# Advanced Calculus and Real Analysis

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> I advise my students to listen carefully the moment they decide to take no more mathematics courses. They might be able to hear the sound of closing doors.
> (James Caballero)

> One difficult decision relates to how much of an effort one should make to acquire basic technique, for example in mathematics and probability theory. One does not wish to be always training to run the race but never running it; however, we do need to train. (E. J. Hannan, 1992)

The beginning masters students at UZH majoring in Business, Data Science, Economics, or Finance, have had a basic course in univariate calculus, and this, during their bachelor studies. Having taught my beginning master's level class in probability theory for 20 years in a row, I know very well the average student level of understanding in calculus, linear algebra, and basic mathematics. It is not very high. If the student's goal is to get more involved in advanced (micro- or macro-) economics, econometrics, quantitative risk management, asset pricing, probability theory, higher level computational-based statistical methods and machine learning, hardcore mathematical finance and financial engineering, etc., then he or she will need to have a much stronger level of mathematics than the typical rudimentary level. Filling this gap is the purpose of this course.

The term "advanced calculus" is often used synonymously with a course in real analysis that focuses on the multivariate case and chronologically follows, obviously, a first course in real analysis. In our course, we will in fact also cover the univariate case, but with more emphasis on "computable, tangible things". The next goal is to cover the important concepts of series of numbers, and, even more relevantly, series of functions, reaching the immensely important topic of Taylor series, which we do in both the univariate and multivariate case. One of the goals of the course is to offer much practice by way of a large number of worked examples, such as "trickier" univariate Riemann integrals. Then, in §3, we turn to a chosen set of topics commonly covered in a course in real analysis, and which are, to some extent, more abstract in nature, notably the study of compactness. This, and the other topics covered there, will be essential when subsequently studying measure theory and the Lebesgue integral.

Then we investigate a select set of topics associated with multivariate calculus, notably vectors and linear algebra, a deep dive into determinants, a detailed presentation of projection and least squares; and then the crucial topics of (partial and total) differentiation, and multivariate Riemann integration (importantly, Fubini's theorem, exchange of derivative and integral, and Leibnitz' rule).

Differing from a typical calculus or advanced calculus course, we will start in the first chapter with material that is highly relevant in general, and notably so with probability theory, namely some more sophisticated combinatorics, generalized binomial theorems, gamma and beta functions, and numerous non-trivial examples invoking this material. This material is leveraged in §2.6.6 to cover Wallis' product and Stirling's approximation. Appendix 7.2 contains material on the so-called digamma and polygamma functions. A further nod to probability theory is showing several ways of computing the univariate integral associated with the Gaussian distribution (Examples 6.21 and 6.22); and §6.7, covering some more elaborate examples of multivariate transformations of random variables.

Throughout the document, many, but not all, results are proven. Understanding proofs is essential in this course, but the main emphasis is on practical examples of a nontrivial nature, going well beyond the trivial examples in a first, undergraduate, course in calculus for students in the social sciences. I occasionally refer to results coming later in the document, e.g., Example 1.10 involves a Riemann integral, and I refer to its linearity property, as stated later, in §2.5.1. The reader is not expected to jump ahead and understand that material; it is there for reference. In my experience, having perused and studied numerous excellent mathematics books, this approach is quite common, because it is nearly unavoidable, notably in presentations such as this one, which cover a variety of topics. The key is to not do it too often!

It is worth mentioning that, in addition to the material herein, students aspiring to learn one or more of the aforementioned topics, e.g., mathematical finance, quant risk management, etc., will also require the *sine qua non* (indispensable and essential action, condition, or ingredient) of linear algebra, advanced statistical methods, measure theory and the Lebesgue integral, and (measure-theoretic) probability theory; not to mention computer coding skills. This set is nicely captured in the preface of the well-received (2004, corrected 2009) book Convex Optimization, by Boyd and Vandenberghe:

> "The only background required of the reader is a good knowledge of advanced calculus and linear algebra. If the reader has seen basic mathematical analysis (e.g., norms, convergence, elementary topology), and basic probability theory, he or she should be able to follow every argument and discussion in the book."

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# 1 Preliminaries

> The point of view that "natural number" cannot be defined would be contested by many mathematicians who would maintain that the concept of "set" is more primitive than that of "number" and who would use it to define "number". Others would contend that the idea of "set" is not at all intuitive and would contend that, in particular, the idea of an *infinite* set is very nebulous. They would consider a definition of "number" in terms of sets to be an absurdity because it uses a difficult and perhaps meaningless concept to define a simple one.
>
> (Harold M. Edwards, 1994, p. 461)

## 1.1 Sets, Supremum, and Functions

It turns out that, mathematically speaking, a precise definition of *set* is problematic. For our purposes, it can be thought of simply as a well-defined collection of objects. This intuitive description cannot be a definition, because the word "collection" is nothing but a synonym for the word set. Nevertheless, in all contexts considered herein, the notion of set will be clear. For example, if $A = \{n \in \mathbb{N} : n < 7\}$, then $A$ is the set of positive integers less than 7, or $A = \{1, 2, \ldots, 6\}$. If $a$ is contained in $A$, then we write $a \in A$; otherwise, $a \notin A$. A set without any objects is called the *empty set* and is denoted $\emptyset$. A set with exactly one element is a *singleton set*.

Let $A$ and $B$ be two sets. The following handful of basic set operations will be used repeatedly throughout:

- the *intersection* of two sets, "$A$ and $B$" (or "$A$ intersect $B$"), denoted $A \cap B$. Each element of $A \cap B$ is contained in $A$, and contained in $B$; $A \cap B = \{x : x \in A, x \in B\}$.

- the *union* of two sets, "$A$ or $B$" (or "$A$ union $B$"), denoted $A \cup B$. An element of $A \cup B$ is either in $A$, or in $B$, or in both.

- set *subsets*, "$A$ is a subset of $B$" or "$A$ is contained in $B$" or "$B$ contains $A$", denoted $A \subset B$ or $B \supset A$. If every element contained in $A$ is also in $B$, then $A \subset B$. Like the ordering symbols $\leq$ and $<$ for the real numbers, it is sometimes useful (if not more correct) to use the notation $A \subseteq B$ to indicate that $A$ and $B$ could be equal, and reserve $A \subset B$ to indicate that $A$ is a *proper subset* of $B$, i.e., $A \subseteq B$ but $A \neq B$; in words, that there is at least one element in $B$ that is not in $A$. Only when this distinction is important will we use $\subseteq$. Also, $\emptyset$ is a subset of every set.

- set *equality*, "$A = B$", which is true if and only if $A \subset B$ and $B \subset A$. To prove two sets are equal, we prove $A \subset B$, and $B \subset A$.

- the *difference*, or *relative complement*, "$B$ setminus $A$", denoted $B \setminus A$ or, sometimes authors write $B - A$. It is the set of elements contained in $B$ but not in $A$.

- If the set $B$ is clear from the context, then it need not be explicitly stated, and the set difference $B \setminus A$ is written as $A^c$, which is the *complement* of $A$. Thus, we can write $B \setminus A = B \cap A^c$.

- the *product* of two sets, $A$ and $B$, consists of all ordered pairs $(a, b)$, such that $a \in A$ and $b \in B$; it is denoted $A \times B$.

The first four previous set operations are extended to more than two sets in a natural way, i.e., for intersection, if $a \in A_1 \cap A_2 \cap \cdots \cap A_n$, then $a$ is contained in each of the $A_i$, and is abbreviated by $a \in \bigcap_{i=1}^n A_i$. A similar notation is used for union. To illustrate this for subsets, let $A_n = [1/n, 1]$, $n \in \{1, 2, \ldots\}$, i.e., $A_1 = \{1\}$ and $A_2 = [1/2, 1] = \{x : 1/2 \leq x \leq 1\}$. Then $A_1 \subset A_2 \subset \cdots$, and $\bigcup_{n=1}^\infty = (0, 1] = \{x : 0 < x \leq 1\}$. In this case, the $A_n$ are said to be *monotone increasing*. If sets $A_i$ are monotone increasing, then

$$\lim_{i \to \infty} A_i = \bigcup_{i=1}^{\infty} A_i. \tag{1.1}$$

Similarly, the sets $A_i$ are *monotone decreasing* if $A_1 \supset A_2 \supset \cdots$, in which case

$$\lim_{i \to \infty} A_i = \bigcap_{i=1}^{\infty} A_i. \tag{1.2}$$

We will also need basic familiarity with the following sets: $\mathbb{N} = \{1, 2, \ldots\}$ is the set of all natural numbers; $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$ is the set of all integers or Zahlen (German for number); $\mathbb{Q} = \{m/n,\, m \in \mathbb{Z},\, n \in \mathbb{N}\}$ is the set of all rational numbers (quotients); $\mathbb{R}$ is the set of all real numbers; $\mathbb{C}$ is the set of complex numbers, and $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. For convenience and clarity, we also define $\mathbb{R}_{>0} = \{x : x \in \mathbb{R},\, x > 0\}$, $\mathbb{R}_{\geq 1} = \{x : x \in \mathbb{R},\, x \geq 1\}$, etc.; if only a range is specified, then the real numbers are assumed, e.g., $x > 0$ is the same as $x \in \mathbb{R}_{>0}$. Also, we take $\mathbb{X} := \mathbb{R} \cup \{-\infty, \infty\}$, which is the *extended real line*. Letting $a \in \mathbb{R}$, properties of $\mathbb{X}$ include $\infty + \infty = \infty + a = \infty$, $a \cdot \infty = \operatorname{sgn}(a) \cdot \infty$, but $\infty - \infty$, $\infty/\infty$, etc., are undefined, as remains $0/0$.

We make use of the common abbreviations $\exists$ ("there exists"), $\nexists$ ("there does not exist"), $\Rightarrow$ ("implies"), iff (if and only if) and $\forall$ ("for all" or, better, "for each"; see Pugh, 2002, p. 5). As an example, $\forall x \in (0, 1)$, $\exists\, y \in (x, 1)$. Also, the notation "$A := B$" means that $A$, or the lhs (left hand side) of the equation, is defined to be $B$, or the rhs (right hand side).

Sets obey certain rules, such as $A \cup A = A$ (idempotent); $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$ (associative); $A \cup B = B \cup A$ and $A \cap B = B \cap A$ (commutative);

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and} \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

(distributive); $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$ (identity); and $(A^c)^c = A$ (involution). Less obvious are De Morgan's laws, after Augustus De Morgan (1806–1871), which state that $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$. More generally,

$$\left( \bigcup_{n=1}^{\infty} A_n \right)^c = \bigcap_{n=1}^{\infty} A_n^c \quad \text{and} \quad \left( \bigcap_{n=1}^{\infty} A_n \right)^c = \bigcup_{n=1}^{\infty} A_n^c. \tag{1.3}$$

**Example 1.1** *Let $B_i := A_i \setminus [A_i \cap (A_1 \cup A_2 \cup \cdots \cup A_{i-1})]$. We wish to demonstrate that*

$$B_i = A_i \setminus (A_1 \cup \cdots \cup A_{i-1}), \quad i \geq 2.$$

*It is useful to take $i = 2$, and draw a Venn diagram, confirming the result in this first case. The general proof is an excuse to practice using basic set theory relations.*

*Use the above rules for sets to get*

$$\begin{aligned}
B_i &= A_i \setminus [A_i \cap (A_1 \cup A_2 \cup \cdots \cup A_{i-1})] = A_i \cap [A_i \cap (A_1 \cup A_2 \cup \cdots \cup A_{i-1})]^c \\
&= A_i \cap [(A_i \cap A_1) \cup (A_i \cap A_2) \cup \cdots \cup (A_i \cap A_{i-1})]^c \\
&= A_i \cap [(A_i \cap A_1)^c \cap (A_i \cap A_2)^c \cap \cdots \cap (A_i \cap A_{i-1})^c] \\
&= A_i \cap (A_i^c \cup A_1^c) \cap (A_i^c \cup A_2^c) \cap \cdots \cap \left(A_i^c \cup A_{i-1}^c\right) \\
&= [A_i \cap (A_i^c \cup A_1^c)] \cap [A_i \cap (A_i^c \cup A_2^c)] \cap \cdots \cap \left[A_i \cap \left(A_i^c \cup A_{i-1}^c\right)\right] \\
&= [(A_i \cap A_i^c) \cup (A_i \cap A_1^c)] \cap \cdots \cap \left[(A_i \cap A_i^c) \cup \left(A_i \cap A_{i-1}^c\right)\right] \\
&= (A_i \cap A_1^c) \cap \cdots \cap \left(A_i \cap A_{i-1}^c\right) = A_i \cap \left(A_1^c \cap \cdots \cap A_{i-1}^c\right) \\
&= A_i \cap (A_1 \cup \cdots \cup A_{i-1})^c = A_i \setminus (A_1 \cup \cdots \cup A_{i-1}) . \qquad \blacksquare
\end{aligned}$$

Two sets are *disjoint*, or *mutually exclusive*, if $A \cap B = \emptyset$, i.e., they have no elements in common. A set $J$ is an *indexing set* if it contains a set of indices, usually a subset of $\mathbb{N}$, and is used to work with a group of sets $A_i$, where $i \in J$. If $A_i$, $i \in J$, are such that $\bigcup_{i \in J} A_i \supset \Omega$, then they are said to *exhaust*, or (form a) *cover* (for) the set $\Omega$. If sets $A_i$, $i \in J$, are nonempty, mutually exclusive and exhaust $\Omega$, then they (form a) *partition* (of) $\Omega$.

**Example 1.2** *Let $A_i$ be monotone increasing sets, i.e., $A_1 \subset A_2 \subset \cdots$. Define $B_1 := A_1$ and $B_i := A_i \setminus (A_1 \cup A_2 \cup \cdots \cup A_{i-1})$. We wish to show that, for $n \in \mathbb{N}$,*

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i. \tag{1.4}$$

*The $B_i$ are clearly disjoint from their definition and such that $B_i$ is the "marginal contribution" of $A_i$ over and above that of $(A_1 \cup A_2 \cup \cdots \cup A_{i-1})$, which follows because the $A_i$ are monotone increasing. Thus, $B_i = A_i \setminus A_{i-1} = A_i \cap A_{i-1}^c$. If $\omega \in \bigcup_{i=1}^n A_i$, then, because the $A_i$ are increasing, either $\omega \in A_1$ (and, thus, in all the $A_i$) or there exists a value $j \in \{2, \ldots, n\}$ such that $\omega \in A_j$ but $\omega \notin A_i$, $i < j$. It follows from the definition of the $B_i$ that $\omega \in B_j$ and thus in $\bigcup_{i=1}^n B_i$, so that (i) $\bigcup_{i=1}^n A_i \subset \bigcup_{i=1}^n B_i$.*

*Likewise, if $\omega \in \bigcup_{i=1}^n B_i$, then, as the $B_i$ are disjoint, $\omega$ is in exactly one of the $B_i$, say $B_j$, $j \in \{1, 2, \ldots, n\}$. From the definition of $B_j$, $\omega \in A_j$, so $\omega \in \bigcup_{i=1}^n A_i$, so that (ii) $\bigcup_{i=1}^n B_i \subset \bigcup_{i=1}^n A_i$. Together, (i) and (ii) imply that $\bigcup_{i=1}^n A_i = \bigcup_{i=1}^n B_i$. Also, for $i > 1$,*

$$\begin{aligned}
B_i &= A_i \setminus (A_1 \cup A_2 \cup \cdots \cup A_{i-1}) = A_i \cap (A_1 \cup A_2 \cup \cdots \cup A_{i-1})^c \\
&= A_i A_1^c A_2^c \cdots A_{i-1}^c = A_i A_{i-1}^c,
\end{aligned}$$

*where the last equality follows from $A_j = \bigcup_{n=1}^j A_n$ (because the $A_i$ are monotone increasing) and, thus, $A_j^c = \bigcap_{n=1}^j A_n^c$.* $\blacksquare$

For $a, b \in \mathbb{R}$ with $a \leq b$, the interval $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ is said to be an *open interval*, while $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ is a *closed interval*. In both cases, the interval has *length* $b - a$. For a set $S \subset \mathbb{R}$, the set of open intervals $\{O_i\}$, for $i \in J$ with $J$ an indexing set, is an *open cover* of $S$ if $\bigcup_{i \in J} O_i$ covers $S$, i.e., if $S \subset \bigcup_{i \in J} O_i$. Let $S \subset \mathbb{R}$ be such that there exists an open cover $\bigcup_{i \in N} O_i$ of $S$ with a finite or countably infinite number of intervals. Denote the length of each $O_i$ as $\ell(O_i)$. If $\forall \epsilon > 0$, there exists a cover $\bigcup_{i \in N} O_i$ of $S$ such that

$$\sum_{i=1}^{\infty} \ell(O_i) < \epsilon, \tag{1.5}$$

then $S$ is said to have *measure zero*. See also §6.2.3.

For our purposes, the most important set with measure zero is any set with a finite or countable number of points. For example, if $f$ and $g$ are functions with domain $I = (a, b) \in \mathbb{R}$, where $a < b$, and such that $f(x) = g(x)$ for all $x \in I$ except for a finite or countably infinite number of points in $I$, then we say that **$f$ and $g$ differ on $I$ by a set of measure zero**. As an example from probability, if $U$ is a continuous uniform random variable on $[0, 1]$, then the event that $U = 1/2$ is not impossible, but it has probability zero, because the point $1/2$ has measure zero, as does any finite collection of points, or any countably infinite set of points on $[0, 1]$, e.g., $\{1/n,\ n \in \mathbb{N}\}$.

Let $S$ be a nonempty subset of $\mathbb{R}$. We say $S$ has an *upper bound* $M$ if $x \leq M$ $\forall x \in S$, in which case $S$ is *bounded above* by $M$. Note that, if $S$ is bounded above, then it has infinitely many upper bounds.

Axiom (The Completeness Axiom): A fundamental property of $\mathbb{R}$ not shared by $\mathbb{Q}$ is that, if $S$ is a nonempty set that has an upper bound $M$, then $S$ possesses a unique *least upper bound*, or *supremum*, denoted $\sup S$. That is, $\exists\, U \in \mathbb{R}$ such that $U$ is an upper bound of $S$, and such that, if $V$ is also an upper bound of $S$, then $V \geq U$.

This axiom can actually be made into a theorem and proven, albeit by assuming different, related properties of the real numbers. See, for example, Stoll, 2021, p. 25 for some discussion on construction of the real numbers and the relation to this axiom.[1]

If $S$ is not bounded above, then $\sup S = \infty$. Also, $\sup \emptyset = -\infty$. Similar terminology applies to the *greatest lower bound*, or *infimum* of $S$, denoted $\inf S$. For example, let $S = \{1/n : n \in \mathbb{N}\}$. Then $\max S = \sup S = 1$ and $\inf S = 0$, but $S$ has no minimum value. Next, let $S$ consist of the truncated values of $\sqrt{2}$ with $n \in \mathbb{N}$ decimal places, i.e., $S = \{1.4, 1.41, 1.414, 1.4142, \ldots\}$. Then $S \subset \mathbb{Q}$ but $\sup S = \sqrt{2} \notin \mathbb{Q}$.[2]

Theorem: If a subset $S$ of $\mathbb{R}$ has a supremum, then it is unique.

> Proof: Let $u$ and $u'$ be two supremums of $S$. Then as $u'$ is an upper bound, and since $u$ is a least upper bound, we must have $u \leq u'$. Similarly, since $u$ is an upper bound, and since $u'$ is a least upper bound, we must also have $u' \leq u$. It now follows that $u = u'$.

Theorem: Let $S$ be a nonempty subset of real numbers that is bounded below. Let $-S$ denote the set of all real numbers $-x$, where $x$ belongs to $S$. Then $\inf(S)$ exists and $\inf(S) = -\sup(-S)$.

> Proof: Let $\ell$ be a lower bound of $S$. Then, for all $x \in S$, $\ell \leq x$. So for all $x \in S$, $-x \leq -\ell$. That is, for all $y \in -S$, $y \leq -\ell$. Thus $-S$ is bounded above because $-\ell$ is an

[1] It also turns out that the nested intervals property (see §3.1) can be taken as the axiom instead of the completeness property. Ash's book Real Variables: With Basic Metric Space Topology, 2007, takes this approach. Bloch's book, The Real Numbers and Real Analysis, is yet more explicit about this; see the preface, and his p. 104: "it turns out that the Heine-Borel Theorem is equivalent to the Least Upper Bound Property, as is discussed in Section 3.5 and proved in Theorem 3.5.4." See also his p. 166, and Theorem 3.5.4, and p. 170. From Stoll, 2021, p. 124, "The nested intervals property can also be used to prove the supremum property of $\mathbb{R}$ (Exercise 21 of his Sec. 3.3, p. 102). Another property of the real numbers that is equivalent to the least upper bound property is the completeness property of $\mathbb{R}$; namely, every Cauchy sequence of real numbers converges."

[2] Observe that any element in $\mathbb{R}$ can be arbitrarily closely approximated by an element in $\mathbb{Q}$, which is the informal description of saying that $\mathbb{Q}$ is *dense* in $\mathbb{R}$. This is of enormous importance when actually working with numerical values in an (unavoidably) finite precision computing world.

upper bound of $-S$. Since $S$ is nonempty, it follows that there exists an element $x \in S$, and so we obtain that $-x \in -S$. Hence $-S$ is nonempty.

