1}{2}(1+i)\right)}{\sqrt{\pi}i!} = \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (i-1)}{i!} = \frac{2^{-i/2}}{(i/2)!} = \frac{(1/2)^{i/2}}{\Gamma(i/2+1)}. \tag{S-10.3}$$

For i odd, note that the second formula in (19.2) reads, for $j = 2i-1$,

$$\Gamma\left(\frac{j+1}{2}+1/2\right) = \frac{1\cdot 3\cdot 5\cdot \cdots \cdot j}{2^{(j+1)/2}}\sqrt{\pi} \quad \text{or} \quad \frac{2^{-(j+1)/2}}{\Gamma(j/2+1)} = \frac{1}{(1\cdot 3\cdot 5\cdot \cdots \cdot j)\sqrt{\pi}}.$$

Thus,

$$\begin{aligned}
\frac{2^{i/2}\Gamma\left(\frac{1}{2}(1+i)\right)}{\sqrt{\pi}i!} &= \frac{\sqrt{2}}{\sqrt{\pi}}\frac{2^{(i-1)/2}\overbrace{\left(\frac{1}{2}(1+i)-1\right)\left(\frac{1}{2}(1+i)-2\right)\cdots(1)}^{(i-1)/2 \text{ terms}}}{i!} \\
&= \frac{\sqrt{2}}{\sqrt{\pi}}\frac{(i-1)(i-3)\cdots(2)}{i!} = \frac{\sqrt{2}}{\sqrt{\pi}}\frac{1}{1\cdot 3\cdot 5\cdot \cdots \cdot i} \\
&= \sqrt{2}\frac{2^{-(i+1)/2}}{\Gamma(i/2+1)} = \frac{(1/2)^{i/2}}{\Gamma(i/2+1)}.
\end{aligned}$$

Solution to Problem 10.12: To derive the density of $R = X/W$, use (19.56) to get

$$\begin{aligned}
f_R(r) &= \int_{-\infty}^{\infty}|w|\, f_{X,W}(rw,w)\,dw = \int_0^{\infty} w f_X(rw)\, f_W(w)\,dw \\
&= \frac{e^{-\theta/2}}{\sqrt{2\pi}}\int_0^{\infty}\exp\left\{-\frac{1}{2}\left[(rw-\mu)^2+w^2\right]\right\}\sum_{i=0}^{\infty}\frac{(\theta/2)^i}{i!}\frac{w^{k+2i}}{2^{k/2+i-1}\Gamma(i+k/2)}dw \\
&= \frac{e^{-\theta/2}}{\sqrt{2\pi}}\int_0^{\infty}\exp\left\{-\frac{1}{2}\left(r^2+1\right)w^2\right\}\exp\left\{-\mu^2/2\right\} \\
&\quad \times\left(\sum_{j=0}^{\infty}\frac{(wr\mu)^j}{j!}\right)\sum_{i=0}^{\infty}\frac{(\theta/2)^i}{i!}\frac{w^{k+2i}}{2^{k/2+i-1}\Gamma(i+k/2)}dw
\end{aligned}$$

or

$$\begin{aligned}
f_R(r) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-\theta/2}}{\sqrt{2\pi}}\exp\left\{-\mu^2/2\right\}(r\mu)^j\frac{(\theta/2)^i}{i!j!}\frac{1}{2^{k/2+i-1}\Gamma(i+k/2)} \\
&\quad \times\int_0^{\infty}\exp\left\{-\frac{1}{2}\left(r^2+1\right)w^2\right\}w^{k+2i+j}dw.
\end{aligned}$$