As $-S$ is nonempty and bounded above, it follows that $\sup(-S)$ exists, by the Least Upper Bound Property of $\mathbb{R}$.

Since $\sup(-S)$ is an upper bound of $-S$, we have that, for all $y \in -S$, $y \leq \sup(-S)$. That is, for all $x \in S$, $-x \leq \sup(-S)$. Hence for all $x \in S$, $-\sup(-S) \leq x$. So $-\sup(-S)$ is a lower bound of $S$.

Next we prove that $-\sup(-S)$ is the greatest lower bound of $S$. Suppose that $\ell'$ is a lower bound of $S$ such that $-\sup(-S) < \ell'$. Then for all $x \in S$, $-\sup(-S) < \ell' \leq x$. That is, for all $x \in S$, $-x \leq -\ell' < \sup(-S)$. Hence, for all $y \in -S$, $y \leq -\ell' < \sup(-S)$. So $-\ell'$ is an upper bound of $-S$, and $-\ell' < \sup(-S)$, which contradicts the fact that $\sup(-S)$ is the least upper bound of $-S$. Hence $\ell' \leq -\sup(-S)$.

Consequently, $\inf S$ exists and $\inf S = -\sup(-S)$.

Theorem: Let $A$ and $B$ be subsets of $\mathbb{R}$. Define

$$A + B = \{a + b : a \in A,\, b \in B\} \quad \text{and} \quad A \cdot B = \{ab : a \in A,\, b \in B\}.$$

If $A$ and $B$ are nonempty and bounded above, then

$$\sup(A + B) = \sup A + \sup B. \tag{1.6}$$

What can you say about $\sup(A \cdot B)$?

Proof of (1.6): For $\sup(A + B) \leq \sup A + \sup B$: Both $A$ and $B$ are non-empty and bounded above, so $\alpha = \sup A$ and $\beta = \sup B$ exist in $\mathbb{R}$. Therefore, $a + b \leq \alpha + \beta$ for all $a \in A$ and $b \in B$. This means $\alpha + \beta$ is an upper bound for $A + B$, and, by definition of sup, $\gamma = \sup(A + B) \leq \alpha + \beta$.

For $\sup(A + B) \geq \sup A + \sup B$: Let $\gamma = \sup(A + B)$, which is an upper bound for $A + B$. Then $a + b \leq \gamma$ for all $a \in A$ and $b \in B$. Let $b \in B$ be arbitrary but fixed. Then $a \leq \gamma - b$ for all $a \in A$. Thus $\gamma - b$ is an upper bound for $A$ and, hence, $\sup A = \alpha \leq \gamma - b$. As this holds for all $b \in B$, we also have $b \leq \gamma - \alpha$ for all $b \in B$. Thus $\sup B = \beta \leq \gamma - \alpha$; i.e., $\alpha + \beta \leq \gamma$.

Let $A = B = [-1, 0]$, which are non-empty and bounded. Then $(\sup A)(\sup B) = 0$, but $\sup(A \cdot B) = 1$. This serves as an example of two nonempty bounded sets $A$ and $B$ for which $\sup(A \cdot B) \neq (\sup A)(\sup B)$. If both $A$ and $B$ consist of nonnegative real numbers, then $\sup(A \cdot B) = \sup(A)\sup(B)$. See, e.g., https://math.stackexchange.com/questions/46738.

Theorem: Let $f$ and $g$ be real-valued functions defined on a nonempty set $X \subset \mathbb{R}$ with bounded ranges. Then

1. $\sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$.

   Proof: Let $\alpha = \sup\{f(x) : x \in X\}$ and $\beta = \sup\{g(x) : x \in X\}$. The ranges of $f$ and $g$ are bounded, so $\alpha$ and $\beta$ are finite, with $f(x) + g(x) \leq \alpha + \beta$ for every $x \in X$. Thus, $\alpha + \beta$ is an upper bound for $\{f(x) + g(x) : x \in X\}$, and $\sup\{f(x) + g(x) : x \in X\} \leq \alpha + \beta$.

2. If $f(x) \leq g(x)$ for all $x \in X$, then

$$\sup\{f(x) : x \in X\} \leq \sup\{g(x) : x \in X\}. \tag{1.7}$$

   This result is easy and important; we can call it monotonicity of the supremum.

> Proof: Let $\alpha = \sup\{g(x) : x \in X\}$, so that $g(x) \leq \alpha$ for all $x \in X$. Thus, by hypothesis, $f(x) \leq \alpha$ for all $x \in X$. Therefore, $\alpha$ is an upper bound for $\{f(x) : x \in X\}$, and, as a consequence, $\sup\{f(x) : x \in X\} \leq \alpha = \sup\{g(x) : x \in X\}$.

3. $\sup\{f(x) + g(y) : x \in X, y \in X\} = \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$.

> Proof: Apply (1.6), taking $A$ to be the range of $f$; and $B$ to be the range of $g$.

As an example for which equality does not hold in (1), let $X = [0, 1]$, $f(x) = x$, and $g(x) = -x$. The lhs is 0; the rhs is $1 + 0$.

Definition: A *relation* between $A$ and $B$ is a subset of $A \times B$.

Definition: If a relation $f$ is such that, for each $a \in A$, there is one and only one $b \in B$ such that $(a, b) \in f$, then $f$ is also a *function* or *mapping*. One writes $f : A \to B$ and $b = f(a)$, with $A$ referred to as the *domain* and $B$ as the *codomain* or *target*.

When $f$ is plotted on the plane in the standard fashion, i.e., with $A$ on the horizontal axis and $B$ on the vertical axis, then a mapping satisfies the "vertical line test".

In the following, let $f$ be a function from $A$ into $B$.

Definition: If $E \subset A$, then $f(E)$ is called *the image of $E$ under $f$*, and is defined by $f(E) := \{f(x) : x \in E\}$.

Definition: For some subset $C \subset B$, the *pre-image* of $C$ is the subset of the domain defined by $f^{-1}(C) := \{a \in A : f(a) \in C\}$.

Definition: The subset of the codomain given by $\{b \in B : \exists a \in A \text{ with } f(a) = b\}$ is the *range* or *image* of $f$.

Definition: A mapping with codomain $B = \mathbb{R}$ is a real-valued *function*.

Definition: Let $f$ be a function with domain $A$ and let $I \in A$ be an interval. If $f$ is such that, $\forall a, b \in A$, $a < b \Rightarrow f(a) < f(b)$, then $f$ is *strictly increasing* on $I$. Likewise, if $a < b \Rightarrow f(a) \leq f(b)$, then $f$ is *(weakly) increasing*. The terms *strictly decreasing* and *(weakly) decreasing* are similarly defined.

Definition: A function that is either increasing or decreasing is said to be *monotone*, while a function that is either strictly increasing or strictly decreasing is *strictly monotone*.

Definition: The mapping $f : A \to B$ is *injective* or *one–to–one* if $f(a_1) = f(a_2)$ implies $a_1 = a_2$. (That is, if a plot of $f$ satisfies the "horizontal line test".)

Definition: A mapping is *surjective* or *onto* if the range is the (whole) codomain.

Definition: A mapping is *bijective* if it is injective and surjective.

Definition: If $f : A \to B$ is bijective, then the *inverse mapping* $f^{-1} : B \to A$ is bijective such that $f^{-1}(b)$ is the (unique) element in $A$ such that $f(a) = b$.

The following simple relations are of occasional great use, and can be proven via induction, but also proven directly, as we do. Let $n \in \mathbb{N}$.

$$x^n - y^n = (x-y)\sum_{j=1}^{n} x^{n-j}y^{j-1}. \tag{1.8}$$

$$x^n + y^n = (x+y)\sum_{j=1}^{n}(-1)^{j-1}x^{n-j}y^{j-1}, \quad \text{for } n \text{ odd.} \tag{1.9}$$

$$x^{-n} - y^{-n} = (y-x)\sum_{j=1}^{n} x^{j-n-1}y^{-j}, \quad \text{if } x \neq 0 \text{ and } y \neq 0. \tag{1.10}$$

$$a^{1/n} - b^{1/n} = (a-b)\left(\sum_{j=1}^{n} a^{1-j/n}b^{(j-1)/n}\right)^{-1}, \quad a, b > 0. \tag{1.11}$$

We use (1.8) in Examples 1.3 and 1.4; and (1.10) in Example 2.7. For (1.8),

$$(x-y)\sum_{j=1}^{n} x^{n-j}y^{j-1} = \sum_{j=1}^{n} x^{n-j+1}y^{j-1} - \sum_{j=1}^{n} x^{n-j}y^{j} = \sum_{j=0}^{n-1} x^{n-j}y^{j} - \sum_{j=1}^{n} x^{n-j}y^{j} = x^n - y^n.$$

For (1.9), replace $y$ in (1.8) by $-y$. For (1.10), replace $x$ and $y$ in (1.8) by $x^{-1}$ and $y^{-1}$, respectively. For (1.11), use (1.8) with $x = a^{1/n}$, $y = b^{1/n}$.

**Example 1.3** *Define $f(x) = x^2$ for $x \geq 0$. Then the function $f : [0, \infty) \to \mathbb{R}$ is strictly increasing, because*

$$u^2 - v^2 = (u-v)(u+v) > 0 \qquad \text{if } u \geq 0, v \geq 0, u > v, \tag{1.12}$$

*from (1.8). Function $f$ is injective, but is not surjective, if the codomain is taken to be $\mathbb{R}$. If the codomain is specified as $\mathbb{R}_{\geq 0}$, then $f$ is surjective, and thus bijective.* ■

**Example 1.4** *Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Then $f$ is strictly increasing: From (1.8),*

$$u^3 - v^3 = (u-v)\left(u^2 + uv + v^2\right) \qquad \text{for all } u, v. \tag{1.13}$$

*If $u$ and $v$ have the same sign, then $uv > 0$; thus $u^2 + uv + v^2 > 0$, so from (1.13), $u^3 > v^3$ if $u > v$. Next, if $u > 0 > v$, then $u^3 > 0 > v^3$. Finally, if $u > v$ and either $u$ or $v$ equals 0, then clearly $u^3 > v^3$. Function $f$ is a bijection.* ■

The above two examples generalize to $f : [0, \infty) \to \mathbb{R}$ for $f(x) = x^n$, for $n \in \mathbb{N}$. Use of (1.8) shows that $f$ is strictly increasing. As shown in §2.2, $f$ is continuous. Further, we can use the Intermediate Value Theorem (2.60) to prove that $f([0, \infty)) = [0, \infty)$; and (2.41) to prove that $f^{-1} : [0, \infty) \to \mathbb{R}$ is continuous.

Theorem: Let $f : X \to Y$ be a function. The following properties hold:

1. For every $A \subseteq X, A \subseteq f^{-1}(f(A))$.
2. If $A_1 \subseteq A_2 \subseteq X$, then $f(A_1) \subseteq f(A_2)$.

3. If $A_1 \subseteq X$ and $A_2 \subseteq X$, then
$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2). \tag{1.14}$$
4. If $A_1 \subseteq X$ and $A_2 \subseteq X$, then $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

> Proof: We show property 1 below. Consider property 4 on the image of an intersection. If $x \in A_1 \cap A_2$, then $f(x) \in f(A_1)$ and $f(x) \in f(A_2)$, so $f(x) \in f(A_1) \cap f(A_2)$. Thus, $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. To see that the other direction does not hold, if there are points $a \neq b$ in $X$ such that $f(a) = f(b)$, then with $A_1 = \{a\}$ and $A_2 = \{b\}$, the intersection $A_1 \cap A_2$ is empty, and hence $f(A_1 \cap A_2)$ is the empty set, but $f(A_1) \cap f(A_2)$ has one element.

Theorem: Let $f : X \to Y$, and let $\{A_i\}$ be a family of (possibly uncountably many) subsets of $Y$. Then
$$f^{-1}\left(\bigcup_i A_i\right) = \bigcup_i f^{-1}(A_i), \quad f^{-1}\left(\bigcap_i A_i\right) = \bigcap_i f^{-1}(A_i) \tag{1.15}$$
and
$$f^{-1}(A_i^c) = \left[f^{-1}(A_i)\right]^c. \tag{1.16}$$

> Proof:
>
> $$\begin{aligned} \bullet\, x \in f^{-1}\left(\bigcup_i A_i\right) &\iff f(x) \in \bigcup_i A_i \iff \exists i : f(x) \in A_i \\ &\iff \exists i : x \in f^{-1}(A_i) \iff x \in \bigcup_i f^{-1}(A_i). \\ \bullet\, x \in f^{-1}\left(\bigcap_i A_i\right) &\iff f(x) \in \bigcap_i A_i \iff \forall i : f(x) \in A_i \\ &\iff \forall i : x \in f^{-1}(A_i) \iff x \in \bigcap_i f^{-1}(A_i). \\ \bullet\, x \in f^{-1}(A_i^c) &\iff f(x) \in A_i^c \iff f(x) \notin A_i \\ &\iff x \notin f^{-1}(A_i) \iff x \in \left[f^{-1} A_i\right]^c. \end{aligned}$$

Let $f : X \to Y$, and $A_1, A_2 \subset X$. We want an example showing that $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$. Let $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be given by $f(x) = x^2$, noting this is onto, but not one-to-one. Let $A_1 = (-1, 0]$ and $A_2 = [0, 1)$, so that $A_1 \cap A_2 = \{0\}$ and $f(A_1 \cap A_2) = 0$; and $f(A_1) = f(A_2) = [0, 1) = f(A_1) \cap f(A_2)$.

Theorem (Results on $f^{-1}[f(A)]$ and $f\,[f^{-1}(B)]$): Let $f : X \to Y$.

1. If $A \subseteq X$, then $A \subset f^{-1}[f(A)]$.

> Proof: $x \in A \Rightarrow f(x) \in f(A) \Rightarrow x \in f^{-1}[f(A)]$.

2. If $B \subseteq Y$, then $f\,[f^{-1}(B)] \subset B$. (Note the strict inequality.)

> Proof: Recall $A \wedge B$ means conditions $A$ and $B$ both hold. ($\vee$ is or).
>
> $$\begin{aligned} y \in f\left[f^{-1}(B)\right] &\Rightarrow \exists x_y : \left[x_y \in f^{-1}(B)\right] \wedge \left[y = f(x_y)\right] \\ &\Rightarrow f(x_y) \in B \Rightarrow y \in B. \end{aligned}$$
>
> As an example, we need a non-onto function, e.g., $X = Y = \mathbb{R}$, $f(x) = x^2$, so with $B = (-1, 1)$, $f^{-1}(B) = [0, 1)$, and $f(f^{-1}(B)) = [0, 1) \subset B$.

3. $\left[f \text{ onto } Y\right] \iff \left[\forall B \subset Y : \, f\left(f^{-1}(B)\right) = B\right]$.

> Proof: Let $f : X \to Y$ be onto $Y$; and let $B \subset Y$. We need to prove $f\left[f^{-1}(B)\right] \supset B$.
>
> $$\begin{aligned} \left[y \in B \subset Y\right] \wedge \left[f \text{ onto}\right] &\Rightarrow \exists x_y \in X : y = f(x_y) \Rightarrow x_y \in \left\{f^{-1}(\{y\})\right\} \\ &\Rightarrow f(x_y) \subset f\left(f^{-1}(\{y\})\right) \subset f\left(f^{-1}(B)\right), \end{aligned} \tag{1.17}$$
>
> i.e., $B \subset f\left[f^{-1}(B)\right]$. Observe in (1.17) we need to write $x_y \in \{f^{-1}(\{y\})\}$ instead of $x_y = f^{-1}(y)$ because $f$ may not be one-to-one. If $f$ is additionally one-to-one, then it is a bijection, and $A = f^{-1}[f(A)]$ and $B = f\left[f^{-1}(B)\right]$.

For mappings $f : A \to B$ and $g : B \to C$, the *composite mapping*, denoted $g \circ f : A \to C$, maps an element $a \in A$ to $g(f(a))$. We next show that, if $f$ and $g$ are injective, then so is $g \circ f$; and if $f$ and $g$ are surjective, then so is $g \circ f$. Thus, if $f$ and $g$ are bijective, then so is $g \circ f$.

Theorem (Composite functions, injectivity, surjectivity)

1. If $f$ and $g$ are injective, then so is $g \circ f$.

> Proof: Both $f$ and $g$ are injective, so $f(x_1) = f(x_2) \implies x_1 = x_2$ and $g(y_1) = g(y_2) \implies y_1 = y_2$. Therefore, $(g \circ f)(x_1) = (g \circ f)(x_2) \implies g(f(x_1)) = g(f(x_2)) \implies f(x_1) = f(x_2) \implies x_1 = x_2$. Hence, $g \circ f$ is also injective.

2. If $f$ and $g$ are surjective, then so is $g \circ f$.

> Proof: Both $f$ and $g$ are surjective, so $\forall y \in Y, \exists x \in X$ such that $f(x) = y$ and $\forall z \in Z, \exists y \in Y$ such that $g(y) = z$. Thus, $\forall z \in Z, \exists x \in X$ such that $g(f(x)) = z$.

3. If $g \circ f$ is injective, then $f$ is injective.

> Proof: $(g \circ f)(x_1) = (g \circ f)(x_2) \implies x_1 = x_2$. Thus $f(x_1) = f(x_2) \implies g(f(x_1)) = g(f(x_2)) \implies (g \circ f)(x_1) = (g \circ f)(x_2) \implies x_1 = x_2$.

4. If $g \circ f$ is surjective, then $g$ is surjective.

> Proof: With $y = f(x)$,
>
> $\forall z \in Z : \, \exists x \in X$ such that $(g \circ f)(x) = z \implies \forall z \in Z, \, \exists y \in Y$ such that $g(y) = z$.

## 1.2 Fundamental Inequalities

If $a \in \mathbb{R}$, then the *absolute value* of $a$ is denoted by $|a|$, and is equal to $a$ if $a \geq 0$, and $-a$ if $a < 0$. Clearly, $a \leq |a|$ and, $\forall a, b \in \mathbb{R}$, $|ab| = |a|\,|b|$. Observe that, for $b \in \mathbb{R}_{>0}$, the inequality $-b < a < b$ is equivalent to $|a| < b$, and, similarly,

$$-b \leq a \leq b \quad \Leftrightarrow \quad |a| \leq b. \tag{1.18}$$

Theorem (One version of Bernoulli's inequality):

$$\forall h \geq 0,\ \forall n \in \mathbb{N},\ \ (1+h)^n \geq 1 + nh. \tag{1.19}$$

Proof: (As in Stoll, p. 18, and noting that (1.19) actually holds for $b > -1$): For $n = 1, (1+h)^1 = 1 + h$. As equality holds, the inequality is certainly valid. Assume that the inequality is true when $n = k, k \geq 1$. Then for $n = k+1$, $(1+h)^{k+1} = (1+h)^k(1+h)$. By the induction hypothesis and the fact that $(1+h) > 0$, we have

$$\begin{aligned} (1+h)^{k+1} &= (1+h)^k(1+h) \\ &\geq (1+kh)(1+h) = 1 + (k+1)h + kh^2 \\ &\geq 1 + (k+1)h. \end{aligned}$$

Thus, by the principle of mathematical induction, (1.19) holds for all $n \in \mathbb{N}$.

Theorem (Triangle Inequality): $\forall x, y \in \mathbb{R}$,

$$|x+y| \leq |x| + |y|\,. \tag{1.20}$$

Proof: Square both sides to get

$$|x+y|^2 = (x+y)^2 = x^2 + 2xy + y^2 \quad \text{and} \quad (|x| + |y|)^2 = x^2 + 2\,|x|\,|y| + y^2.$$

Note $xy \leq |xy| = |x|\,|y|$. Alternatively, note that, $\forall a \in \mathbb{R}$, $-|a| \leq a \leq |a|$, so adding $-|x| \leq x \leq |x|$ to $-|y| \leq y \leq |y|$ gives $-(|x| + |y|) \leq x + y \leq |x| + |y|$, which, from (1.18) with $a = x + y$ and $b = |x| + |y|$, is equivalent to $|x+y| \leq |x| + |y|$. Also, with $z = -y$, the triangle inequality states that, $\forall x, z \in \mathbb{R}$, $|x - z| \leq |x| + |z|$.

Theorem (Reverse Triangle Inequality): $\forall a, b \in \mathbb{R}$,

$$||a| - |b|| \leq |a+b|, \quad \text{or} \quad ||a| - |b|| \leq |b-a|. \tag{1.21}$$

Proof: Write $|a| = |(a+b) + (-b)| \leq |a+b| + |b|$, or $|a| - |b| \leq |a+b|$. Switching $a$ and $b$ gives $|b| - |a| \leq |a+b|$ or $||a| - |b|| \leq |a+b|$. Replacing $b$ with $-b$ gives $||a| - |b|| \leq |a - b| = |b - a|$.

Theorem (Cauchy-Schwarz Inequality): For any points $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, $n \in \mathbb{N}$,

$$|x_1y_1 + \cdots + x_ny_n| \leq (x_1^2 + \cdots + x_n^2)^{1/2}\,(y_1^2 + \cdots + y_n^2)^{1/2}. \tag{1.22}$$

It is named after Augustin Louis Cauchy (1789–1857) and Hermann Schwarz (1843–1921), and also referred to as Cauchy's inequality or the Schwarz inequality. It was first published by Cauchy in 1821.

Proof: Let $f(r) = \sum_{i=1}^{n} (rx_i + y_i)^2 = Ar^2 + Br + C$, where $A = \sum_{i=1}^{n} x_i^2$, $B = 2\sum_{i=1}^{n} x_i y_i$ and $C = \sum_{i=1}^{n} y_i^2$. As $f(r) \geq 0$, the quadratic $Ar^2 + Br + C$ has one or no real roots, so that its discriminant $B^2 - 4AC \leq 0$, i.e., $B^2 \leq 4AC$ or, substituting, $\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$, which is (1.22) after taking square roots.

The Cauchy-Schwarz inequality is used to show the generalization of (1.20):

Theorem (Triangle Inequality): For any points $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, $n \in \mathbb{N}$,

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|, \tag{1.23}$$

where

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2} \tag{1.24}$$

is the *Euclidean norm* of $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. There are other important norms for $\mathbf{x} \in \mathbb{R}^n$; see §3.2.

Proof: Using the above notation for $A, B$ and $C$,

$$\|\mathbf{x} + \mathbf{y}\|^2 = \sum_{i=1}^{n} (x_i + y_i)^2 = \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2 = A + B + C$$

and, as $B^2 \leq 4AC$, $A + B + C \leq A + 2\sqrt{AC} + C = \left(\sqrt{A} + \sqrt{C}\right)^2$. Taking square roots gives $\|\mathbf{x} + \mathbf{y}\| = \sqrt{A + B + C} \leq \sqrt{A} + \sqrt{C} = \|\mathbf{x}\| + \|\mathbf{y}\|$.

Remark: In §3.2, we will see that the Cauchy-Schwarz inequality can be generalized to *Hölder's inequality*, and the triangle inequality can be generalized to *Minkowski's inequality*. There are also analogous Hölder and Minkowski inequalities for integrals.

## 1.3 Binomial and Generalized Binomial Theorems

The number of ways that $n \in \mathbb{N}$ distinguishable objects can be ordered is given by

$$n\,(n-1)\,(n-2)\ldots 2\cdot 1 =: n!, \qquad 0! := 1,$$

pronounced "$n$ factorial". The number of ways that $k$ objects can be chosen from $n$, $0 \le k \le n$, when order is relevant, is

$$n\,(n-1)\ldots(n-k+1) =: n_{[k]} = \frac{n!}{(n-k)!}, \tag{1.25}$$

which is referred to as the *falling, or descending factorial*.[3]

If the order of the $k$ objects is irrelevant, then $n_{[k]}$ is adjusted by dividing by $k!$, the number of ways of arranging the $k$ chosen objects. Thus, the total number of ways is

$$\frac{n\,(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!\,k!} =: \binom{n}{k}, \quad \binom{n}{0} = 1, \tag{1.26}$$

which is pronounced "$n$ choose $k$" and referred to as a *binomial coefficient* for reasons which will become clear below. Notice that, both algebraically and intuitively,

$$\binom{n}{k} = \binom{n}{n-k}. \tag{1.27}$$

**Example 1.5** *For $k$ even, let $A(k) = 2\cdot 4\cdot 6\cdot 8\cdot\cdots\cdot k$. Then*

$$A(k) = (1\cdot 2)\,(2\cdot 2)\,(3\cdot 2)\,(4\cdot 2)\cdots\left(\frac{k}{2}\cdot 2\right) = 2^{k/2}\left(\frac{k}{2}\right)!.$$

*With $m$ odd and $C(m) = 1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot m$,*

$$C(m) = \frac{(m+1)!}{(m+1)\,(m-1)\,(m-3)\cdots 6\cdot 4\cdot 2} = \frac{(m+1)!}{A(m+1)} = \frac{(m+1)!}{2^{(m+1)/2}\left(\frac{m+1}{2}\right)!}.$$

*Thus,*

$$C(2i-1) = 1\cdot 3\cdot 5\cdots(2i-1) = \frac{(2i)!}{2^i i!}, \qquad i \in \mathbb{N}, \tag{1.28}$$

*a simple result that we will use below.* ■

A very useful identity is

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \quad k < n, \tag{1.29}$$

which follows because

$$\begin{aligned}\binom{n}{k} &= \frac{n!}{(n-k)!\,k!}\cdot 1 = \frac{n!}{(n-k)!\,k!}\cdot\left(\frac{n-k}{n} + \frac{k}{n}\right)\\ &= \frac{(n-1)!}{(n-k-1)!\,k!} + \frac{(n-1)!}{(n-k)!\;(k-1)!} = \binom{n-1}{k} + \binom{n-1}{k-1}.\end{aligned}$$

[3] Similarly, we denote the *rising, or ascending factorial*, by $n^{[k]} = n\,(n+1)\ldots(n+k-1)$. There are other notational conventions for expressing the falling factorial; for example, William Feller's influential volume I (first edition, 1950, p. 28) advocates $(n)_k$, while Norman L. Johnson (1975) and the references therein (Johnson being the author and editor of numerous important statistical encyclopediae) give reasons for supporting $n^{(k)}$ for the falling factorial (and $n^{[k]}$ for the rising). One still sees the rising factorial denoted by $(n)_k$, which is referred to as the *Pochhammer symbol*, after Leo August Pochhammer, 1841–1920. It will be made clear from context what is meant, so there will be no notational confusion.

**Example 1.6** *Consider the sum $S_n = \sum_{k=0}^{n} \binom{n}{k}$ for $n \in \mathbb{N}$. Imagine that the objects under consideration are the bits in computer memory; they can each take on the value 0 or 1. Among $n$ bits, observe that there are $2^n$ possible signals that can be constructed. But this is what $S_n$ also gives, because, for a given $k$, $\binom{n}{k}$ is the number of ways of choosing which of the $n$ bits are set to one, and which are set to zero, and we sum this up over all possible $k$ (0 to $n$) so that it gives all the possible signals that $n$ binary bits can construct. (That $S_n = 2^n$ also follows directly from the binomial theorem, which is discussed below.) To prove the result via induction, assume it holds for $n-1$, so that, from (1.29),*

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \left[ \binom{n-1}{k} + \binom{n-1}{k-1} \right].$$

*Using the fact that $\binom{m}{i} = 0$ for $i > m$,*

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n-1} \binom{n-1}{k} + \sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1} + \sum_{k=1}^{n} \binom{n-1}{k-1},$$

*and, with $j = k-1$, the latter term is*

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1},$$

*so that $\sum_{k=0}^{n} \binom{n}{k} = 2^{n-1} + 2^{n-1} = 2\left(2^{n-1}\right) = 2^n$.* ■

**Example 1.7** *To prove the identity*

$$\frac{1}{2} = \sum_{i=0}^{n-1} \binom{n+i-1}{i} \left(\frac{1}{2}\right)^{n+i} =: P_n, \quad n \in \mathbb{N}, \tag{1.30}$$

*first note that $P_1 = 1/2$ and assume $P_n = 1/2$. Then,*

$$\begin{aligned}
2P_{n+1} &= \sum_{i=0}^{n} \binom{n+i}{i} \left(\frac{1}{2}\right)^{n+i} \\
&= \sum_{i=0}^{n} \binom{n+i-1}{i} \left(\frac{1}{2}\right)^{n+i} + \sum_{i=0}^{n} \binom{n+i-1}{i-1} \left(\frac{1}{2}\right)^{n+i} \\
&= \underbrace{\sum_{i=0}^{n-1} \binom{n+i-1}{i} \left(\frac{1}{2}\right)^{n+i}}_{=P_n=\frac{1}{2}} + \binom{2n-1}{n} \left(\frac{1}{2}\right)^{2n} + \sum_{i=1}^{n} \binom{n+i-1}{i-1} \left(\frac{1}{2}\right)^{n+i} \\
&\overset{j=i-1}{=} \frac{1}{2} + \binom{2n-1}{n} \left(\frac{1}{2}\right)^{2n} + \sum_{j=0}^{n-1} \binom{n+j}{j} \left(\frac{1}{2}\right)^{n+j+1} \\
&= \frac{1}{2} + \binom{2n-1}{n} \left(\frac{1}{2}\right)^{2n} + P_{n+1} - \binom{2n}{n} \left(\frac{1}{2}\right)^{2n+1}.
\end{aligned}$$

*Now note that*

$$\begin{aligned}
\binom{2n-1}{n} \left(\frac{1}{2}\right)^{2n} &= \frac{(2n-1)(2n-2)\cdots n}{n!} \left(\frac{1}{2}\right)^{2n} \\
&= \frac{n}{2n} \frac{2n(2n-1)(2n-2)\cdots(n+1)}{n!} \left(\frac{1}{2}\right)^{2n} = \binom{2n}{n} \left(\frac{1}{2}\right)^{2n+1},
\end{aligned}$$

*or*

$$2P_{n+1} = \frac{1}{2} + P_{n+1} \Leftrightarrow P_{n+1} = \frac{1}{2}.$$

*We will use this result in the next example; and prove (1.30) in a different way, in Example 1.24 below.* ■

**Example 1.8** *Prove, for $N \in \mathbb{N}$,*

$$1 = \sum_{k=0}^{N} \binom{2N-k}{N} \left(\frac{1}{2}\right)^{2N-k}. \tag{1.31}$$

*This is equivalent to (substitute $i = N-k$, so $k = N-i$, and $2N-k = 2N-(N-i) = N+i$)*

$$2^{2N} = \sum_{k=0}^{N} \binom{2N-k}{N} 2^k = \sum_{i=0}^{N} \binom{N+i}{N} 2^{N-i} = 2^N \sum_{i=0}^{N} \binom{N+i}{N} \left(\frac{1}{2}\right)^i,$$

*or*

$$2^N = \sum_{i=0}^{N} \binom{N+i}{N} \left(\frac{1}{2}\right)^i.$$

*But this holds, because, from (1.30), it follows that*

$$2^{n-1} = \sum_{i=0}^{n-1} \binom{n+i-1}{i} \left(\frac{1}{2}\right)^i,$$

*and taking $N = n-1$. We will use (1.31) to prove (7.22) is a valid pmf.* ■

By applying (1.29) recursively,

$$\begin{aligned}
\binom{n}{k} &= \binom{n-1}{k} + \binom{n-1}{k-1} \\
&= \binom{n-1}{k} + \binom{n-2}{k-1} + \binom{n-2}{k-2} \\
&= \binom{n-1}{k} + \binom{n-2}{k-1} + \binom{n-3}{k-2} + \binom{n-3}{k-3} \\
&\vdots \\
&= \sum_{i=0}^{k} \binom{n-i-1}{k-i}, \quad k < n,
\end{aligned}$$

i.e.,

$$\binom{n}{k} = \sum_{i=0}^{k} \binom{n-i-1}{k-i}, \quad k < n. \tag{1.32}$$

In (1.32), replace $n$ with $n+r$, set $k = n$ and rearrange to get

$$\binom{n+r}{n} = \sum_{i=0}^{n} \binom{n+r-i-1}{n-i} = \sum_{i=0}^{n} \binom{i+r-1}{i}, \tag{1.33}$$

which we will require in §7.3.

Theorem (Binomial Theorem): The relation

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \tag{1.34}$$

is simple, yet fundamental result that arises in numerous applications. Examples include

$$(x+(-y))^2 = x^2 - 2xy + y^2, \qquad (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$

$$0 = (1-1)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i}, \qquad 2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i}.$$

Proof: We use induction. Observe first that (1.34) holds for $n=1$. Then, assuming it holds for $n-1$,

$$\begin{aligned}(x+y)^n &= (x+y)(x+y)^{n-1} = (x+y) \sum_{i=0}^{(n-1)} \binom{(n-1)}{i} x^i y^{(n-1)-i} \\ &= \sum_{i=0}^{n-1} \binom{n-1}{i} x^{i+1} y^{n-(i+1)} + \sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-1-i+1}.\end{aligned}$$

Then, with $j = i+1$,

$$\begin{aligned}(x+y)^n &= \sum_{j=1}^{n} \binom{n-1}{j-1} x^j y^{n-j} + \sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-i} \\ &= x^n + \sum_{j=1}^{n-1} \binom{n-1}{j-1} x^j y^{n-j} + \sum_{i=1}^{n-1} \binom{n-1}{i} x^i y^{n-i} + y^n \\ &= x^n + \sum_{i=1}^{n-1} \left\{ \binom{n-1}{i-1} + \binom{n-1}{i} \right\} x^i y^{n-i} + y^n \\ &= x^n + \sum_{i=1}^{n-1} \binom{n}{i} x^i y^{n-i} + y^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}\end{aligned}$$

proving the theorem.

Recall Bernoulli's inequality (1.19), namely, $\forall h \geq 0,\ \forall n \in \mathbb{N},\ (1+h)^n \geq 1+nh$, which we proved via induction. It also follows from the binomial theorem (1.34):

$$(1+h)^n = \sum_{i=0}^{n} \binom{n}{i} h^i = 1 + nh + \binom{n}{2} h^2 + \cdots + nh^{n-1} + h^n \geq 1 + nh. \qquad \square$$

The binomial theorem can be used for proving the following result, which, in turn, will be used for proving (2.114) below.

Theorem: Let $a \in \mathbb{R}_{>1}$ and $k \in \mathbb{N}$. Then

$$\lim_{n\to\infty} a^n / n^k = \infty. \tag{1.35}$$

Proof: As in Lang, Undergraduate analysis, 2nd ed., 1997, p. 55: Write $a = 1 + b$, so

$$(1+b)^n = 1 + nb + \cdots + \frac{n(n-1)\cdots(n-k)}{(k+1)!} b^{k+1} + \cdots .$$

All the terms in this expansion are positive. The coefficient of $b^{k+1}$ can be written in the form

$$\frac{n^{k+1}}{(k+1)!} + \text{ terms with lower powers of } n.$$

For example, with $k = 3$,

$$\frac{n\,(n-1)\,(n-2)\,(n-3)}{(3+1)!} = \frac{n^{3+1}}{(3+1)!} + \left(-\frac{1}{4}n + \frac{11}{24}n^2 - \frac{1}{4}n^3\right).$$

Hence,

$$\frac{(1+b)^n}{n^k} \geqq \frac{n}{(k+1)!}\left(1 + \frac{c_1}{n} + \cdots + \frac{c_{k+1}}{n^{k+1}}\right) b^{k+1},$$

where $c_1, \ldots, c_{k+1}$ are numbers depending only on $k$ but not on $n$. Hence when $n \to \infty$, it follows that the expression on the right also $\to \infty$, by the rule for the limit of a product with one factor $n/(k+1)! \to \infty$, while the other factor has the limit $b^{k+1}$ as $n \to \infty$.

**Example 1.9** *Let $f$ and $g$ denote functions whose $n$th derivatives exist. Then, by using the usual product rule for differentiation and an induction argument, we can show that*

$$[fg]^{(n)} = \sum_{j=0}^{n} \binom{n}{j} f^{(j)} g^{(n-j)}, \tag{1.36}$$

*where $f^{(j)}$ denotes the $j$th derivative of $f$. This is sometimes (also) referred to as Leibniz' rule. This is not the binomial theorem* per se, *though it has an obvious association.* ■

**Example 1.10** *Consider computing $I = \int_{-1}^{1} (x^2-1)^j \, dx$, for any $j \in \mathbb{N}$. From the binomial theorem and the basic linearity property (2.167) of the Riemann integral,*

$$I = \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} \int_{-1}^{1} x^{2k} dx = \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} \frac{2}{2k+1},$$

*which is simple to program and compute as a function of $j$. In fact, as shown in the next example, integral $I$ can also be expressed as*

$$I = \frac{(-1)^j 2^{2j+1}}{\binom{2j}{j}(2j+1)}, \tag{1.37}$$

*thus implying the charming and non-obvious combinatoric identity*

$$\sum_{k=0}^{j} \binom{j}{k} \frac{(-1)^k}{2k+1} = \frac{2^{2j}}{\binom{2j}{j}(2j+1)}, \quad j \in \mathbb{N},$$

*as $(-1)^{-k} = (-1)^k$ and cancelling a 2 and $(-1)^j$ from both sides.* ■

**Example 1.11** *We wish to show that*

$$\int_{-1}^{1}\left(x^2-1\right)^j dx = \frac{(-1)^j 2^{2j+1}}{\binom{2j}{j}(2j+1)},$$

*thus proving identity (1.37). Using integration by parts (stated and proven below, in (2.191)),*

$$\begin{aligned}
\int_{-1}^{1}\left(x^2-1\right)^j dx &= \int_{-1}^{1}(x-1)^j(x+1)^j dx \\
&= \int_{-1}^{1}(x-1)^j d\left(\frac{(x+1)^{j+1}}{j+1}\right) \\
&= \left[\frac{1}{j+1}(x-1)^j(x+1)^{j+1}\right]_{-1}^{1} - \int_{-1}^{1}\frac{j}{j+1}(x-1)^{j-1}(x+1)^{j+1}dx \\
&= (-1)\frac{j}{j+1}\int_{-1}^{1}(x-1)^{j-1}(x+1)^{j+1}dx.
\end{aligned}$$

*Repeating this,*

$$\begin{aligned}
\int_{-1}^{1}\left(x^2-1\right)^j dx &= (-1)\frac{j}{j+1}\int_{-1}^{1}(x-1)^{j-1}d\left(\frac{(x+1)^{j+2}}{j+2}\right) \\
&= -\left[\frac{j}{(j+1)(j+2)}(x-1)^{j-1}(x+1)^{j+2}\right]_{-1}^{1} \\
&+ (-1)^2\int_{-1}^{1}\frac{j(j-1)}{(j+1)(j+2)}(x-1)^{j-2}(x+1)^{j+2}dx \\
&= (-1)^2\frac{j(j-1)}{(j+1)(j+2)}\int_{-1}^{1}(x-1)^{j-2}(x+1)^{j+2}dx \\
&\vdots \\
&= (-1)^j\frac{j!}{(2j)!/j!}\int_{-1}^{1}(x+1)^{2j}dx \\
&= (-1)^j\frac{1}{\binom{2j}{j}}\left[\frac{(x+1)^{2j+1}}{2j+1}\right]_{-1}^{1} = \frac{(-1)^j 2^{2j+1}}{\binom{2j}{j}(2j+1)}. \qquad \blacksquare
\end{aligned}$$

A generalization of the left hand sides of (1.25) and (1.26) is obtained by relaxing the positive integer constraint on the upper term in the binomial coefficient:

Definition: For $r \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$\binom{r}{k} := \frac{r\,(r-1)\cdots(r-k+1)}{k!}, \qquad \binom{r}{0} := 1. \tag{1.38}$$

The calculations clearly still go through, but the result will, in general, be a real number. Notice that $r$ can be negative, and $k$ can exceed $r$. Listing 1 gives code for computing (1.38).

Theorem: For $n \in \mathbb{N}$,

$$\binom{-n}{k} = (-1)^k\binom{n+k-1}{k}. \tag{1.39}$$

Note that, for $n=1$, this reduces to $(-1)^k$.

```
function c=c(n,k)

if any(n~=round(n)) | any(n<0),  c=cgeneral(n,k); return,  end

vv=find( (n>=k) & (k>=0) );  if length(vv)==0, c=0; return, end
if length(n)==1, nn=n; else nn=n(vv); end
if length(k)==1, kk=k; else kk=k(vv); end

c=zeros(1,max(length(n),length(k)));
t1 = gammaln(nn+1); t2=gammaln(kk+1); t3=gammaln(nn-kk+1);
c(vv)=round( exp ( t1-t2-t3  )   );

function c=cgeneral(nvec,kvec)
% assumes nvec and kvec have equal length and kvec are positive integers.
c=zeros(length(nvec),1);
for i=1:length(nvec)
  n=nvec(i); k=kvec(i);
  p=1; for j=1:k, p=p*(n-j+1); end
  c(i) = p/gamma(k+1);
end
```

**Program Listing 1:** Computes (1.38) for possible vector values of $n$ and $k$.

> Proof: From (1.38),
>
> $$\binom{-n}{k} = \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!} = (-1)^k \frac{(n)(n+1)\cdots(n+k-1)}{k!}$$
> $$= (-1)^k \binom{n+k-1}{k}.$$

This next example gives a useful result using (1.39), but requires using a Taylor series expansion, which we will develop below in §2.6.11.[4]

**Example 1.12** *Let $f(x) = (1-x)^t$, $t \in \mathbb{R}$, and $|x| < 1$. With*

$$f'(x) = -t(1-x)^{t-1}, \quad f''(x) = t(t-1)(1-x)^{t-2},$$

*and, in general, $f^{(j)}(x) = (-1)^j t_{[j]} (1-x)^{t-j}$, the Taylor series expansion (2.325) of $f(x)$ around zero is given by*

$$(1-x)^t = f(x) = \sum_{j=0}^{\infty} (-1)^j t_{[j]} \frac{x^j}{j!} = \sum_{j=0}^{\infty} \binom{t}{j} (-x)^j, \quad |x| < 1, \tag{1.40}$$

*or $(1+x)^t = \sum_{j=0}^{\infty} \binom{t}{j} x^j$, $|x| < 1$. For $t = -1$, (1.40) and (1.39) yield the familiar*

[4] It is inspired from having seen it in my first class in statistics, using the book by Mood, Graybill, Boes, Introduction to the Theory of Statistics, 3rd ed., 1976, p. 533; the latter author having been my instructor.