Then, as in Problem 19.10, as $w > 0$, let

$$u = \frac{(r^2+1)w^2}{2}, \quad w = \left(\frac{2u}{r^2+1}\right)^{1/2}, \quad dw = 2^{-1/2}\left(r^2+1\right)^{-1/2}u^{-1/2}du$$

so that

$$
\begin{aligned}
&= \int_0^\infty \exp\left\{-\frac{1}{2}\left(r^2+1\right)w^2\right\} w^{k+2i+j}dw \\
&= \left(\frac{2}{r^2+1}\right)^{(k+2i+j)/2} \int_0^\infty \exp\{-u\}\, u^{(k+2i+j)/2} 2^{-1/2}\left(r^2+1\right)^{-1/2} u^{-1/2}du \\
&= \left(r^2+1\right)^{-(k+2i+j+1)/2} 2^{(k+2i+j-1)/2}\Gamma\left(\left(k+2i+j+1\right)/2\right),
\end{aligned}
$$

or

$$
f_R(r) = \frac{e^{-\left(\theta+\mu^2\right)/2}}{\sqrt{\pi}} \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(\theta/2)^i}{i!j!} \frac{\Gamma\left(\left(k+2i+j+1\right)/2\right)}{\Gamma\left(i+k/2\right)} \left(r\mu\sqrt{2}\right)^j \left(1+r^2\right)^{-(k+2i+j+1)/2}.
$$

Finally, the density of $T = \sqrt{k}R$ involves a straightforward scale transformation yielding

$$
\begin{aligned}
f_T(t) &= f_R\left(tk^{-1/2}\right)k^{-1/2} \\
&= \frac{e^{-\left(\theta+\mu^2\right)/2}}{\sqrt{\pi k}} \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(\theta/2)^i}{i!j!} \frac{\Gamma\left(\left(k+2i+j+1\right)/2\right)}{\Gamma\left(i+k/2\right)} \frac{\left(t\mu\sqrt{2/k}\right)^j}{\left(1+t^2/k\right)^{(k+2i+j+1)/2}}.
\end{aligned}
$$

For (19.66), $f_T(t)$ is

$$
\begin{aligned}
& \frac{e^{-\left(\theta+\mu^2\right)/2}}{\sqrt{\pi k}} \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(\theta/2)^i}{i!j!} \frac{\Gamma\left(\left(k+2i+j+1\right)/2\right)}{\Gamma\left(i+k/2\right)} \left(t\mu\sqrt{2/k}\right)^j \left(1+t^2/k\right)^{-(k+2i+j+1)/2} \\
&= \frac{e^{-\left(\theta+\mu^2\right)/2}}{\sqrt{\pi k}} \sum_{j=0}^\infty \frac{1}{j!} \frac{\left(t\mu\sqrt{2/k}\right)^j}{\left(1+t^2/k\right)^{(k+j+1)/2}} \sum_{i=0}^\infty \frac{\Gamma\left(\left(k+2i+j+1\right)/2\right)}{\Gamma\left(i+k/2\right)} \frac{(\theta/2)^i}{i!} \left(1+t^2/k\right)^{-i} \\
&= \frac{e^{-\left(\theta+\mu^2\right)/2}}{\sqrt{\pi k}} \sum_{j=0}^\infty \frac{1}{j!} \frac{\left(t\mu\sqrt{2/k}\right)^j}{\left(1+t^2/k\right)^{(k+j+1)/2}} \frac{\Gamma\left(\frac{k+j+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} {}_1F_1\left(\frac{k+j+1}{2}, \frac{k}{2}, \frac{\theta}{2\left(1+t^2/k\right)}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
{}_1F_1\left(\frac{k+j+1}{2}, \frac{k}{2}, \frac{\theta}{2\left(1+t^2/k\right)}\right) &= \sum_{i=0}^\infty \frac{\left(\frac{k+j+1}{2}\right)^{[i]}}{\left(\frac{k}{2}\right)^{[i]}} \frac{1}{i!} \left(\frac{\theta}{2\left(1+t^2/k\right)}\right)^i \\
&= \sum_{i=0}^\infty \frac{\Gamma\left(\frac{k+j+1+2i}{2}\right)\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k+j+1}{2}\right)\Gamma\left(\frac{k+2i}{2}\right)} \frac{(\theta/2)^i}{i!} \left(1+t^2/k\right)^{-i},
\end{aligned}
$$

using the fact that $a^{[m]} = \Gamma(a+m)/\Gamma(a)$.