*$(1-x)^{-1} = \sum_{j=0}^{\infty} x^j$, while for $t=-n$, $n \in \mathbb{N}$, they imply*

$$(1-x)^{-n} = \sum_{j=0}^{\infty} \binom{-n}{j} (-x)^j = \sum_{j=0}^{\infty} \binom{n+j-1}{j} x^j, \quad |x| < 1. \tag{1.41}$$

*Taylor's theorem and properties of the gamma function are used to prove the convergence of these expressions. Some references include Protter and Morrey, 1991, pp. 238-9; Hijab, 1997, p. 91; and Stoll, 2001, Thm. 8.8.4.* ■

We will use (1.41) below in Example 1.13.

Theorem: For $n \in \mathbb{N}$,

$$\binom{2n}{n} = (-1)^n \, 2^{2n} \binom{-\frac{1}{2}}{n}. \tag{1.42}$$

We will need this for proving (7.25).

> Proof: From (1.38), (1.42) follows from
>
> $$\begin{aligned}\binom{-\frac{1}{2}}{n} &= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-n+\frac{1}{2}\right)}{n!} = \left(-\frac{1}{2}\right)^n \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{n!} \\ &= \left(-\frac{1}{2}\right)^n \frac{1}{n!} \frac{(2n)!}{(2n)(2n-2)\cdots 4\cdot 2} = (-1)^n \left(\frac{1}{2}\right)^n \frac{1}{n!} \frac{(2n)!}{2^n \, n!} \\ &= \left(\frac{1}{2}\right)^{2n} (-1)^n \binom{2n}{n}. \end{aligned} \tag{1.43}$$
>
> Numerically checking (always a good idea), for $n=3$, both sides numerically resolve to $-5/16$; while for $n=4$, we get, for both sides, $35/128$. Similarly and easier, from (1.38),
>
> $$(-1)^n \binom{-\frac{1}{2}}{n} = \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(n-\frac{1}{2}\right)}{n!} = \binom{n-\frac{1}{2}}{n}.$$
>
> Indeed, for $n=3$, both sides numerically reduce to $5/16$, while for $n=4$, both sides give $35/128$. Thus, multiplying (1.43) by $(-1)^n$, we also have
>
> $$\binom{n-\frac{1}{2}}{n} = (-1)^n \binom{-\frac{1}{2}}{n} = \left(\frac{1}{2}\right)^{2n} \binom{2n}{n}.$$

**Example 1.13** *Consider proving the identity*

$$\sum_{s=0}^{\infty} \frac{m}{m+s} \binom{m+s}{s} (1-\theta)^s = \theta^{-m}, \quad m \in \mathbb{N}, \quad 0 < \theta < 1. \tag{1.44}$$

*The result for $m=1$ is simple: Recall from (1.27), $\binom{1+s}{s} = \binom{1+s}{1} = 1+s$. Then*

$$\sum_{s=0}^{\infty} \frac{1}{1+s} \binom{1+s}{s} (1-\theta)^s = \sum_{s=0}^{\infty} (1-\theta)^s = \theta^{-1}.$$

*For the general case, observe that*

$$\frac{m}{m+s}\frac{(m+s)!}{m!s!} = \frac{(m+s-1)!}{(m-1)!s!} = \binom{m+s-1}{s} = (-1)^s \binom{-m}{s}$$

*from (1.39). Using this and (1.41) implies that (1.44) is*

$$\sum_{s=0}^{\infty} \binom{-m}{s} \left(-(1-\theta)\right)^s = (1-(1-\theta))^{-m} = \theta^{-m}. \qquad \blacksquare$$

A non-obvious extension of the binomial theorem is

$$(x+y)^{[n]} = \sum_{i=0}^{n} \binom{n}{i} x^{[i]} y^{[n-i]}, \quad x^{[n]} := \prod_{j=0}^{n-1} (x+ja), \tag{1.45}$$

for $n = 0, 1, 2, \ldots$, and $x, y, a$ are real numbers. It holds trivially for $n = 0$, and is easy to see for $n = 1$ and $n = 2$, but otherwise appears difficult to verify, and induction gets messy and doesn't (seem to) lead anywhere. Perhaps somewhat surprisingly, the general proof involves calculus; it is proven below in (1.70). Taking $a = -1$ results in a very important special case. Assuming for now the validity of (1.45), we obtain the following. (Paolella, Fundamental Probability, gives three direct proofs of (1.46).)

Theorem (Vandermonde): For $x, y, n \in \mathbb{N}$,

$$\binom{x+y}{n} = \sum_{i=0}^{n} \binom{x}{i} \binom{y}{n-i}. \tag{1.46}$$

We require this for the next example; but more relevantly, it is the justification that the probability mass function of a hypergeometric random variable indeed sums to one.

> Proof: With $a = -1$,
>
> $$k^{[n]} = (k)(k-1)(k-2)\cdots(k-(n-1)) = \frac{k!}{(k-n)!} = \binom{k}{n} n!,$$
>
> so that (1.45) yields, with $k = x + y$,
>
> $$k^{[n]} = \binom{x+y}{n} n! = \sum_{i=0}^{n} \binom{n}{i} \binom{x}{i} i! \binom{y}{n-i} (n-i)! = n! \sum_{i=0}^{n} \binom{x}{i} \binom{y}{n-i}.$$
>
> Cancelling the $n!$ yields the result.

**Example 1.14** *We wish to prove*

$$\binom{r_1+r_2+y-1}{y} = \sum_{i=0}^{y} \binom{r_1+i-1}{i} \binom{r_2+y-i-1}{y-i}, \tag{1.47}$$

*which we will invoke below for proving (1.70). From (1.39), it follows that*

$$\binom{r_1+i-1}{i} = (-1)^i \binom{-r_1}{i} \quad \textit{and} \quad \binom{r_2+y-i-1}{y-i} = (-1)^{y-i} \binom{-r_2}{y-i},$$

*so that the rhs of the desired equation is*

$$S = \sum_{i=0}^{y} \binom{-r_1}{i} \binom{-r_2}{y-i} (-1)^y$$
$$\overset{(1.46)}{=} (-1)^y \binom{-(r_1+r_2)}{y} \overset{(1.39)}{=} \binom{r_1+r_2+y-1}{y}. \qquad \blacksquare$$

Definition: If a set of $n$ distinct objects is to be divided into $k$ distinct groups, whereby the size of each group is $n_i, i = 1, \ldots, k$, and $\sum_{i=1}^{k} n_i = n$, then the number of possible divisions is given by

$$\binom{n}{n_1, n_2, \ldots, n_k} := \binom{n}{n_1} \binom{n-n_1}{n_2} \binom{n-n_1-n_2}{n_3} \cdots \binom{n_k}{n_k} = \frac{n!}{n_1! n_2! \cdots n!}.$$

Note how this reduces to the familiar combinatoric when $k = 2$.

Theorem (Multinomial Theorem): For $r, n \in \mathbb{N}$,

$$\left(\sum_{i=1}^{r} x_i\right)^n = \sum_{\mathbf{n}:\, n_\bullet = n,\, n_i \geq 0} \binom{n}{n_1, \ldots, n_r} \prod_{i=1}^{r} x_i^{n_i}, \tag{1.48}$$

where $\mathbf{n}$ denotes the vector $(n_1, \ldots, n_r)$, and $n_\bullet = \sum_{i=1}^{r} n_i$.

In words, the sum is taken over all nonnegative integer solutions to $\sum_{i=1}^{r} n_i = n$, the number of which $\binom{n+r-1}{n}$, obtained using the usual "stars and bars" trick; see, e.g., Paolella, Fundamental Probability, Ch. 2. With $n = 2$, this is just $(\sum_{i=1}^{r} x_i)^2 = \sum_{i=1}^{r} \sum_{j=1}^{r} x_i x_j$.

**Example 1.15** *The expression $(x_1 + x_2 + x_3)^4$ corresponds to $r = 3$ and $n = 4$, so that its expansion will have $\binom{6}{4} = 15$ terms, starting with*

$$\begin{aligned}(x_1+x_2+x_3)^4 &= \binom{4}{4,0,0} x_1^4 x_2^0 x_3^0 + \binom{4}{0,4,0} x_1^0 x_2^4 x_3^0 + \binom{4}{0,0,4} x_1^0 x_2^0 x_3^4 \\ &\quad + \binom{4}{3,1,0} x_1^3 x_2^1 x_3^0 + \binom{4}{3,0,1} x_1^3 x_2^0 x_3^1 + \cdots,\end{aligned}$$

*or in full (and obtained fast and reliably from a symbolic computing environment)*

$$\begin{aligned}(x_1+x_2+x_3)^4 &= x_1^4 + x_2^4 + x_3^4 \\ &\quad + 4x_1^3 x_2 + 4x_1^3 x_3 + 4x_1 x_2^3 + 4x_1 x_3^3 + 4x_2^3 x_3 + 4x_2 x_3^3 \\ &\quad + 6x_1^2 x_2^2 + 6x_1^2 x_3^2 + 6x_2^2 x_3^2 \\ &\quad + 12x_1^2 x_2 x_3 + 12x_1 x_2^2 x_3 + 12x_1 x_2 x_3^2, \qquad \blacksquare\end{aligned}$$

The proof of (1.48) is by induction on $r$. For $r = 1$, the theorem clearly holds. Assuming it holds for $r = k$, observe that, with $S = \sum_{i=1}^{k} x_i$,

$$\begin{aligned}\left(\sum_{i=1}^{k+1} x_i\right)^n &= (S + x_{k+1})^n = \sum_{i=0}^{n} \frac{n!}{i!\,(n-i)!} x_{k+1}^i S^{n-i} \\ &= \sum_{i=0}^{n} \frac{n!}{i!\,(n-i)!} \sum_{n_1+\cdots+n_k=n-i} \frac{(n-i)!}{n_1! \cdots n_k!} \prod_{i=1}^{k} x_i^{n_i}\, x_{k+1}^i,\end{aligned}$$

from the binomial theorem and (1.48) for $r = k$. By setting $n_{k+1} = i$, this becomes

$$\left(\sum_{i=1}^{k+1} x_i\right)^n = \sum_{n_1+\cdots+n_{k+1}=n} \frac{n!}{n_1!\cdots n_{k+1}!} \prod_{i=1}^{k} x_i^{n_i}\, x_{k+1}^i,$$

as the sum $\sum_{i=0}^{n} \sum_{n_1+\cdots+n_k=n-i}$ is equivalent to $\sum_{n_1+\cdots+n_{k+1}=n}$ for nonnegative $n_i$. This is precisely (1.48) for $r = k+1$, proving the theorem.

**Example 1.16** *Using the power series expression of the exponential function*

$$f(x) = \exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \tag{1.49}$$

*we wish to attempt to show that* $[\exp(x)]^n = \exp(nx)$. *Applying the multinomial theorem* (1.48) *to* (1.49), *and using* (2.48), *i.e., that* $[f(x)]^n$ *is a composition of two continuous functions, thus allowing the passing of the limit, gives*

$$[f(x)]^n = \left(\lim_{r\to\infty} \sum_{s=0}^{r} \frac{x^s}{s!}\right)^n = \lim_{r\to\infty} \left(\sum_{s=0}^{r} \frac{x^s}{s!}\right)^n. \tag{1.50}$$

*Consider first the case with* $r = 2$. *Because the sum over* $s$ *starts with a zero instead of a one, we use the terms* $n_0, n_1, \ldots,$ *instead of starting with* $n_1$. *We begin by explicitly isolating the terms that give rise to* $x^0$, $x^1$, *and* $x^2$ *in the expansion. In particular, as* $1^{n_0} = 1$ *for* $n_0 \geq 1$, *we seek the terms of the form* $x^{n_1}$ *and* $(x^2)^{n_2}$ *such that we get products of the form* $x^0$, $x^1$, *and* $x^2$. *We let* $k$ *indicate the power of* $x$ *of this product, so the term with* $k = 0$ *will be* $1^{n_0} x^{n_1} (x^2)^{n_2}$ *with* $n_0 = n$, $n_1 = n_2 = 0$; *the* $k = 1$ *term will be* $1^{n_0} x^{n_1} (x^2)^{n_2}$ *with* $n_0 = n-1$, $n_1 = 1$, $n_2 = 0$; *and finally, for* $k = 2$, $1^{n_0} x^{n_1} (x^2)^{n_2}$ *with either* $n_0 = n-1$, $n_1 = 0$, $n_2 = 1$, *or* $n_0 = n-2$, $n_1 = 2$, $n_2 = 0$. *The expansion, showing explicitly the* $k = 0, 1, 2$ *cases, is*

$$\begin{aligned}
\left(1 + x + \frac{x^2}{2}\right)^n &= \sum_{\mathbf{n}:\, n_\bullet = n,\, n_i \geq 0} \binom{n}{n_0, n_1, n_2} 1^{n_0} \left(\frac{x}{1!}\right)^{n_1} \left(\frac{x^2}{2!}\right)^{n_2} \\
&= \binom{n}{n, 0, 0} 1^n \left(\frac{x}{1!}\right)^0 \left(\frac{x^2}{2!}\right)^0 + \binom{n}{n-1, 1, 0} 1^{n-1} \left(\frac{x}{1!}\right)^1 \left(\frac{x^2}{2!}\right)^0 \\
&\quad + \binom{n}{n-1, 0, 1} 1^{n-1} \left(\frac{x}{1!}\right)^0 \left(\frac{x^2}{2!}\right)^1 + \binom{n}{n-2, 2, 0} 1^{n-2} \left(\frac{x}{1!}\right)^2 \left(\frac{x^2}{2!}\right)^0 \\
&\quad + \sum_{n_1 + 2n_2 = 3} \binom{n}{n_0, n_1, n_2} 1^{n_0} \left(\frac{x}{1!}\right)^{n_1} \left(\frac{x^2}{2!}\right)^{n_2} + \cdots.
\end{aligned}$$

*The last line gathers terms that give rise to powers of* $x$ *of* $k = 3$, *i.e., we require, as always,* $n_0 + n_1 + n_2 = n$, *and also* $n_1 + 2n_2 = 3$. *The remaining terms not shown are for* $k = 4$, $k = 5$, *etc.. Simplifying, we get*

$$\begin{aligned}
\left(1 + x + \frac{x^2}{2}\right)^n &= 1 + \frac{n!}{(n-1)!} x + \frac{n!}{(n-1)!} \frac{x^2}{2} + \frac{n!}{(n-2)!2!} x^2 + \cdots + C_k x^k + \cdots \\
&= 1 + nx + \frac{nx^2}{2} + \frac{n(n-1)}{2} x^2 + \cdots \\
&= 1 + (nx) + \frac{(nx)^2}{2} + \cdots,
\end{aligned}$$

*which indeed begins to look like* $f\,(nx)$ *from* (1.49).

*Of course, more work is required to actually determine that*

$$C_k = \sum_{n_1+2n_2+\cdots rn_r=k} \binom{n}{n_0,\ldots,n_r} \prod_{i=1}^{r} (i!)^{-n_i} = \frac{n^k}{k!}.$$

*This might be amenable to induction; we do not attempt it here. For* $k=3$*, the solutions to* $n_1+2n_2+\cdots rn_r=3$ *(with* $n_0$ *such that* $\sum_{i=0}^{r} n_r = n$*) are* $(n_0,\ldots,n_r) = (n_0,3,0,\ldots,0)$, $(n_0,1,1,0,\ldots,0)$*, and* $(n_0,0,0,1,0,\ldots,0)$*. Thus,*

$$C_3 = \frac{n!}{(n-3)!3!}\frac{1}{1^3} + \frac{n!}{(n-2)!}\frac{1}{1^1}\frac{1}{2^1} + \frac{n!}{(n-1)!}\frac{1}{6^1} = \frac{n^3}{3!},$$

*suggesting this method of proof is viable.* ■

## 1.4 Gamma and Beta Functions

> Young people today love luxury. They have bad manners, despise authority, have no respect for older people, and chatter when they should be working.
>
> (Socrates, 470–399 BC)

There exist (infinitely many) elementary functions $f$ such that there is no elementary function $F(x)$ satisfying $F'(x) = f(x)$. Important (because of their application to real problems) examples of such functions $f$ include the gamma function discussed here, the beta function discussed below, and the Gauss error function $\exp(x^2)$. Perhaps surprisingly, it is true also for $e^x/x$ and $1/\ln x$.

The gamma function can be expressed as

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt, \quad x \in \mathbb{R}_{>0}. \tag{1.51}$$

Being an improper integral, we need to confirm its existence. This requires use of the comparison test for improper integrals, given in (2.232):

> Proof: In view of the inequality $t^{x-1}e^{-t} \leq t^{x-1}$ for $t > 0$, the existence of the integral $\int_0^1 t^{x-1}e^{-t}dt$ follows from that of $\int_0^1 t^{x-1}dt$, provided that $x > 0$. Next, since $t^{x+1}e^{-t} \to 0$ as $t \to +\infty$, we have, for some $H > 0$, $t^{x-1}e^{-t} \leq Ht^{-2}$ for $t \geq 1$. Hence, the existence of $\int_1^\infty t^{x-1}e^{-t}dt$ follows from that of $\int_1^\infty t^{-2}dt$.
>
> The convergence of the improper integral from 1 to infinity is also addressed in Example 2.62.

As mentioned above, an expression for $\Gamma(x)$ in terms of "elementary" functions does not exist for general $x$. However, a basic integration by parts (see (2.191)) shows that

$$\Gamma(x) = (x-1)\Gamma(x-1), \quad x \in \mathbb{R}_{>1}. \tag{1.52}$$

Thus, for $n \in \mathbb{N}$,

$$\Gamma(n) = (n-1)!. \tag{1.53}$$

As in Andrews, Askey and Roy (1999, pp. 2-3; see also p. 35), suppose that $x \geq 0$ and $n \geq 0$ are integers. Write

$$x! = \frac{(x+n)!}{(x+1)_n}, \tag{1.54}$$

where $(a)_n$ denotes the rising factorial (using Pochhammer's notation; see above)

$$(a)_n = a(a+1)\cdots(a+n-1) \text{ for } n > 0, \quad (a)_0 = 1,$$

and $a$ is any real number. Rewrite (1.54) as

$$x! = \frac{n!(n+1)_x}{(x+1)_n} = \frac{n!n^x}{(x+1)_n} \cdot \frac{(n+1)_x}{n^x}.$$

Since

$$\lim_{n\to\infty} \frac{(n+1)_x}{n^x} = 1,$$

we conclude that

$$x! = \lim_{n\to\infty} \frac{n!n^x}{(x+1)_n}.$$

This, along with (1.53), is (also) used to *define* the gamma function as

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)}, \quad x > 0, \tag{1.55}$$

known as the Gauss product formula. The equivalence of (1.51) and (1.55) is proven in Appendix 7.1. We will require (1.55) below in Example 7.3. Appendix §7.1 contains further results on the gamma (and beta) functions. We will also require the identity (we will need it directly below, and for deriving the important relationship (1.64) between the gamma and beta functions)

$$\Gamma(a) = 2\int_0^\infty u^{2a-1}e^{-u^2}du. \tag{1.56}$$

This follows directly by using the substitution $u = x^{1/2}$ in (1.51) (recall $x$ is positive), so that $x = u^2$ and $dx = 2udu$. Another useful fact that follows from (1.56) and Example 6.21 (and letting, say, $v = \sqrt{2}u$) is that

$$\Gamma(1/2) = \sqrt{\pi}, \tag{1.57}$$

which we will use often.

**Example 1.17** *Recall the Gaussian probability density function. For $Z \sim \mathrm{N}(0,1)$, we wish to compute the even positive moments, $\mathbb{E}[Z^{2r}]$, for $r \in \mathbb{N}$. With $u = z^2/2$, $z = (2u)^{1/2}$ (because $z$ is positive), and $dz = (2u)^{-1/2}du$,*

$$\begin{aligned}\mathbb{E}\left[Z^{2r}\right] &= \int_{-\infty}^{\infty} z^{2r} f_Z(z)\,dz = \frac{2}{\sqrt{2\pi}}\int_0^\infty z^{2r}e^{-\frac{1}{2}z^2}dz \\ &= \frac{2^{r+1-1/2}}{\sqrt{2\pi}}\int_0^\infty u^{r-1/2}e^{-u}du = \frac{2^r\Gamma\left(r+\frac{1}{2}\right)}{\sqrt{\pi}}.\end{aligned}$$

*That is, for $s = 2r$ and recalling that $\Gamma(a+1) = a\Gamma(a)$ and $\Gamma(1/2) = \sqrt{\pi}$,*

$$\mathbb{E}[Z^s] = \frac{1}{\sqrt{\pi}}2^{s/2}\Gamma\left(\frac{1}{2}(1+s)\right) = (s-1)(s-3)(s-5)\cdot\cdots\cdot 3\cdot 1. \tag{1.58}$$

*This can also be written*

$$\mathbb{E}[Z^s] = \mathbb{E}\left[Z^{2r}\right] = \frac{(2r)!}{2^r r!}, \tag{1.59}$$

*which follows because (in the numerator, note $(2r)! = (2r)(2r-1)(2r-2)!$)*

$$M(r) := \frac{(2r)!}{2^r r!} = \left[\frac{(2r-1)\,2r}{2r}\right]\frac{(2(r-1))!}{2^{r-1}(r-1)!} = (2r-1)\,M(r-1),$$

*e.g.,*

$$\mathbb{E}[Z^6] = \mathbb{E}[Z^{2\cdot 3}] = M(3) = 5\cdot M(2) = 5\cdot 3\cdot M(1) = 5\cdot 3\cdot 1.$$

*With $X = \sigma Z + \mu \sim \mathrm{N}(\mu, \sigma^2)$,*

$$\mathbb{E}\left[(X-\mu)^{2r}\right] = \sigma^{2r}\mathbb{E}\left[Z^{2r}\right] = \left(2\sigma^2\right)^r\pi^{-1/2}\Gamma\left(r+\frac{1}{2}\right). \tag{1.60}$$

*(The reader should directly check that (1.60) reduces to $3\sigma^4$ for $r = 2$.) An expression for the even raw moments of $X$ can be obtained via (1.59) and the binomial formula applied to $(\sigma Z + \mu)^{2r}$.*

*For odd moments, similar calculations give*[5]

$$\begin{aligned}\int_{-\infty}^{\infty} z^{2r+1} f_Z(z)\,dz &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} z^{2r+1} e^{-\frac{1}{2}z^2}dz + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} z^{2r+1} e^{-\frac{1}{2}z^2}dz \\ &= -\frac{2^r\Gamma(r+1)}{\sqrt{2\pi}} + \frac{2^r\Gamma(r+1)}{\sqrt{2\pi}} = 0.\end{aligned}$$

*Thus, for example, the skewness and kurtosis of $X = \sigma Z + \mu$ is zero and three, respectively, recalling that those measures are location and scale invariant.*

*To calculate $\mathbb{E}|Z| := \mathbb{E}\big[|Z|\big]$, use the same $u$ substitution as above to give*

$$\begin{aligned}\mathbb{E}|Z| &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |z|\, f_Z(z)\,dz = \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} z e^{-\frac{1}{2}z^2}dz \\ &= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} (2u)^{1/2} e^{-u} (2u)^{-1/2} du = \sqrt{\frac{2}{\pi}},\end{aligned} \tag{1.61}$$

*where $\int_0^\infty e^{-u}du = 1$.* ■

The beta function is an integral expression of two parameters, denoted $B(\cdot,\cdot)$ and defined to be

$$B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}\,dx, \quad a,b \in \mathbb{R}_{>0}. \tag{1.62}$$

By substituting $x = \sin^2\theta$ into (1.62) we obtain (and as directly used below) that

$$B(a,b) = \int_0^{\pi/2} \left(\sin^2\theta\right)^{a-1}\left(\cos^2\theta\right)^{b-1} 2\sin\theta\cos\theta d\theta = 2\int_0^{\pi/2} (\sin\theta)^{2a-1}(\cos\theta)^{2b-1}\,d\theta. \tag{1.63}$$

Closed-form expressions do not exist for general $a$ and $b$; however, the identity

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{1.64}$$

can be used for its evaluation in terms of the gamma function. There are several ways of proving this. Here is one. Using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$ (see (6.42) in §6.6) along with (1.56) and (1.63),

$$\begin{aligned}\Gamma(a)\Gamma(b) &= 4\int_0^\infty\int_0^\infty x^{2a-1}y^{2b-1}e^{-\left(x^2+y^2\right)}dxdy \\ &= 4\int_0^{2\pi}\int_0^\infty r^{2(a+b)-2+1}e^{-r^2}(\cos\theta)^{2a-1}(\sin\theta)^{2b-1}drd\theta \\ &= 4\left(\int_0^\infty r^{2(a+b)-1}e^{-r^2}dr\right)\left(\int_0^{2\pi}(\cos\theta)^{2a-1}(\sin\theta)^{2b-1}d\theta\right) \\ &= \Gamma(a+b)\,B(a,b).\end{aligned}$$

[5] It should be clear from the symmetry of $f_Z$ that $\int_{-k}^{k} z^{2r+1} f_Z(z)\,dz = 0$ for any $k > 0$ (see §2.5.3). Recall also from §2.5.3 that, for a general density $f_X$, in order to claim that $\int_{-\infty}^{\infty} x^{2r+1} f_X(x)\,dx = 0$, both $\lim_{k\to\infty}\int_0^k x^{2r+1} f_X(x)\,dx$ and $-\lim_{k\to\infty}\int_{-k}^0 x^{2r+1} f_X(x)\,dx$ must converge to the same finite value.

A direct proof without the use of polar coordinates can be found in Hijab (1997, p. 193). If $a = b$, then, from symmetry (or use the substitution $y = 1 - x$) and use of (1.64), it follows that

$$\int_0^{1/2} x^{a-1}(1-x)^{a-1}\,dx = \int_{1/2}^{1} x^{a-1}(1-x)^{a-1}\,dx = \frac{1}{2}\frac{\Gamma^2(a)}{\Gamma(2a)}, \tag{1.65}$$

where $\Gamma^2(a)$ is just a shorthand notation for $[\Gamma(a)]^2$. We will use this result now:

Theorem (Legendre's duplication formula):

$$\Gamma(2a) = \frac{2^{2a-1}}{\sqrt{\pi}}\Gamma(a)\Gamma\left(a+\frac{1}{2}\right). \tag{1.66}$$

Proof: Use (1.65) with $u = 4x(1-x)$ (and, as $0 \le x \le 1/2$, $x = \left(1-\sqrt{1-u}\right)/2$ and $dx = 1/(4\sqrt{1-u})du$) to get

$$\begin{aligned}\frac{\Gamma^2(a)}{\Gamma(2a)} &= 2\int_0^{1/2} x^{a-1}(1-x)^{a-1}\,dx = \frac{2}{4^{a-1}}\int_0^{1/2}(4x(1-x))^{a-1}\,dx \\ &= \frac{2}{4^{a-1}}\int_0^1 u^{a-1}\frac{1}{4}(1-u)^{-1/2}\,du = 2^{1-2a}\frac{\Gamma(a)\Gamma(1/2)}{\Gamma(a+1/2)}.\end{aligned}$$

As $\Gamma(1/2) = \sqrt{\pi}$, the result follows.

**Example 1.18** *From Legendre's duplication formula (1.66) with $i \in \mathbb{N}$ and using (1.28), we obtain (but also note that the result follows directly from (1.53) and (1.57))*

$$\begin{aligned}\Gamma\left(i+\frac{1}{2}\right) &= \frac{\sqrt{\pi}\Gamma(2i)}{2^{2i-1}\Gamma(i)} = \frac{\sqrt{\pi}}{2^{2i-1}}\frac{(2i-1)!}{(i-1)!} = \frac{\sqrt{\pi}}{2^{2i-1}}\frac{i}{2i}\frac{(2i)!}{i!} \\ &= \frac{\sqrt{\pi}}{2^{2i}}\frac{(2i)!}{i!} = \frac{\sqrt{\pi}}{2^{2i}}2^i C(2i-1) \\ &= \frac{1\cdot 3\cdot 5\cdots(2i-1)}{2^i}\sqrt{\pi},\end{aligned} \tag{1.67}$$

*which is required, for example, when deriving properties of the noncentral Student's $t$ and related distributions; see, e.g., Paolella, Intermediate Probability.* ■

**Example 1.19** *To express $\int_0^1 \sqrt{1-x^4}\,dx$ in terms of the beta function, let $u = x^4$ and $dx = (1/4)u^{1/4-1}\,du$, so that*

$$\int_0^1 \sqrt{1-x^4}\,dx = \frac{1}{4}\int_0^1 u^{-3/4}(1-u)^{1/2}\,du = \frac{1}{4}B\left(\frac{1}{4},\frac{3}{2}\right). \quad ■$$

**Example 1.20** *To compute*

$$I = \int_0^s x^a(s-x)^b\,dx, \qquad s \in (0,1), \quad a,b > 0,$$

*use $u = 1 - x/s$ (so that $x = (1-u)s$ and $dx = -s\,du$) to get*

$$\begin{aligned}I &= \int_0^s x^a(s-x)^b\,dx = -s\int_1^0 ((1-u)s)^a(s-(1-u)s)^b\,du \\ &= s^{a+b+1}\int_0^1 (1-u)^a u^b\,du = s^{a+b+1}B(b+1,a+1). \quad ■\end{aligned}$$

**Example 1.21** *To compute*

$$I=\int_{-1}^{1}\left(1-x^2\right)^a(1-x)^b\,dx,$$

*use* $1-x^2=(1-x)(1+x)$ *and* $u=(1+x)/2$ $(x=2u-1,\ dx=2du)$ *to get*

$$I=2^{2a+b+1}\int_0^1 u^a(1-u)^{a+b}\,du=2^{2a+b+1}B(a+1,a+b+1).\qquad\blacksquare$$

**Example 1.22** *The moment generating function (m.g.f.) of a location-zero, scale-one logistic random variable is (with* $y=(1+\mathrm{e}^{-x})^{-1}$*), for* $|t|<1$,

$$\begin{aligned}\mathbb{M}_X(t)=\mathbb{E}\left[e^{tX}\right]&=\int_{-\infty}^{\infty}\left(e^{-x}\right)^{1-t}\left(1+e^{-x}\right)^{-2}dx\\&=\int_0^1\left(\frac{1-y}{y}\right)^{1-t}y^2y^{-1}(1-y)^{-1}dy=\int_0^1(1-y)^{-t}y^tdy\\&=B(1-t,1+t)=\Gamma(1-t)\Gamma(1+t).\end{aligned}$$

*If, in addition,* $t\neq 0$*, the m.g.f. can also be expressed as*

$$\mathbb{M}_X(t)=t\Gamma(t)\Gamma(1-t)=t\frac{\pi}{\sin\pi t},\tag{1.68}$$

*where the second identity is called* Euler's reflection formula*: Andrews, Askey and Roy (1999, pp. 9-10) provide four different methods for proving Euler's reflection formula; see also Jones (2001, pp. 217-18), Havil (2003, p. 59), Schiff (1999, p. 174), and Duren, Invitation to Classical Analysis, 2012, §9.5. Notice also, from (1.68) with* $t=1/2$*, it follows that* $\Gamma(1/2)=\sqrt{\pi}$. $\blacksquare$

**Example 1.23** *An interesting relation both theoretically and computationally is given by*

$$\sum_{j=k}^{n}\binom{n}{j}p^j(1-p)^{n-j}=\frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)}\int_0^p x^{k-1}(1-x)^{n-k}\,dx,\tag{1.69}$$

*for* $0\le p\le 1$ *and* $k=1,2,\ldots,$ *where* $\binom{n}{j}$ *is a binomial coefficient, and can be proven by repeated integration by parts. To motivate this, take* $k=1$*. From the binomial theorem (1.34) with* $x=p=1-y$*, it follows directly that the lhs of (1.69) is* $1-(1-p)^n$*. The rhs is, with* $y=1-x$,

$$\frac{n!}{(n-1)!}\int_0^p(1-x)^{n-1}\,dx=-n\int_1^{1-p}y^{n-1}dy=\left.y^n\right|_{1-p}^{1}=1-(1-p)^n.$$

*For* $k=2$*, the lhs of (1.69) is easily seen to be* $1-(1-p)^n-np(1-p)^{n-1}$*, while the rhs is, using* $y=1-x$,

$$\begin{aligned}\frac{n!}{1!\,(n-2)!}\int_0^p x(1-x)^{n-2}\,dx&=-n(n-1)\int_1^{1-p}(1-y)\,y^{n-2}dy\\&=n(n-1)\left[\left.\frac{y^{n-1}}{n-1}\right|_{1-p}^{1}-\left.\frac{y^n}{n}\right|_{1-p}^{1}\right]\\&=1-np(1-p)^{n-1}-(1-p)^n,\end{aligned}$$

*after some rearranging.* $\blacksquare$

**Example 1.24** *By substituting $2n-1$ for $n$; $n$ for $k$; and taking $p=1/2$ in (1.69), we get*

$$\sum_{i=n}^{2n-1}\binom{2n-1}{i}\left(\frac{1}{2}\right)^{2n-1}=\frac{\Gamma(2n)}{\Gamma^2(n)}\int_0^{1/2}u^{n-1}(1-u)^{n-1}\,du=\frac{1}{2}$$

*from (1.65), directly showing (1.30) in Example 1.7.* ■

Theorem (Binomial Theorem Extension): For $n=0,1,2,\ldots$, and $x,y,a\in\mathbb{R}$, define

$$x^{[n]}:=\prod_{j=0}^{n-1}(x+ja).$$

Then

$$(x+y)^{[n]}=\sum_{i=0}^{n}\binom{n}{i}x^{[i]}y^{[n-i]}, \tag{1.70}$$

as was first stated above in (1.45), without proof.

Proof: As

$$\begin{aligned}\prod_{j=0}^{k}(x+ja) &= (x)(x+a)\cdots(x+ka)\\ &= a^{k+1}\left(\frac{x}{a}\right)\left(\frac{x}{a}+1\right)\cdots\left(\frac{x}{a}+k\right)=a^{k+1}\frac{\Gamma(k+1+x/a)}{\Gamma(x/a)},\end{aligned}$$

(1.70) can be expressed as the conjecture

$$\begin{aligned}(x+y)^{[n]} &\stackrel{?}{=} \sum_{i=0}^{n}\binom{n}{i}x^{[i]}y^{[n-i]}\\ \prod_{j=0}^{n-1}(x+y+ja) &\stackrel{?}{=} \sum_{i=0}^{n}\binom{n}{i}\left(\prod_{j=0}^{i-1}(x+ja)\right)\left(\prod_{j=0}^{n-i-1}(y+ja)\right)\\ a^n\frac{\Gamma(n+(x+y)/a)}{\Gamma((x+y)/a)} &\stackrel{?}{=} \sum_{i=0}^{n}\binom{n}{i}\left(a^i\frac{\Gamma(i+x/a)}{\Gamma(x/a)}\right)\left(a^{n-i}\frac{\Gamma(n-i+y/a)}{\Gamma(y/a)}\right)\end{aligned}$$

or

$$\frac{\Gamma(n+(x+y)/a)}{\Gamma((x+y)/a)}\stackrel{?}{=}\sum_{i=0}^{n}\binom{n}{i}\frac{\Gamma(i+x/a)}{\Gamma(x/a)}\frac{\Gamma(n-i+y/a)}{\Gamma(y/a)}$$

or

$$\frac{\Gamma(x/a)\Gamma(y/a)}{\Gamma((x+y)/a)}\stackrel{?}{=}\sum_{i=0}^{n}\binom{n}{i}\frac{\Gamma(i+x/a)\Gamma(n-i+y/a)}{\Gamma(n+(x+y)/a)}, \tag{1.71}$$

or, equivalently, using (1.64),

$$B\left(\frac{x}{a},\frac{y}{a}\right)\stackrel{?}{=}\sum_{i=0}^{n}\binom{n}{i}B\left(\frac{x}{a}+i,\frac{y}{a}+n-i\right). \tag{1.72}$$

We now need to prove (1.71) and (1.72). This will be done using results from probability theory. In turn, this proves (1.70).

Let $X_i \overset{\text{ind}}{\sim} \operatorname{Gam}(a_i, c)$, $i = 1, 2$, and define $S = X_1 + X_2$, which follows a $\operatorname{Gam}(a_1 + a_2, c)$ distribution. The linearity of expectation, and the binomial theorem, imply

$$\mathbb{E}\big[S^k\big] = \mathbb{E}\big[(X_1 + X_2)^k\big] = \sum_{i=0}^{k} \binom{k}{i} \mathbb{E}\left[X_1^i\right] \mathbb{E}\big[X_2^{k-i}\big],$$

or, using that, for $X \sim \operatorname{Gam}(\alpha, \beta)$, $\mathbb{E}\left[X^k\right] = \frac{\Gamma(k+\alpha)}{\beta^k \Gamma(\alpha)}$ for $k > -\alpha$,

$$\frac{\Gamma(k + a_1 + a_2)}{c^k \Gamma(a_1 + a_2)} = \sum_{i=0}^{k} \binom{k}{i} \frac{\Gamma(i + a_1)}{c^i \Gamma(a_1)} \frac{\Gamma(k - i + a_2)}{c^{k-i} \Gamma(a_2)}. \tag{1.73}$$

That is, noting the $c$-terms cancel,

$$\frac{(k + a_1 + a_2 - 1)!}{(a_1 + a_2 - 1)! k!} = \sum_{i=0}^{k} \frac{(i + a_1 - 1)!}{i! (a_1 - 1)!} \frac{(k - i + a_2 - 1)!}{(k - i)! (a_2 - 1)!},$$

or

$$\binom{k + a_1 + a_2 - 1}{k} = \sum_{i=0}^{k} \binom{i + a_1 - 1}{i} \binom{k - i + a_2 - 1}{k - i},$$

which is precisely (1.47). Rearranging (1.73) gives (1.71), i.e.,

$$\frac{\Gamma(a_1) \Gamma(a_2)}{\Gamma(a_1 + a_2)} = \sum_{i=0}^{k} \binom{k}{i} \frac{\Gamma(i + a_1) \Gamma(k - i + a_2)}{\Gamma(k + a_1 + a_2)}.$$

Using (1.64), this can be expressed as

$$B(a_1, a_2) = \sum_{i=0}^{k} \binom{k}{i} B(a_1 + i, a_2 + k - i),$$

which gives (1.72). The latter result can also be obtained, faster, by letting $X \sim \operatorname{Beta}(a_1, a_2)$. In particular, and using the binomial theorem,

$$\begin{aligned}
1 &= \int_0^1 f_X(x)\, dx = \frac{1}{B(a_1, a_2)} \int_0^1 (x + 1 - x)^k x^{a_1 - 1} (1 - x)^{a_2 - 1}\, dx \\
&= \frac{1}{B(a_1, a_2)} \sum_{i=0}^{k} \binom{k}{i} \int_0^1 x^{a_1 + i - 1} (1 - x)^{a_2 - 1 + k - i}\, dx \\
&= \frac{1}{B(a_1, a_2)} \sum_{i=0}^{k} \binom{k}{i} B(a_1 + i, a_2 + k - i).
\end{aligned}$$

# 2 Univariate Calculus

Leibniz never married; he had considered it at the age of fifty; but the person he had in mind asked for time to reflect. This gave Leibniz time to reflect, too, and so he never married. (Bernard Le Bovier Fontenelle)

## 2.1 Sequences and Limits

We begin with some basic definitions and results associated with sequences of real numbers.

Definition: A *sequence* is a function $f : \mathbb{N} \to \mathbb{R}$, with $f(n)$, $n \in \mathbb{N}$, being the $n$th *term* of $f$. We often denote the sequence of $f$ as $\{a_n\}$, where $a_n = f(n)$.

We will use the following notation. If $\{a_k\}$ is a sequence such that, for all $k \in \mathbb{N}$, $a_k \in A \subset \mathbb{R}$, then we say that $\{a_k\}$ is a sequence in $A$, or, in short, $\{a_k\} \subset A$.

Definition: The sequence $\{a_k\} \subset \mathbb{R}$ *converges* to $a \in \mathbb{R}$ if:

$$\forall \epsilon > 0, \ \exists K \in \mathbb{N} \ \text{ such that } \ \forall k > K, \ |a_k - a| < \epsilon. \tag{2.1}$$

Point $a$ is the *limit* of $\{a_k\}$ if $\{a_k\}$ converges to $a$, in which case one writes $\lim_{k\to\infty} a_k = a$. If $\{a_n\}$ does not converge, then it is said to *diverge*.

Definition: Sequence $\{s_n\}$ is *strictly increasing* if $s_{n+1} > s_n$, and *increasing* if $s_{n+1} \geq s_n$. The sequence is *bounded from above* if $\exists c \in \mathbb{R}$ such that $s_n \leq c$ for all $n$. Similar definitions apply to *decreasing*, *strictly decreasing*, and *bounded from below*.

Theorem: A convergent sequence has a unique limit.

> Proof: Suppose $a_n \to a$ as well as $a_n \to b$. If $b \neq a$, let $\epsilon := |a - b|$. Since $a_n \to a$, there is $n_1 \in \mathbb{N}$ such that $|a_n - a| < \epsilon/2$ for all $n \geq n_1$, and since $a_n \to b$, there is $n_2 \in \mathbb{N}$ such that $|a_n - b| < \epsilon/2$ for all $n \geq n_2$. Let $n_0 := \max\{n_1, n_2\}$. Then
>
> $$|a - b| \leq |a - a_{n_0}| + |a_{n_0} - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = |a - b|. \tag{2.2}$$
>
> This contradiction shows that $b = a$.