Solution to Problem 10.13: The density of $Y \sim \chi^2(k,\theta)$ is $f_Y(y) \sum_{i=0}^\infty \omega_{i,\theta}\, g_{k+2i}(y)$

so that, with $Z = \sqrt{Y/k}$,

$$\begin{aligned}
f_Z(z) &= 2zkf_Y\left(z^2k\right) = 2zk\sum_{i=0}^{\infty}\omega_{i,\theta}\, g_{k+2i}\left(z^2k\right) \\
&= 2zk\sum_{i=0}^{\infty}\frac{2^{-(k+2i)/2}}{\Gamma\left((k+2i)/2\right)}\left(z^2k\right)^{(k+2i)/2-1} e^{-\left(z^2k\right)/2} \\
&= e^{-\left(z^2k\right)/2}\sum_{i=0}^{\infty}\frac{2^{1-(k+2i)/2}k^{(k+2i)/2}}{\Gamma\left((k+2i)/2\right)} z^{(k+2i)-1}
\end{aligned}$$

and, with $y = z^2k$, $z = k^{-1/2}y^{1/2}$, $dz = k^{-1/2}\left(1/2\right)y^{-1/2}dy$,

$$\begin{aligned}
F_T(t;\mu,k,\theta) &= \Pr(T \le t) \\
&= \Pr\left(X \le t\sqrt{Y/k}\right) = \Pr(X \le tZ) = \int_0^{\infty} F_X(tz)\, f_Z(z)\, dz \\
&= \sum_{i=0}^{\infty}\omega_{i,\theta}\int_0^{\infty}\Phi(tz-\mu)\frac{2^{1-(k+2i)/2}k^{(k+2i)/2}}{\Gamma\left((k+2i)/2\right)} z^{(k+2i)-1}e^{-\left(z^2k\right)/2}dz \\
&= \sum_{i=0}^{\infty}\omega_{i,\theta}\int_0^{\infty}\Phi\left(t\left(\frac{y}{k}\right)^{1/2}-\mu\right)\frac{2^{-(k+2i)/2}}{\Gamma\left((k+2i)/2\right)} y^{(k+2i)/2-1}e^{-y/2}dy \\
&= \sum_{i=0}^{\infty}\omega_{i,\theta}\int_0^{\infty}\Phi\left(t\left(\frac{y}{k}\right)^{1/2}-\mu\right) g_{k+2i}(y)\, dy.
\end{aligned}$$

For $\theta = 0$, this reduces to

$$F_T(t;\mu,k,0) = \int_0^{\infty}\Phi\left(t\left(\frac{y}{k}\right)^{1/2}-\mu\right) g_k(y)\, dy.$$

Thus,

$$\begin{aligned}
F_T(t;\mu,k,\theta) &= \sum_{i=0}^{\infty}\omega_{i,\theta}\int_0^{\infty}\Phi\left(t\left(\frac{y}{k}\right)^{1/2}-\mu\right) g_{k+2i}(y)\, dy \\
&= \sum_{i=0}^{\infty}\omega_{i,\theta}\int_0^{\infty}\Phi\left(t\left(\frac{k+2i}{k}\right)^{1/2}\left(\frac{y}{k+2i}\right)^{1/2}-\mu\right) g_{k+2i}(y)\, dy \\
&= \sum_{i=0}^{\infty}\omega_{i,\theta}F_T\left(t\left(\frac{k+2i}{k}\right)^{1/2}; \mu, k+2i, 0\right).
\end{aligned}$$

Solution to Problem 10.14:

a) As in (10.44),

$$\begin{aligned}
\mathbb{M}_X(\bar{s}) &= \mathbb{E}\left[e^{\bar{s}X}\right] = \mathbb{E}\left[e^{aX-ibX}\right] = \mathbb{E}\left[e^{aX}e^{-ibX}\right] \\
&= \mathbb{E}\left[e^{aX}\left(\cos(-bX) + i\sin(-bX)\right)\right] \\
&= \mathbb{E}\left[e^{aX}\left(\cos(bX) - i\sin(bX)\right)\right] = \bar{\mathbb{M}}_X(s)
\end{aligned}$$

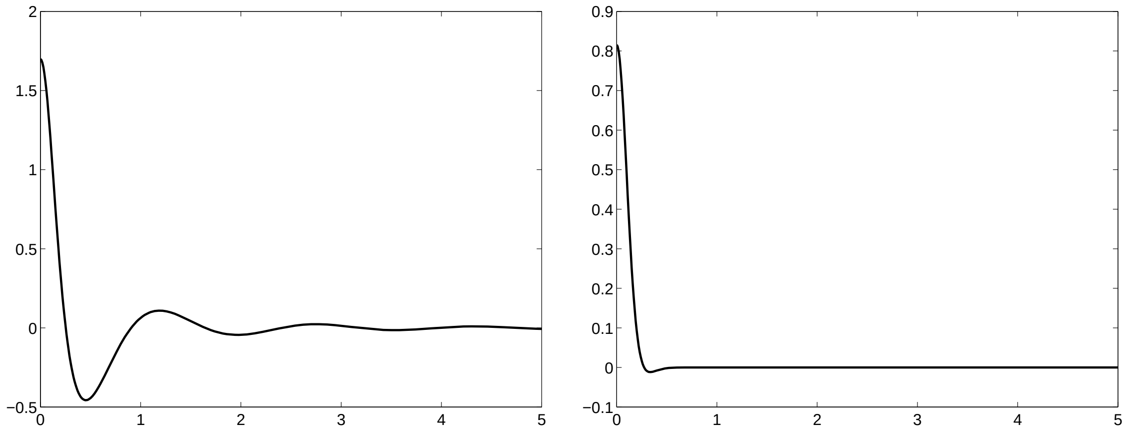

Figure S-10.1: The integrands in (19.80) and (19.84). For the left panel, use call Program S-10.1 with `helstrom(4,[1 -1],[1 1],[0 0],1,0.25,0,5)`. For the right panel, use `helstrom(4,[1 -1],[1 1],[0 0],1)`.

for all values of s such that the expectation exists. Thus,

$$\begin{aligned}\exp\{I(\bar{s})\} &= \frac{\mathbb{M}_X(\bar{s})\exp\{-x\bar{s}\}}{\operatorname{sgn}(c)\bar{s}} = \frac{\bar{\mathbb{M}}_X(s)\overline{\exp\{-xs\}}}{\operatorname{sgn}(c)\bar{s}} = \frac{\overline{\mathbb{M}_X(s)\exp\{-xs\}}}{\operatorname{sgn}(c)\bar{s}} \\ &= \overline{\exp\{I(s)\}},\end{aligned}$$

as $e^{z} = \overline{e^{\bar{z}}}$ and $\bar{z}_1\bar{z}_2 = \overline{z_1 z_2}$.

b) Note that with $C = 0$ and $s_0 = c$, the integral in (19.80) is the same as the one in (19.84), so that program S-10.1 can be used to produce the plot in the left panel of Figure S-10.1.

c) The cgf of Q is

$$\mathbb{K}_Q(s) = \ln \mathbb{M}_Q(s) = -\frac{1}{2}\sum_j \nu_j \ln(1-2\lambda_j s) + \sum_j \frac{\lambda_j\theta_j s}{1-2\lambda_j s}.$$

The saddlepoint equation (19.82) becomes

$$\sum_j \frac{\lambda_j\nu_j}{1-2\lambda_j s_0} + \sum_j \frac{\lambda_j\theta_j}{(1-2\lambda_j s_0)^2} - x - \frac{1}{s_0} = 0,$$

and the curvature of the parabola (19.83) is $C = \dfrac{I^{(3)}(x_0)}{3I^{(2)}(s_0)}$, where

$$\begin{aligned}I^{(2)}(s_0) =& 2\sum_j \frac{\lambda_j^2\nu_j}{(1-2\lambda_j s_0)^2} + 4\sum_j \frac{\lambda_j^2\theta_j}{(1-2\lambda_j s_0)^3} + \frac{1}{s_0^2}, \text{ and} \\ I^{(3)}(s_0) =& 8\sum_j \frac{\lambda_j^3\nu_j}{(1-2\lambda_j s_0)^3} + 24\sum_j \frac{\lambda_j^3\theta_j}{(1-2\lambda_j s_0)^4} - \frac{2}{s_0^3}.\end{aligned}$$