Theorem:

$$\text{Every convergent sequence is bounded.} \tag{2.3}$$

> Proof: Recall from the definition that a sequence is a function $f : \mathbb{N} \to \mathbb{R}$, i.e., $f(n) \in \mathbb{R}$, as opposed to the extended real line, i.e., $f(n)$ cannot be plus or minus infinity. Let $\{a_n\}$ be a sequence that converges to the number $a$. Taking $\epsilon = 1$, it follows from the definition of convergence that we can select an index $N$ such that $|a_n - a| < 1$ for all indices $n \geq N$. Observe that we have $a_n = (a_n - a) + a$, so that by the Triangle Inequality, $|a_n| = |(a_n - a) + a| \leq |a_n - a| + |a|$. Thus, by the choice of the index $N$, $|a_n| \leq 1 + |a|$ for all indices $n \geq N$. Define $M \equiv \max\{1 + |a|, |a_1|, \ldots, |a_{N-1}|\}$. Then $|a_n| \leq M$ for every index $n$. Thus, the sequence $\{a_n\}$ is bounded.

A simple but fundamental result is that, if $\{s_n\}$ is bounded from above and increasing, or bounded from below and decreasing, then it is convergent:

Theorem (The Monotone Convergence Theorem):

$$\text{A monotone sequence converges if and only if it is bounded.} \tag{2.4}$$

Moreover, the bounded monotone sequence $\{a_n\}$ converges to

i. $\sup_{n\in\mathbb{N}}\{a_n\}$ if it is monotonically increasing, and to

ii. $\inf_{n\in\mathbb{N}}\{a_n\}$ if it is monotonically decreasing.

> Proof: We have already proven that a convergent sequence is bounded, so it remains to be shown that if the monotone sequence $\{a_n\}$ is bounded, then it converges to limits determined by (i) and (ii). We first suppose that the sequence $\{a_n\}$ is monotonically increasing. Then if we define $S = \{a_n \mid n \text{ in } \mathbb{N}\}$, by assumption, the set $S$ is bounded above. According to the Completeness Axiom, $S$ has a least upper bound. Define $\ell \equiv \sup S$. We claim that the sequence $\{a_n\}$ converges to $\ell$. Indeed, let $\epsilon > 0$. We need to find an index $N$ such that
>
> $$|a_n - \ell| < \epsilon \quad \text{for all indices } n \geq N;$$
>
> that is,
>
> $$\ell - \epsilon < a_n < \ell + \epsilon \quad \text{for all indices } n \geq N. \tag{2.5}$$
>
> Since the number $\ell$ is an upper bound for the set $S$, we have
>
> $$a_n \leq \ell < \ell + \epsilon \quad \text{for every index } n. \tag{2.6}$$
>
> On the other hand, since $\ell$ is the least upper bound for $S$, the number $\ell - \epsilon$ is not an upper bound for $S$, so there is an index $N$ such that $\ell - \epsilon < a_N$. However, the sequence $\{a_n\}$ is monotonically increasing, so
>
> $$\ell - \epsilon < a_N \leq a_n \quad \text{for all indices } n \geq N. \tag{2.7}$$
>
> From the inequalities (2.6) and (2.7) follows the required inequality (2.5). Thus, the sequence $\{a_n\}$ converges to $\ell$. We leave it to the reader to construct a similar proof when the sequence is monotonically decreasing.

Theorem: Let $\{a_n\}, \{b_n\} \subset \mathbb{R}$ be sequences such that $\{b_n\}$ is bounded and $\lim a_n = 0$. Then

$$\lim a_n b_n = 0. \tag{2.8}$$

> Proof: As $\{b_n\}$ is bounded, $\exists M \in \mathbb{R}_+$ such that, $\forall n \in \mathbb{N}$, $|b_n| \leq M$. As $\{a_n\}$ is convergent, $\exists N \in \mathbb{N}$ such that, for any given $\epsilon > 0$, $|a_n| < \epsilon/M$. Then $0 \leq |a_n b_n| = |a_n| \times |b_n| < \epsilon$.

Theorem (Squeeze Theorem): Suppose $\{a_n\}, \{b_n\}, \{c_n\} \subset \mathbb{R}$ are sequences for which there exists $n_o \in \mathbb{N}$ such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$, $n \geq n_o$, and that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$. Then

$$\textbf{the sequence } \{b_n\} \textbf{ converges and } \lim_{n\to\infty} b_n = L. \tag{2.9}$$

Proof: For any $\epsilon > 0$, $\exists n_a \in \mathbb{N}$ such that, $\forall n \geq n_a, a_n \in N_\epsilon(L)$; and $\exists n_c \in \mathbb{N}$ such that, $\forall n \geq n_c, c_n \in N_\epsilon(L)$. Set $N = \max(n_o, n_a, n_c)$, so that $\forall n \geq N, \{a_n, c_n\} \in N_\epsilon(L)$ and $a_n \leq b_n \leq c_n$. Then $b_n \in N_\epsilon(L)$ for $n \geq N$, which is the definition of convergence of sequence $\{b_n\}$.

Theorem: Let $a \in \mathbb{R}$. Then

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0. \tag{2.10}$$

Proof: Choose $m \in \mathbb{N}$ such that $|a| < m$. Then for $n > m$,

$$0 \leq \left|\frac{a^n}{n!}\right| = \frac{|a|^n}{m!}\left(\prod_{j=m+1}^{n} \frac{1}{j}\right) < \frac{|a|^n}{m!}\left(\frac{1}{m^{n-m}}\right) = \frac{m^m}{m!}\left(\frac{|a|}{m}\right)^n.$$

Since $m$ is a constant and $|a| < m$, $(|a|/m)^n \to 0$. The Squeeze Theorem (2.9) implies that $|a^n/n!| \to 0$, and thus, from definition (2.1), $a^n/n! \to 0$.

Theorem: Let $E$ be a nonempty subset of $\mathbb{R}$. Assume $E$ is bounded above and $a := \sup E$. Then there exists a sequence $\{a_n\}$ such that $a_n \in E$ for all $n \in \mathbb{N}$ and $a_n \to a$.

Suppose $E$ is bounded above. Let $a := \sup\ E$. Then for every $n \in \mathbb{N}$, there is $a_n \in E$ such that $a_n > a - (\frac{1}{n})$. Also since $a_n \leq a$ for all $n \in \mathbb{N}$, by the Squeeze Theorem (2.9), we see that $a_n \to a$.

Theorem: Let $x > 0$. We wish to show

$$0 < \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} < \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{2.11}$$

where (not importantly right now) the latter expression is equal to $e^x$, as will be shown later.

Proof: Inequality (2.11) is obvious for $0 < x \leq 1$. Next, for each fixed $x > 1$, the ratio of the two terms in the sums is $(x^n/n!)/(x^{2n}/(2n)!) = (2n)!/n! \times x^{-n}$. From the reciprocal result of (2.10), the ratio of the two terms from (2.11) goes to infinity as $n$ increases. As the sums are infinite, the inequality must hold.

We will use result (2.11) in Example 2.95.

Theorem: Let $\{a_n\}$ be a convergent sequence with $a_n \to a \in \mathbb{R}$. Then

$$|a_n| \to |a|. \tag{2.12}$$

Proof: The reverse triangle inequality (1.21) implies $0 \leq ||a_n| - |a|| \leq |a_n - a|$. This holds $\forall n \in \mathbb{N}$, so, as $a_n \to a$, the Squeeze Theorem (2.9) implies $|a_n| \to |a|$.

Observe how (2.12) holds for $a_n = -1$ and $a = 1$, but $a_n \not\to a$, i.e., the converse of (2.12) is not true.

Theorem: Let $\{a_n\}, \{b_n\} \subset \mathbb{R}$ and let $a, b \in \mathbb{R}$ such that $a_n \to a$ and $b_n \to b$. Then

$$\max\{a_n, b_n\} \to \max\{a, b\} \quad \text{and} \quad \min\{a_n, b_n\} \to \min\{a, b\}. \tag{2.13}$$

Proof: Let $\epsilon > 0$ be given. Then $\exists n_1, n_2 \in \mathbb{N}$ such that

$$\forall n \geq n_1,\ \ a - \epsilon < a_n < a + \epsilon, \quad \text{and} \quad \forall n \geq n_2,\ \ b - \epsilon < b_n < b + \epsilon.$$

Let $n_0 = \max\{n_1, n_2\}$. Then $\forall n \geq n_0$, $\max\{a-\epsilon, b-\epsilon\} < \max\{a_n, b_n\} < \max\{a+\epsilon, b+\epsilon\}$. As $\max\{a-\epsilon, b-\epsilon\} = \max\{a, b\} - \epsilon$ and $\max\{a+\epsilon, b+\epsilon\} = \max\{a, b\} + \epsilon$, it follows that $\max\{a_n, b_n\} \to \max\{a, b\}$. The proof for min is similar.

Theorem: Let $x_n$ and $y_n$ be sequences such that $\lim_{n\to\infty}\ x_n = x$ and $\lim_{n\to\infty}\ y_n = y$. Then:

$$\textbf{If } x_n \leq y_n \textbf{ for all } n \textbf{ sufficiently large, then } x \leq y. \tag{2.14}$$

Proof: $|x_n - x| < \epsilon$, so $-\epsilon < x_n - x < \epsilon$, implying $-x > -x_n - \epsilon$. Similarly, $|y_n - y| < \epsilon$, or $-\epsilon < y_n - y < \epsilon$, implying $y > y_n - \epsilon$. Adding the two inequalities gives

$$y - x \geq (y_n - \epsilon) - (x_n + \epsilon) = (y_n - x_n) - 2\epsilon \geq -2\epsilon.$$

As $\epsilon$ is arbitrary, it follows that $y - x \geq 0$.

Theorem: Let $\{a_n\}$ be a convergent sequence with $\lim a_n = a$, and suppose that $a_n \geq 0$ for all $n \in \mathbb{N}$. Then

$$a \geq 0. \tag{2.15}$$

Proof: This follows directly from (2.14). We can also argue as follows: Suppose to the contrary that $a < 0$, and let $\varepsilon = |a|/2$. The interval $(a - \varepsilon, a + \varepsilon)$ contains no $a_n$, i.e., if $a_n \in (a - \varepsilon, a + \varepsilon)$ then $a_n < a + \varepsilon < 0$ which is a contradiction. Thus, $a \geq 0$.

The next definition and results are of utmost importance and we will use them through the remainder of the text.

Definition: Sequence $\{s_n\}$ is termed a *Cauchy sequence* if, for a given $\epsilon \in \mathbb{R}_{>0}$, $\exists\, N \in \mathbb{N}$ such that $\forall n, m \geq N$, $|s_m - s_n| < \epsilon$.

Proposition: Every convergent sequence is Cauchy.

Proof: Suppose that $\{a_n\}$ is a sequence that converges to the number $a$. Let $\epsilon > 0$. We need to find an index $N$ such that $|a_n - a_m| < \epsilon$ if $n \geq N$ and $m \geq N$. But since $\{a_n\}$ converges to $a$, we can choose an index $N$ such that $|a_k - a| < \epsilon/2$ for every index $k \geq N$. Thus, if $n \geq N$ and $m \geq N$, setting $a_n - a_m = (a_n - a) + (a - a_m)$, by the Triangle Inequality,

$$|a_n - a_m| = |(a_n - a) + (a - a_m)| \leq |a_n - a| + |a_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{2.16}$$

Lemma: Every Cauchy sequence is bounded.

> Proof: Suppose that $\{a_n\}$ is a Cauchy sequence. For $\epsilon = 1$, we can choose an index $N$ such that $|a_n - a_m| < 1$ if $n \geq N$ and $m \geq N$. In particular, we have $|a_n - a_N| < 1$ if $n \geq N$. But, setting $a_n = a_N + (a_n - a_N)$, by the Triangle Inequality, $|a_n| = |a_N + (a_n - a_N)| \leq |a_N| + |a_n - a_N|$. Consequently, we see that $|a_n| \leq |a_N| + 1$ if $n \geq N$. Define $M = \max\{|a_N| + 1, |a_1|, |a_2|, \ldots, |a_{N-1}|\}$. Then $|a_n| \leq M$ for every index $n$.

Theorem (The Cauchy Convergence Criterion for Sequences):

$$\textbf{A sequence } \{s_n\} \textbf{ converges} \iff \{s_n\} \textbf{ is a Cauchy sequence.} \tag{2.17}$$

Half of the proof is given in (2.16), while the other half is given (after we develop the required machinery) in (3.60).

We now turn to the limit of a function. Informally, the limit of a function at a particular point, say $x$, is the value that $f(x)$ approaches, but need not assume at $x$. For example, $\lim_{x\to 0} (\sin x)/x = 1$, even though the ratio is not defined at $x = 0$. Formally, as instigated in 1821 by Cauchy:

Definition (The $\delta$-$\epsilon$ definition of right- and left-hand limits of functions): The function $f : A \subset \mathbb{R} \to \mathbb{R}$ has the *right-hand limit $L$ at $c \in \mathbb{R}$*, if $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$x \in (c, c+\delta) \cap A \;\Rightarrow\; |f(x) - L| < \epsilon, \tag{2.18}$$

for which we write $L = \lim_{x\to c^+} f(x)$. Likewise, $f$ has the *left-hand limit $L$ at $c \in \mathbb{R}$*, if $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$x \in (c-\delta, c) \cap A \;\Rightarrow\; |f(x) - L| < \epsilon, \tag{2.19}$$

denoted $L = \lim_{x\to c^-} f(x)$. Observe in both (2.18) and (2.19), point $c$ is not necessarily a member of domain $A$.

There are equivalent, and equally important, definitions of left- and right-hand limits of functions. We state these first, and then prove their equivalence.

Definition (The sequential definition of right- and left-hand limits of functions): The function $f : A \subset \mathbb{R} \to \mathbb{R}$ has the *right-hand limit $L$ at $c \in \mathbb{R}$*, if, for any monotone decreasing sequence $\{t_k\}_{k\in\mathbb{N}} \subset A$ and $t_k \to c$ as defined in (2.1), then $\lim_{k\to\infty} f(t_k) = L$ (as in (2.1)). Likewise, $f$ has the *left-hand limit $L$ at $c \in \mathbb{R}$*, if, for any monotone increasing sequence $\{t_k\}_{k\in\mathbb{N}} \subset A$ and $t_k \to c$, then $\lim_{k\to\infty} f(t_k) = L$.

Theorem: Let $c$ be a real number, and let $f$ be a real-valued function whose domain includes an open interval $(c, d)$ for some $c, d \in \mathbb{R}$ with $c < d$. The following two statements are equivalent:

(a) For every $\varepsilon > 0$ there exists a $\delta$ with $0 < \delta < d$ such that $|L - f(t)| < \varepsilon$ whenever $c < t < \delta$. That is, from definition (2.18),

$$\lim_{t\to c^+} f(t) = L. \tag{2.20}$$

(b) If $\{t_k\}_{k\in\mathbb{N}}$ is a monotone decreasing sequence contained in $(c, d)$ and $t_k \to c$, then

$$\lim_{k\to\infty} f(t_k) = L. \tag{2.21}$$

The reader should formulate the counterpart involving $\lim_{t\to d^-} f(t)$; and $\lim_{k\to\infty} f(t_k)$ for $\{t_k\}_{k\in\mathbb{N}}$ a monotone increasing sequence contained in $(c, d)$ and $t_k \to d$.

Proof: (Heil, Measure Theory for Scientists and Engineers, 2025 (forthcoming), Exercise # 1.9.30)

(a) $\Rightarrow$ (b). Suppose that $f(t) \to L$ as $t \to c^+$, and let $\{t_k\}_{k\in\mathbb{N}}$ be any monotone decreasing sequence contained in $(c,d)$ such that $t_k \to c$. Fix $\varepsilon > 0$. Then there exists $0 < \delta < d-c$ such that $|f(t) - L| < \varepsilon$ whenever $c < t < \delta$. Since $t_k \to c$, there exists an $N > 0$ such that $c < t_k < \delta$ for $k \geq N$. Hence $|L - f(t_k)| < \varepsilon$ for all $k \geq N$, so $f(t_k) \to L$ as $k \to \infty$.

(b) $\Rightarrow$ (a). We use a contrapositive argument for this direction. Suppose that statement (a) fails; that is, $f(t)$ does not converge to $L$ as $t \to c$. Then there exists an $\varepsilon > 0$ such that:

(*) for every $0 < \delta < d$ there is a real number $t$ with $c < t < \delta$ such that $|L - f(t)| \geq \varepsilon$.

Let
$$\delta = c + \frac{d-c}{2}.$$
Then, by hypothesis $(*)$, there must exist a real number $c < t_1 < \delta$ such that $|L - f(t)| \geq \varepsilon$. Next, consider
$$\delta = \min\left\{t_1, c + \frac{d-c}{4}\right\}.$$
By hypothesis $(*)$, there must exist a real number $c < t_2 < \delta$ such that $|L - f(t)| \geq \varepsilon$. In particular, $t_2 < t_1$. Continuing in this way we obtain numbers $t_1 > t_2 > \cdots$ such that
$$c < t_k < c + \frac{d-c}{2^k},$$
and therefore $t_k \to c$, yet $|L - f(t_k)| \geq \varepsilon$ for every $k$. Therefore statement (b) fails.

With both one-sided limits defined, we can now define the limit of a function, doing so again in two ways, and then proving their equivalence.

Definition (The $\delta$-$\epsilon$ definition of limit of a function): From (2.18) and (2.19) for right-hand and left-hand limits, if $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist and coincide, then $L$ is the *limit of $f$ at $c$*, denoted $L = \lim_{x\to c} f(x)$.

Definition (The sequential definition of limit of a function): For function $f : A \subset \mathbb{R} \to \mathbb{R}$ and sequence $\{x_n\} \subset A$ such that $x_n \to c$, if sequence $\{f(x_n)\}$ converges as in (2.1), so that $L = \lim_{n\to\infty} f(x_n)$ exists, then $L$ is the *limit of $f$ at $c$*.

Theorem: The $\delta$-$\epsilon$ formulation, and the sequential limit formulation, of limit of a function, are equivalent. That is, for $c \in \mathbb{R}$; for function $f : D \subset \mathbb{R} \to \mathbb{R}$, where $D = I \setminus \{c\}$ for $I$ an open interval; and $\{u_k\} \subset D$ such that $u_k \to c$,
$$L = \lim_{u\to c} f(u) \iff L = \lim_{k\to\infty} f(u_k). \tag{2.22}$$

Proof: (Heil, p. 32)

$\Rightarrow$ Suppose that $f(u) \to L$ as $u \to c$, and let $(u_k)_{k\in\mathbb{N}}$ be any sequence contained in $I\setminus\{c\}$ such that $u_k \to c$. Fix $\varepsilon > 0$. Then there exists a $\delta > 0$ such that $(c-\delta, c+\delta) \subseteq I$ and $|f(u) - L| < \varepsilon$ whenever $0 < |u - c| < \delta$. Since $u_k \to c$, there exists an $N \in \mathbb{N}$ such that $|u_k - c| < \delta$ for $k \geq N$. Also, $|u_k - c| > 0$ for every $k$ by hypothesis, so $|L - f(u_k)| < \varepsilon$ for all $k \geq N$. Therefore $f(u_k) \to L$ as $k \to \infty$.

$\Leftarrow$ We use a contrapositive argument for this direction. Let $N$ be large enough that $\left(c - \frac{1}{N}, c + \frac{1}{N}\right) \subseteq I$. Suppose that $f(u)$ does not converge to $L$ as $u \to c$. Then there exists an $\varepsilon > 0$ such that for each integer $k \geq N$ there is a real number $v_k$ with $0 < |c - v_k| < 1/k$ such that $|L - f(v_k)| \geq \varepsilon$. Therefore $\{v_k\}_{k \geq N}$ is a sequence of real numbers in $I \backslash \{c\}$ such that $v_k \to c$, but $f(v_k) \not\to c$ as $k \to \infty$. By reindexing (that is, setting $u_k = v_{k+N-1}$) we obtain a sequence $\{u_k\}_{k \in \mathbb{N}}$ in $I \backslash \{c\}$ such that $u_k \to c$, but $f(u_k) \not\to c$ as $k \to \infty$.

Of course, not all limits are finite. We write $\lim_{x \to c^+} f(x) = \infty$ if, $\forall M \in \mathbb{R}$, $\exists\, \delta > 0$ such that $f(x) > M$ for every $x \in (c, c+\delta)$; and $\lim_{x \to c^-} f(x) = \infty$ if, $\forall M \in \mathbb{R}$, $\exists\, \delta > 0$ such that $f(x) > M$ for every $x \in (c - \delta, c)$. Similar definitions hold for $\lim_{x \to c^+} f(x) = -\infty$ and $\lim_{x \to c^-} f(x) = -\infty$. As with a finite limit, if $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = \pm\infty$, then we write $\lim_{x \to c} f(x) = \pm\infty$. Lastly, we write $\lim_{x \to \infty} f(x) = L$ if, for each $\epsilon > 0$, $\exists\, x_0$ such that $|f(x) - L| < \epsilon$ for all $x > x_0$, and $\lim_{x \to -\infty} f(x) = L$ if, for each $\epsilon > 0$, $\exists\, x_0$ such that $|f(x) - L| < \epsilon$ for all $x < x_0$. As a shorthand, let $f(\infty) := \lim_{x \to \infty} f(x)$ and $f(-\infty) := \lim_{x \to -\infty} f(x)$. If $f(\infty) = f(-\infty)$, then we take $f(\pm\infty) := f(\infty) = f(-\infty)$.

Theorem: Let $f$ and $g$ be functions whose domain contains the point $c$ and such that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. Then, for constant values $k_1, k_2 \in \mathbb{R}$,

$$\lim_{x \to c} [k_1 f(x) + k_2 g(x)] = k_1 L + k_2 M, \tag{2.23}$$

$$\lim_{x \to c} f(x) g(x) = LM, \tag{2.24}$$

$$\lim_{x \to c} f(x)/g(x) = L/M, \text{ if } M \neq 0, \tag{2.25}$$

$$\text{if } g(x) \leq f(x), \text{ then } M \leq L. \tag{2.26}$$

The proof of (2.23) is a simple application of the triangle inequality (1.20). For (2.25), see, e.g., Stoll, Thm 3.2.1(c)). The proof of (2.26) follows from (2.14). For (2.24), the core of the proof involves considering sequences, say $\{a_n\}$ and $\{b_n\}$, that converge to $a$ and $b$ respectively, and writing

$$0 \leq |a_n b_n - ab| = |(a_n b_n - a_n b) + (a_n b - ab)| \leq |a_n|\,|b_n - b| + |b|\,|a_n - a|\,. \tag{2.27}$$

By taking $n$ large enough, and recalling (2.3), both of the terms on the rhs of (2.27) can be made arbitrarily small. Now use the sequential limit definition just above (2.22) and the Squeeze Theorem (2.9).

Theorem (Squeeze Theorem for Functions): As in Giv, Thm 3.34)[6] let $f, g$, and $h$ be functions defined on $E \subset \mathbb{R}$ such that $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, where $a$ is a limit point of $E$. If for every $x \in E \backslash \{a\}$ which is sufficiently close to $a$,

$$g(x) \leq f(x) \leq h(x), \tag{2.28}$$

then

$$\lim_{x \to a} f(x) = L. \tag{2.29}$$

[6]Hossein Giv, Mathematical Analysis and Its Inherent Nature, 2016.

We begin by noting that, when Giv writes (correctly, and as other authors also do, such at Mattuck, but not in the most precise way) that "If for every $x \in E \backslash \{a\}$ which is sufficiently close to $a$," this means:

$$\exists \delta_0 > 0 \text{ such that, } \forall x \in (B_{\delta_0}(a) \backslash \{a\}) \cap E, \;\; g(x) \leq f(x) \leq h(x). \tag{2.30}$$

• Proof I: By the equivalence of the $\epsilon - \delta$ and sequential limit criterion of function limits, to prove (2.29), it is sufficient to show that, for $\{a_n\}$ any sequence in $E \backslash \{a\}$ that converges to $a$,

$$\lim_{n \to \infty} f(a_n) = L. \tag{2.31}$$

From (2.30) and that $a_n \to a$, $\exists N \in \mathbb{N}$ such that, $\forall n \geq N$, $a_n \in B_{\delta_0}(a)$. Thus, $\forall n \geq N$, $g(a_n) \leq f(a_n) \leq h(a_n)$, and (2.31) follows from the Squeeze Theorem for sequences (2.9).

• Proof II (Ralf): Let $\epsilon > 0$. Then (as $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$) $\exists \delta_g, \delta_h > 0$ such that

$$\begin{aligned} &\forall x \in E \text{ such that } 0 < d(x,a) < \delta_g, \;\; |g(x) - L| < \epsilon; \text{ and} \\ &\forall x \in E \text{ such that } 0 < d(x,a) < \delta_h, \;\; |h(x) - L| < \epsilon. \end{aligned}$$

Let $\delta = \min\{\delta_0, \delta_h, \delta_g\}$. Then

$$\forall x \in E \text{ such that } 0 < d(x,a) < \delta, \quad -\epsilon < g(x) - L \leq f(x) - L \leq h(x) - L < \epsilon,$$

which implies $\lim_{x \to a} f(x) = L$.

For the limit of a composition of functions, let $b = \lim_{x \to a} f(x)$ and $L = \lim_{y \to b} g(y)$. Then

$$\lim_{x \to a} g(f(x)) = L. \tag{2.32}$$

**Example 2.1** *Compute*

$$\lim_{h \to 0} \frac{(e^h - 1)(\sin t^2)}{h^2}.$$

*Using* (2.24)*, we can separately compute* $\lim_{h \to 0}(e^h - 1)/h$ *and* $\lim_{t \to 0}(\sin t^2)/t$. *For the former, use of l'Hôpital's rule* (2.75) *yields the limit to be 1. Alternatively, from power series expansion of the exponential function* (2.272) *applied to the numerator,* $\lim_{h \to 0}(e^h - 1)/h = \lim_{h \to 0}(1 + h/2 + \cdots)$*, yielding 1. For the latter, with* $x = t^2 > 0$, $\lim_{t \to 0}(\sin t^2)/t = \lim_{x \to 0} \sin(x)/\sqrt{x} = \lim_{x \to 0} \left[\sin(x)/x\right] \times \left[x/\sqrt{x}\right] = \lim_{x \to 0}(\sin(x)/x) \times \lim_{x \to 0} \sqrt{x} = 1 \times 0 = 0$, *having used* (2.87)*. The desired limit is thus* $1 \times 0$. ■

Definition: A *deleted (or punctured) neighborhood* of $\xi$ is an interval $(a, b)$ with the point $\xi$, $a < \xi < b$, removed.

Theorem (Monotonicity of limits of functions): Let $A \subset \mathbb{R}$ and $f, g : A \to \mathbb{R}$ such that

$$\lim_{\substack{x \to \alpha \\ x \in A}} f(x) = b \quad \text{and} \quad \lim_{\substack{x \to \alpha \\ x \in A}} g(x) = c.$$

1. (Laczkovich and Sós, Thm 10.30) If $b < c$, then there exists a punctured neighborhood $\dot{U}$ of $\alpha$, denoted $\dot{U}(\alpha)$, such that

$$\forall x \in \dot{U}(\alpha) \cap A, \;\; f(x) < g(x). \tag{2.33}$$

> Proof: Let $\epsilon := (c-b)/2$. Then $\exists \dot{U}_1(\alpha)$ such that, for $x \in A \cap \dot{U}_1(\alpha)$, $|f(x)-b| < \epsilon$; and $\exists \dot{U}_2(\alpha)$ such that, for $x \in A \cap \dot{U}_2(\alpha)$, $|g(x)-c| < \epsilon$. Let $\dot{U}(\alpha) = \dot{U}_1(\alpha) \cap \dot{U}_2(\alpha)$. Then
>
> $$x \in A \cap \dot{U}(\alpha) \implies f(x) < b + \epsilon = b + \frac{c-b}{2} = \frac{b+c}{2} = c - \frac{c-b}{2} = c - \epsilon < g(x).$$
>
> Note that the converse is not true: If $f(x) < g(x)$ holds on a punctured neighborhood of $\alpha$, then we cannot conclude that $\lim_{x\to\alpha} f(x) < \lim_{x\to\alpha} g(x)$. If, for example, $f(x) = 0$ and $g(x) = |x|$, then $f(x) < g(x)$ for all $x \neq 0$, but $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$.

2. (Laczkovich and Sós, Thm 10.31) Now assume $f(x) \leq g(x)$ holds for all $x \in A \cap \dot{U}(\alpha)$. Then $b \leq c$.

> Proof: Let $\dot{U}(\alpha)$ be such that, $\forall x \in A \cap \dot{U}(\alpha)$, $f(x) \leq g(x)$. Suppose that $b > c$. Then by part (1), $\exists \dot{V}(\alpha)$ such that, $\forall x \in A \cap \dot{V}(\alpha)$, $f(x) > g(x)$. This, however, is impossible, because the set $A \cap \dot{U}(\alpha) \cap \dot{V}(\alpha)$ is nonempty, and
>
> $$\forall x \in \big(A \cap \dot{U}(\alpha) \cap \dot{V}(\alpha)\big), \quad f(x) \leq g(x).$$
>
> The converse is not true: If $\lim_{x\to\alpha} f(x) \leq \lim_{x\to\alpha} g(x)$, then we cannot conclude that $f(x) \leq g(x)$ holds in a punctured neighborhood of $\alpha$. If, for example, $f(x) = |x|$ and $g(x) = 0$, then $\lim_{x\to 0} f(x) \leq \lim_{x\to 0} g(x) = 0$, but $f(x) > g(x)$ for all $x \neq 0$.

The next result comes from, e.g., Stoll, Thm 4.1.8, and p. 143, exercise #12. The reference to a metric space can be ignored for now, and the reader can just take $X = \mathbb{R}$. Later, have a look at §3.2.

Theorem: Suppose $E$ is a subset of a metric space $X$, $p$ is a limit point of $E$, and $f, g$ are real-valued functions on $E$. Let $g$ be bounded on $E$, and $\lim_{x\to p} f(x) = 0$. Then $\lim_{x\to p} f(x)g(x) = 0$.

> Proof: We can either use the sequential criterion or the definition of the limit of a function. Function $g$ is bounded on $E \implies \exists M > 0$ such that $\forall x \in E$, $|g(x)| \leq M$; while $\lim_{x\to p} f(x) = 0 \implies \forall \varepsilon > 0$, $\exists \delta > 0$ such that, $\forall x \in E$, $0 < d(x,p) < \delta \implies |f(x)| < \epsilon$. Then,
>
> $$\forall x \in E,\ \ 0 < d(x,p) < \delta \implies |f(x)g(x)| = |f(x)||g(x)| < M\epsilon.$$
>
> Thus, $\lim_{x\to p} f(x)g(x) = 0$.

The next result (from, e.g., Stoll, 2021, p. 144, exercise #15) is such that its proof (at least the one we give) invokes the Bolzano-Weierstrass Theorem, which we prove in (3.58). This latter theorem simply says: Every bounded sequence in $\mathbb{R}$ has a convergent subsequence. Thus, we also need to invoke subsequences, whose use we otherwise delay until §3.5. We give the definition also now, but the reader may skip this and return to it later.

Definition: Consider a sequence $\{a_n\}$. Let $\{n_k\}$ be a sequence of natural numbers that is strictly increasing; that is, $n_1 < n_2 < n_3 < \cdots$. Then the sequence $\{b_k\}$ defined by $b_k = a_{n_k}$, for every index $k$, is called a *subsequence* of the sequence $\{a_n\}$.

Theorem: Let $E$ be a subset of a metric space, and let $p$ a limit point of $E$. Suppose $f$ is a bounded real-valued function on $E$ having the property that $\lim_{x \to p} f(x)$ does not exist. Then there exist distinct sequences $\{p_n\}$ and $\{q_n\}$ in $E$ with $p_n \to p$ and $q_n \to p$ such that $\lim_{n \to \infty} f(p_n)$ and $\lim_{n \to \infty} f(q_n)$ exist, but are not equal.

We begin with two illustrations, and then provide a proof.

1. Take a bounded function on an interval with a jump discontinuity for some $p \in (a, b)$, but is otherwise continuous. Let sequence $\{p_n\}$ approach $p$ from the left, and let sequence $\{q_n\}$ approach $p$ from the right.

2. Let $E$ be any nonempty interval of $\mathbb{R}$, and let $f : E \to \mathbb{R}$ be the Dirichlet function, namely $f(x) = \chi_{\mathbb{Q}}(x)$. Let $\{p_n\} \in \mathbb{Q}$ with $p_n \to p$; and $\{q_n\} \in \mathbb{R} \setminus \mathbb{Q}$ with $q_n \to p$. Then $\lim_{n \to \infty} f(p_n) = \lim_{n \to \infty} 1 = 1$, while $\lim_{n \to \infty} f(q_n) = \lim_{n \to \infty} 0 = 0$.

Proof: Recall the equivalence between the sequential criterion of the limit of a function and the formulation in terms of distances using $\epsilon$-$\delta$:

- Let $(X, d)$ be a metric space, $E$ be a subset of $X$ and $f$ a real-valued function with domain $E$. Suppose that $p$ is a limit point of $E$. The function $f$ has a limit at $p$ if there exists a number $L \in \mathbb{R}$ such that given any $\epsilon > 0$, there exists a $\delta > 0$ for which $|f(x) - L| < \epsilon$ for all points $x \in E$ satisfying $0 < d(x, p) < \delta$. If this is the case, we write

$$\lim_{x \to p} f(x) = L \quad \text{or} \quad f(x) \to L \quad \text{as} \quad x \to p.$$

- Let $E$ be a subset of a metric space $X$, $p$ a limit point of $E$, and $f$ a real-valued function defined on $E$. Then

$$\lim_{x \to p} f(x) = L \quad \text{if and only if} \quad \lim_{n \to \infty} f(p_n) = L$$

for every sequence $\{p_n\}$ in $E$, with $p_n \neq p$ for all $n$, and $\lim_{n \to \infty} p_n = p$.

Since $f : E \to \mathbb{R}$ does not have a limit at the limit point $p \in E$, by the negation of the above, either:

1. for any sequence $\{p_n\}$ in $E$ converging to $p$ with $p_n \neq p$ $\forall n \in \mathbb{N}$, $\lim_{n \to \infty} f(p_n)$ exists but there exist different sequences $\{q_n\}, \{r_n\}$ in $E$ converging to $p$ (again with $q_n \neq p$ and $r_n \neq p$ $\forall n \in \mathbb{N}$), for which $\lim_{n \to \infty} f(q_n) \neq \lim_{n \to \infty} f(r_n)$;

2. there exists at least one sequence $\{p_n\}$ in $E$ converging to $p$ with $p_n \neq p$ $\forall n \in \mathbb{N}$, for which $\lim_{n \to \infty} f(p_n)$ fails to exist.

In the first case, we can take the two sequences $\{q_n\}, \{r_n\}$ and arrive at the conclusion of the theorem.

In the second case, we use the boundedness of $f$ and the Bolzano-Weierstrass Theorem to produce two such sequences. As $f$ and therefore $\{f(p_n)\}$ is bounded, there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $\{f(p_{n_k})\}$ converges to some $a \in \mathbb{R}$. Since $\{f(p_n)\}$ does not converge to $a$, $\exists \epsilon > 0$ such that for any given $m \in \mathbb{N}$, $\exists l \geq m$ with $|f(p_l) - a| \geq \epsilon$. For each $m \in \mathbb{N}$, choose such an index $l$. This way, we construct a subsequence $\{p_l\}$ such that $\{f(p_l)\}$ never enters $N_\epsilon(a)$. By the above corollary, as $\{f(p_l)\}$ is still bounded, there exists a subsequence $\{p_{l_k}\}$ of $\{p_l\}$ such that $\{f(p_{l_k})\}$ converges to some point $b \in \mathbb{R}$. Clearly $a \neq b$ so that $\{p_{l_k}\}, \{p_{n_k}\}$ are two sequences in $E$ converging to $p$ with $\lim_{k \to \infty} f(p_{l_k}) \neq \lim_{k \to \infty} f(p_{n_k})$.

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If $\lim_{x\to 0} f(x)$ exists. Then

1. $\lim_{x\to 0} f(x) = 0$, and
2. $\lim_{x\to p} f(x)$ exists for every $p \in \mathbb{R}$.

> Proof: For (1),
>
> $$\lim_{x\to 0} f(x) = \lim_{x\to 0} f\left(\frac{x}{2} + \frac{x}{2}\right) = \lim_{x\to 0} f\left(\frac{x}{2}\right) + \lim_{x\to 0} f\left(\frac{x}{2}\right) = 2\lim_{x\to 0} f(x),$$
>
> which implies $\lim_{x\to 0} f(x) = 0$.
>
> For (2), let $p \in \mathbb{R}$, and note that $f(p+x) = f(p) + f(x)$. Take the limit as $x \to 0$, which gives (via part (1))
>
> $$\lim_{x\to p} f(x) = \lim_{x\to 0} f(p+x) = \lim_{x\to 0} f(p) + \lim_{x\to 0} f(x) = f(p) + \lim_{x\to 0} f(x).$$
>
> The rhs exists, so the lhs exists. As $p \in \mathbb{R}$ was arbitrary, the lhs exists for every $p \in \mathbb{R}$.

Recall definitions (2.18) and (2.19) for right and left limits of functions. We repeat these formulations here, introducing a bit more notation, and then consider discontinuities, and monotone functions.

Definition ($\delta$-$\epsilon$ definition of right- and left-hand limits of functions (again)): Let $E \subset \mathbb{R}$ and let $f$ be a real-valued function defined on $E$. Suppose $p$ is a limit point of $E \cap (p, \infty)$. The function $f$ has a *right limit* at $p$ if there exists a number $L \in \mathbb{R}$ such that given any $\epsilon > 0$, there exists a $\delta > 0$ for which

$$|f(x) - L| < \epsilon \text{ for all } x \in E \text{ satisfying } p < x < p + \delta.$$

The right limit of $f$, if it exists, is denoted by $f(p+)$, and we write

$$f(p+) = \lim_{x\to p^+} f(x) = \lim_{\substack{x\to p \\ x>p}} f(x).$$

Similarly, if $p$ is a limit point of $E \cap (-\infty, p)$, the *left limit* of $f$ at $p$, if it exists, is denoted by $f(p-)$, and we write

$$f(p-) = \lim_{x\to p^-} f(x) = \lim_{\substack{x\to p \\ x<p}} f(x).$$

In the next section, continuity of a function $f : D \to \mathbb{R}$ at point $p \in D$ means $f(p+) = f(p-) = f(p)$. In the next definition, it suffices for the definition of the interior of a set to be just for intervals of the real line, given by $\mathrm{Int}(I) = (a, b)$ for $I = [a, b]$, or $I = (a, b]$, or $I = [a, b)$, for $a < b$.

Definition: Let $f$ be a real valued function defined on an interval I. The function $f$ has a jump discontinuity at $p \in \mathrm{Int}(I)$ if $f(p+)$ and $f(p-)$ both exist, but $f$ is not continuous at $p$. If $p \in I$ is a left (right) endpoint of $I$, then $f$ has a jump discontinuity at $p$ if $f(p+)$ $(f(p-))$ exists, but $f$ is not continuous at $p$.

Definition: Jump discontinuities are also referred to as *simple discontinuities*, or *discontinuities of the first kind*. All other discontinuities are said to be of *second kind*.

If $f(p+)$ and $f(p-)$ both exist, but $f$ is not continuous at $p$, then either (a) $f(p+) \neq f(p-)$, or (b) $f(p+) = f(p-) \neq f(p)$. In case (a), $f$ has a jump discontinuity at $p$. In case (b), the discontinuity is *removable*. All discontinuities for which $f(p+)$ or $f(p-)$ does not exist are discontinuities of the second kind.

Theorem: Let $I \subset \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be monotone increasing on $I$. Then $f(p+)$ and $f(p-)$ exists for every $p \in I$ and

$$\sup_{x<p} f(x) = f(p-) \leq f(p) \leq f(p+) = \inf_{p<x} f(x). \tag{2.34}$$

Furthermore,

$$\text{if } p < q,\ p, q \in I, \text{ then } f(p+) \leq f(q-). \tag{2.35}$$

Proof: Fix $p \in I$. Since $f$ is increasing on $I$, $\{f(x) : x < p, x \in I\}$ is bounded above by $f(p)$. (Since $I$ is open and $p$ in $I$, this set is nonempty. So, along with being bounded, its sup exists.) Let $A = \sup\{f(x) : x < p,\ x \in I\}$. Then $A \leq f(p)$. We now show that $\lim_{x \to p^-} f(x) = A$. Let $\epsilon > 0$ be given. Since $A$ is the least upper bound of $\{f(x) : x < p\}$, there exists $x_o < p$ such that $A - \epsilon < f(x_o) \leq A$. Thus, if $x_o < x < p$, then $A - \epsilon < f(x_o) \leq f(x) \leq A$. Therefore, $|f(x) - A| < \epsilon$, for all $x, x_o < x < p$. Thus, by definition, $\lim_{x \to p^-} f(x) = A$. Similarly

$$f(p) \leq f(p+) = \inf\{f(x) : p < x,\ x \in I\}.$$

Finally, suppose $p < q$. Then

$$\begin{aligned} f(p+) &= \inf\{f(x) : x > p, x \in I\} \leq \inf\{f(x) : p < x < q\} \\ &\leq \sup\{f(x) : p < x < q\} \leq \sup\{f(x) : x < q, x \in I\} = f(q-). \end{aligned}$$

Notice that, for a given set $S$, $\inf(S) \leq \sup(S)$, so the trick for the previous equation is to determine the set $S = \{f(x) : p < x < q\}$.

Corollary: If $f$ is monotone on an open interval $I$, then the set of discontinuities of $f$ is at most countable.

Proof: (Based on Stoll, 2021, Coro 4.4.8, with added detail.)

(Step 1): Let $E = \{p \in I : f \text{ is discontinuous at } p\}$. Suppose $f$ is monotone increasing on $I$. Then

$$p \in E \quad \text{if and only if} \quad f(p-) < f(p+).$$

Indeed, a function $f$ is continuous at $p \in (a, b)$ if and only if (a) $f(p+)$ and $f(p-)$ both exist; and (b) $f(p+) = f(p-) = f(p)$. Thus, the statement follows from the contrapositive.

(Step 2): For each $p \in E$, choose $r_p \in \mathbb{Q}$ such that

$$f(p-) < r_p < f(p+).$$

Indeed, this choice can be made because $\mathbb{Q}$ is dense in $\mathbb{R}$. See §3.1.