The singularities of the integrand lie on the real axis at $1/(2\lambda_j)$, $j \in 1, 2, \ldots, n$. Therefore, when the weights (and, thus, the singularities) are all positive, the search for the saddlepoint has to be conducted between the left-most singularity and the origin, and between the right-most one and the origin otherwise. When there are both positive and negative weights, there is a saddlepoint on either side of the origin. In that case, use $s_0 > 0$ if $q > \mathbb{E}[Q]$, and $s_0 < 0$ otherwise. The Matlab code used to produce the right panel of Figure S-10.1 is given in Listing S-10.1. Note that the integrand dies off much more quickly, and is less oscillatory.

```
function F = helstrom(gvec,a,eta,theta,doplot,x0,K,upper)
% cdf of sum_j a_j X_j, X_j~Chi^2(eta,theta), at the values in gvec
% if x0 and K are not passed, the Helstrom (1996) values are used
% upper is the limit of integration
if nargin<5, doplot=0;  end
if nargin<3, eta=a*0+1; end
if nargin<4, theta=a*0; end tol=1e-8;
Ff=zeros(size(gvec));
expect=sum(a.*(eta+theta));
for loop=1:length(gvec)
  g=gvec(loop);
  if nargin<7,
    if min(a)<0 & max(a) >0
      if g>=expect
          left=1e-15;right=(1/(2*max(a)))-1e-15;
      else
          right=-1e-15;left=(1/(2*min(a)))+1e-15;
      end;
    else
      if min(a)<0
          right=-1e-15;left=(1/(2*min(a)))+1e-15;
      else
          left=1e-15;right=(1/(2*max(a)))-1e-15;
      end;
    end;
    x0=fzero(@helspe,[left,right],optimset,g,a,eta,theta);
    phipppx0=8*sum((eta.*a.^3)./((1-2*x0*a).^3))...
        +24*sum(((a.^3).*theta)./(1-2*a*x0).^4)-2*x0^-3;
    phippx0=2*sum((eta.*a.^2)./((1-2*x0*a).^2))...
        +4*sum(((a.^2).*theta)./(1-2*a*x0).^3)+x0^-2;
    K=(phipppx0/(3*phippx0));
  end
```

Program Listing S-10.1: Program to evaluate (19.84). Continued in Listing S-10.2

```
  if nargin<8
    upper=tol;
    while abs(helintegrand(upper,g,a,eta,theta,x0,K))>1e-7,
      upper=upper*2;
    end
    upper=upper*2;
  end
  F(loop) = (x0>0) - (1/pi) * ...
       quadl(@helintegrand,0,upper,tol,[],g,a,eta,theta,x0,K);
end; if doplot
  fplot(@helintegrand,[0 upper],[],[],[],g,a,eta,theta,x0,K);
end


function I = helintegrand(yvec,g,a,eta,theta,x0,K);
% Called by function helstrom.
I=zeros(size(yvec)); for loop=1:length(yvec)
  y=yvec(loop);  z=x0+0.5*K*y^2+i*y;
  phiz=-0.5*sum(eta.*log(1-2*z*a))+ ...
      z*sum((a.*theta)./(1-2*z*a))-g*z-log(z);
  I(loop)=real(exp(phiz)*(1-i*K*y));
end;

function x0 = helspe(zvec,g,a,eta,theta)
% saddlepoint equation; called by function helstrom.
x0=zeros(size(zvec)); for zloop=1:length(zvec)
  z=zvec(zloop);
  x0(zloop) = sum((eta.*a)./(1-2*z*a)) ...
            + sum((a.*theta)./(1-2*a*z).^2)-g-z^-1;
end;
```

Program Listing S-10.2: Continuation of Listing S-10.1