(Step 3): If $p < q$, then $f(p+) \leq f(q-)$. This is (2.35).

> (Step 4): Therefore, if $p, q \in E$, we have $r_p \neq r_q$; and thus the function $p \to r_p$ is a one-to-one map of $E$ into $\mathbb{Q}$.
>
> Indeed recall: $f : X \to Y$ is one-to-one if: $\forall x_1, x_2 \in X$, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.
>
> (Step 5): Therefore, $E$ is equivalent to a subset of $\mathbb{Q}$ and thus is at most countable.
>
> To understand this, we need to know: Two sets $A$ and $B$ are said to be *equivalent* (or to have the same cardinality), denoted $A \sim B$, if there exists a one-to-one function of $A$ onto $B$. Here, function $p \to r_p$ is a one-to-one map of $E$ <u>into</u> $\mathbb{Q}$. Then, as indicated in part 3 of the subsequent lemma, $E$ is equivalent to <u>a subset of</u> $\mathbb{Q}$ and thus is at most countable.

The last part of the above proof is part of a bigger set of results, which we state here, without proof. A proof can be found in, e.g., Heil, Measure Theory for Scientists and Engineers, 2025 (forthcoming), p. 13.

<u>Lemma</u>: Let $X$ and $Y$ be sets.

1. If $X$ is countable and $Y \subseteq X$, then $Y$ is countable.
2. If $X$ is uncountable and $Y \supseteq X$, then $Y$ is uncountable.
3. If $X$ is countable and there exists a one-to-one function $f : Y \to X$, then $Y$ is countable.
4. If $X$ is uncountable and there exists a one-to-one function $f : X \to Y$, then $Y$ is uncountable.
5. If $Y$ is uncountable and there exists an onto function $f : X \to Y$, then $X$ is uncountable.

We take up continuity in the next section. We end this section with a result that holds without assuming continuity, and which parallels (2.46) in the next section, in which we assume continuity. It was taken from an exercise from Stoll, 2021, p. 143, #9.

<u>Theorem</u>: Suppose $f : (a, b) \to \mathbb{R}$, $p \in [a, b]$, and $\lim_{x \to p} f(x) > 0$. Then, $\exists \delta > 0$ such that, $\forall x \in (a, b)$ with $0 < |x - p| < \delta$, $f(x) > 0$. (Notice continuity is not assumed.)

> Proof: Let $L = \lim_{x \to p} f(x) > 0$. Take $\epsilon = L/2$. From the definition of limit, $\exists \delta > 0$ such that, $\forall x \in B_\delta(p) \cap ((a, b) \setminus \{p\})$,
>
> $$L/2 < f(x) < 3L/2 \Leftrightarrow -L/2 < f(x) - L < L/2 \Leftrightarrow 0 < |f(x) - L| < L/2 = \epsilon.$$

## 2.2 Function Continuity and Uniform Continuity

Definition: Let $f$ be a function with domain $A \subset \mathbb{R}$ and $a \in A$. (Note that, without specification of the codomain, it is understood to be $\mathbb{R}$, which is sometimes also stated as saying "let $f$ be a real-valued function".) If $\lim_{x \to a^+} f(x) = f(a)$, then $f$ is said to be *continuous on the right at* $a$; and if $\lim_{x \to a^-} f(x) = f(a)$, then $f$ is *continuous on the left at* $a$. We have *continuity at point* $a$ when both of these conditions hold:

$$f \textbf{ is continuous at } a \textbf{ if } \lim_{x \to a} f(x) = f(a)\textbf{.} \tag{2.36}$$

The function $f : A \to \mathbb{R}$ is said to be *continuous* provided that it is continuous at every point in $A$.

Recall (the equivalence of) (2.20) and (2.21); and also recall (2.22). These imply that an equivalent definition of continuity is as follows.

Definition: A function $f : A \to \mathbb{R}$ is said to be continuous at the point $x_0$ in $A$ provided that, whenever $\{x_n\}$ is a sequence in $A$ that converges to $x_0$, the image sequence $\{f(x_n)\}$ converges to $f(x_0)$.

Definition: If $f$ is continuous at each point $a \in S \subset A \subset \mathbb{R}$, then $f$ is *continuous on* $S$, in which case we also say that $f$ is of *class* $\mathcal{C}^0$ on $S$, or $f \in \mathcal{C}^0(S)$. Often, subset $S$ will be an interval, say $(a, b)$ or $[a, b]$, in which case we write $f \in \mathcal{C}^0(a, b)$ and $f \in \mathcal{C}^0[a, b]$, respectively. If $f$ is continuous on (its whole domain) $A$, then we say $f$ is continuous, or that $f$ is of class $\mathcal{C}^0$, or $f \in \mathcal{C}^0$.

From the above definitions, we can express the limit result for continuous functions as follows. If $f : A \subset \mathbb{R} \to \mathbb{R}$ and $f \in \mathcal{C}^0(S)$ for $S \subset A$, then

$$\forall a \in S, \qquad \lim_{x \to a} f(x) = f\left(\lim_{x \to a} x\right) = f(a)\,. \tag{2.37}$$

Given two functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$, we define the sum $f + g : D \to \mathbb{R}$ and the product $fg : D \to \mathbb{R}$ by $(f + g)(x) \equiv f(x) + g(x)$ and $(fg)(x) \equiv f(x)g(x)$, $\forall x \in D$. Moreover, if $g(x) \neq 0$ for all $x$ in $D$, the quotient $f/g : D \to \mathbb{R}$ is defined by

$$(f/g)(x) \equiv \frac{f(x)}{g(x)} \quad \text{for all } x \text{ in } D.$$

The following theorem is an analog, and also a consequence, of the sum, product, and quotient properties of convergent sequences.

Theorem: Suppose that the functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are continuous at the point $x_0$ in $D$. Then the sum

$$f + g : D \to \mathbb{R} \textbf{ is continuous at } x_0; \tag{2.38}$$

the product

$$fg : D \to \mathbb{R} \textbf{ is continuous at } x_0; \tag{2.39}$$

and, if $g(x) \neq 0$ for all $x$ in $D$, the quotient

$$f/g : D \to \mathbb{R} \textbf{ is continuous at } x_0. \tag{2.40}$$

Proof: Let $\{x_n\}$ be a sequence in $D$ that converges to $x_0$. From definition (2.37), and the sequential limit definition (2.22), $\lim_{n\to\infty} f(x_n) = f(x_0)$ and $\lim_{n\to\infty} g(x_n) = g(x_0)$. Now observe that (2.23) implies

$$\lim_{n\to\infty} [f(x_n) + g(x_n)] = f(x_0) + g(x_0);$$

(2.24) implies

$$\lim_{n\to\infty} [f(x_n) g(x_n)] = f(x_0) g(x_0);$$

and, if $g(x) \neq 0$ for all $x$ in $D$, (2.25) implies

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{f(x_0)}{g(x_0)}.$$

Results (2.38), (2.39), and (2.40) follow.

Definition: For a nonnegative integer $k$ and numbers $c_0, c_1, \ldots, c_k$, the function $p : \mathbb{R} \to \mathbb{R}$, defined by

$$p(x) = \sum_{i=0}^{k} c_i x^i \quad \text{for all } x \text{ in } \mathbb{R}$$

is called a polynomial. If $c_k \neq 0$, then $p$ is said to have degree $k$.

Corollary (Polynomial functions are continuous): Let $f : A \to \mathbb{R}$ be an $n$th order polynomial, $n \in \mathbb{N}$. Then $f$ is continuous on $A$.

Proof: First let $f : I \to \mathbb{R}$ be given by $f(x) = k$, for some $k \in \mathbb{R}$ and $I$ an open interval. Then, for $a \in A$, (2.36) implies $\lim_{x\to a} f(x) = \lim_{x\to a} k = k = f(a)$, i.e., constant functions are continuous. Now let $f : I \to \mathbb{R}$ be given by $f(x) = x$, so that $\lim_{x\to a} f(x) = \lim_{x\to a} x = a = f(a)$, so that $f(x) = x$ is continuous. It follows from (2.38) and (2.39) that polynomials are continuous.

**Example 2.2** *Let $\{a_n\} \subset \mathbb{R}$ be a convergent sequence with $\lim a_n = a$. We wish to prove that $\lim a_n^2 = a^2$. The result follows from continuity of polynomials, which in this case is $f : \mathbb{R} \to [0, \infty)$; $f(x) = x^2$. Now consider using (2.36). We need show that, for any $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that, for $n > N$, $|a_n^2 - a^2| < \epsilon$. Note $a_n^2 - a^2 = (a_n + a)(a_n - a)$. As $\{a_n\}$ is convergent, it is bounded, so $\exists M \in \mathbb{R}_+$ such that, $\forall n \in \mathbb{N}$, $|a_n| \leq M$. Further, for any $\epsilon_1 > 0$, $\exists N \in \mathbb{N}$ such that, for $n > N$, $|a_n - a| < \epsilon_1$. The (regular) triangle inequality then implies that, for $n > N$,*

$$|a_n^2 - a^2| \leq |a_n + a| \times |a_n - a| \leq (|a_n| + |a|) \times |a_n - a| < (M + |a|)\epsilon_1.$$

*Taking $\epsilon_1 = \epsilon/(M + |a|)$ shows $|a_n^2 - a^2| < \epsilon$.* ■

Theorem: Let $I$ be an interval and let $f : I \to \mathbb{R}$ be a function that is strictly monotonic on $I$. Then

$$f^{-1} : f(I) \to \mathbb{R} \text{ is continuous.} \tag{2.41}$$

Proof: (Ghorpade and Limaye, p. 75) Since $f$ is strictly monotonic on $I$, we see that $f$ is one-one and its inverse $f^{-1}: f(I) \to \mathbb{R}$ is well-defined. Consider $d \in f(I)$. Then there is a unique $c \in I$ such that $f(c) = d$.

Assume first that $f$ is strictly increasing on $I$. Let $\epsilon > 0$ be given. Suppose that $c$ is neither the left endpoint nor the right endpoint of the interval $I$. Then there are $c_1, c_2 \in I$ such that

$$c - \epsilon < c_1 < c < c_2 < c + \epsilon. \tag{2.42}$$

Let $d_1 := f(c_1)$ and $d_2 := f(c_2)$. Since $f$ is strictly increasing on $I$, we see that $d_1 < d < d_2$, and since $f^{-1}$ is also strictly increasing on $f(I)$, we obtain

$$y \in f(I), \quad d_1 < y < d_2 \implies c_1 = f^{-1}(d_1) < f^{-1}(y) < f^{-1}(d_2) = c_2.$$

From this, (2.42), and that $f(c) = d$, $f^{-1}(d) = c$, $f^{-1}(d) - \epsilon < f^{-1}(y) < f^{-1}(d) + \epsilon$. Thus if we let $\delta := \min\{d - d_1, d_2 - d\}$, we see that $\delta > 0$ and

$$y \in f(I), |y - d| < \delta \Longrightarrow \left|f^{-1}(y) - f^{-1}(d)\right| < \epsilon.$$

Hence $f^{-1}$ is continuous at $d$. See Ghorpade and Limaye for the case of the left and right endpoints; and the case when $f$ is strictly decreasing.

Corollary: Let $f : [0, \infty) \to \mathbb{R}$ be given by $f(x) = x^n$, for $n \in \mathbb{N}$. As $f$ is strictly monotonically increasing,

$$f^{-1}: [0, \infty) \to \mathbb{R}, \text{ with } f^{-1}(y) = y^{1/n} \text{ is continuous.} \tag{2.43}$$

**Example 2.3** *Let $\{a_n\} \in \mathbb{R}_{\geq 0}$ be a convergent sequence with $\lim a_n = a$. We wish to prove that $\lim \sqrt{a_n} = \sqrt{a}$. The result follows because $f : [0, \infty) \to [0, \infty)$; $f(x) = x^{1/2}$ is continuous. Consider now using the sequence convergence formulation of continuity. We need show that, for any $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that, for $n > N$, $|\sqrt{a_n} - \sqrt{a}| < \epsilon$. From (2.14), $a \geq 0$. Choose any $\epsilon > 0$. If $a = 0$, choose $N$ such that $a_n < \epsilon^2$ for all $n \geq N$. If $a > 0$, choose $N$ such that $|a_n - a| < \epsilon\sqrt{a}$ for all $n \geq N$. For such $n$,*

$$|\sqrt{a_n} - \sqrt{a}| = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} \leq \frac{|a_n - a|}{\sqrt{a}} < \epsilon. \qquad \blacksquare$$

Definition: For $x > 0$ and rational number $r = m/n$, where $m$ and $n$ are integers with $n$ positive, we define $x^r \equiv (x^m)^{1/n}$.

Theorem: For $x > 0$ and integers $m$ and $n$ with $n$ positive, $(x^m)^{1/n} = \left(x^{1/n}\right)^m$.
See, e.g., Fitzpatrick, p. 79 for proof.

Theorem: For $r$ a rational number, define $f(x) = x^r$, for $x \geq 0$. The function $f : [0, \infty) \to \mathbb{R}$ is continuous.

Proof: Let $m, n$ be integers such that $n > 0$ and $r = m/n$. Define $g(x) = x^{1/n}$ and $h(x) = x^m$, for $x \geq 0$. Then $f(x) = g(h(x)) = (g \circ h)(x)$ for $x \geq 0$. Being a polynomial, $h : [0, \infty) \to \mathbb{R}$ is continuous. By (2.41), $g : [0, \infty) \to \mathbb{R}$ is continuous. From (subsequent) (2.48), $f : [0, \infty) \to \mathbb{R}$ is continuous.

Corollary: Let $f : (0, \infty) \to \mathbb{R}$ be given by $f(x) = x^r$, for $r \in \mathbb{Q}$. Then $f$ is strictly monotone and continuous, and

$$f^{-1} : (0, \infty) \to \mathbb{R}, \text{ with } f^{-1}(y) = y^{1/r} \text{ is continuous.} \tag{2.44}$$

Theorem: Let $D \subset \mathbb{R}$, and let $f, g : D \to \mathbb{R}$ be functions continuous at $c \in D$. Then

(i)

$$|f| \text{ is continuous at } c. \tag{2.45}$$

(ii) $\max\{f, g\}$ and $\min\{f, g\}$ are continuous at $c$.

Proof:

(i) Let $\{x_n\}$ be a sequence in $D$ such that $x_n \to c$. From the continuity of $f$ and the sequential limit definition (2.22), $f(x_n) \to f(c)$. Result (2.12) implies $|f(x_n)| \to |f(c)|$.

(ii) Let $\{x_n\}$ be a sequence in $D$ such that $x_n \to c$. We have $f(x_n) \to f(c)$ and $g(x_n) \to g(c)$, so by (2.13),

$$\max\{f(x_n), g(x_n)\} \to \max\{f(c), g(c)\} \quad \text{and} \quad \min\{f(x_n), g(x_n)\} \to \min\{f(c), g(c)\}.$$

Alternatively, use the fact that we can write

$$\max\{f(x), g(x)\} = \frac{1}{2}(f(x) + g(x) + |f(x) - g(x)|),$$

along with (2.12) and (2.38).

Theorem: Let $f : D \to \mathbb{R}$ be a continuous function on (nonempty) interval $D = (a, b) \in \mathbb{R}$. Let $c \in (a, b)$ such that $f(c) > 0$. Then

$$\exists \delta > 0 \textbf{ such that } f(x) > 0 \textbf{ for } x \in (c - \delta, c + \delta)\textbf{.} \tag{2.46}$$

Likewise, if $f(c) < 0$, then there is $\delta > 0$ such that $f(x) < 0$ whenever $x \in D$ and $|x - c| < \delta$.

Proof I (sequential argument): Suppose to the contrary that no such $\delta$ exists. Then, for every $\delta > 0$, there exists $x \in (c-\delta, c+\delta)$ such that $f(x) \leq 0$. If we take $\delta = 1/n$, we obtain a sequence $x_n$ in $(c - 1/n, c + 1/n)$ with $f(x_n) \leq 0$. The inequality $|x_n - c| < 1/n$ shows that the sequence $x_n$ converges to $c$, and the continuity of $f$ implies that the sequence $f(x_n)$ converges to $f(c)$. From (2.14) and that $f(x_n) \leq 0$, $f(c) \leq 0$. This contradicts the assumption that $f(c) > 0$.

Proof II ($\epsilon$-$\delta$ argument): Let $\epsilon := f(c) > 0$. By continuity, $\exists \delta > 0$ such that

$$f(x) - f(c)| < \underbrace{f(c)}_{=\epsilon}, \quad \forall x \in (a, b) \text{ with } d(x, c) < \delta.$$

That is, $\forall x \in (a, b)$ with $d(x, c) < \delta$, we have $0 = f(c) - f(c) < f(x)$.

Proof III (using existing results): Function $f$ is continuous on $(a, b)$ and thus at $c \in (a, b)$. Since $c$ is a limit point of $(a, b)$, $\lim_{x \to c} f(x) = f(c) > 0$. The result now follows from (2.33). More specifically, let $g(x) \equiv 0$, and $\lim_{x \to c} f(x) = f(c) > 0 = \lim_{x \to c} g(x)$.

We wish to devise an example to show that the converse of the previous theorem is not true. Hint: This means you need to demonstrate a continuous function $f$ such that

$$\forall x \in \{x : 0 < |x - c| < \delta\}, \quad f(x) > 0, \quad f(c) \leq 0.$$

An example is: Take $f(x) = x^2$, $c = 0$.

Theorem: Let $f : D \to \mathbb{R}$ be a continuous function at a point $c$ in $(a, b) \subset D$. Prove:

$$\textbf{If } f(x) \geq 0 \textbf{ on } (a, c) \cup (c, b) \textbf{ then } f(c) \geq 0. \tag{2.47}$$

> Proof: Let $\{a_n\}$ be a sequence in $(a, b)$ converging to $c$, and $a_n \neq c$. Then $f(a_n) \geq 0$, and the result (2.15) implies that $\lim f(a_n) \geq 0$. Since $f$ is continuous at $c$, this means that $f(c) \geq 0$.

An important result is the continuity of composite functions: Let $f : A \to B$ and $g : B \to C$ be continuous. Then $g \circ f : A \to C$ is continuous. More precisely, if $f$ is continuous at $a \in A$, and $g$ is continuous at $b = f(a) \in B$, then

$$\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right). \tag{2.48}$$

We defined continuity of function $f : A \subset \mathbb{R} \to \mathbb{R}$ at point $a \in A$ in (2.36) as being when $\lim_{x \to a} f(x) = f(a)$. As proven below, here is an equivalent definition.

Definition: Let $f$ be a function with domain $A \subset \mathbb{R}$. Function $f$ is continuous at $a \in A$ if, given $\epsilon > 0$, $\exists \delta > 0$ such that

$$x \in A \text{ and } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon. \tag{2.49}$$

Its value is seen when contrasting it with a definition of *uniform continuity*:

Definition: Let $f$ be a function with domain $A$. Function $f$ is uniformly continuous on $A$ if the condition holds: For a given $\epsilon > 0$, $\exists \delta > 0$ such that

$$x, y \in A \text{ and } |x - y| < \delta \implies |f(x) - f(y)| < \epsilon. \tag{2.50}$$

Note crucially that, with uniform continuity, $\delta$ does not depend on the choice of $x \in [a, b]$.

As perhaps expected, there is a comparable, equivalent definition of uniform continuity in terms of limits of sequences.

Definition: Let $D \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$ be a function. We say that $f$ is uniformly continuous on $D$ if, for any sequences $\{x_n\}, \{y_n\} \subset D$,

$$x_n - y_n \to 0 \implies f(x_n) - f(y_n) \to 0. \tag{2.51}$$

The equivalences of the two forms of definitions, for both continuity, and uniform continuity, are proved in most all beginning books on real analysis. The proofs are instructive, and we give them here, as presented in (the magnificent) Ghorpade and Limaye (2018, Propositions 3.8 and 3.22), along with some examples and further results.

Theorem: Let $D \subseteq \mathbb{R}$, $c \in D$, and let $f : D \to \mathbb{R}$ be a function. Then $f$ is continuous at $c$ as in definition (2.36) if and only if condition (2.49) holds, i.e., $f$ satisfies the following $\epsilon - \delta$ condition: $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$x \in D \text{ and } |x - c| < \delta \implies |f(x) - f(c)| < \epsilon. \tag{2.52}$$

Proof: Let $f$ be continuous at $c$. Suppose the $\epsilon - \delta$ condition does not hold. This means that there is $\epsilon > 0$ such that for every $\delta > 0$, there is $x \in D$ satisfying

$$|x - c| < \delta, \quad \text{but} \quad |f(x) - f(c)| \geq \epsilon.$$

Then there is a sequence $(x_n)$ in $D$ such that $|x_n - c| < 1/n$, but $| f(x_n) - f(c) | \geq \epsilon$ for all $n \in \mathbb{N}$. But then $x_n \to c$ and $f(x_n) \nrightarrow f(c)$. This contradicts the continuity of $f$ at $c$.

Conversely, assume the $\epsilon - \delta$ condition. Let $(x_n)$ be any sequence in $D$ such that $x_n \to c$. Let $\epsilon > 0$ be given. Then there is $\delta > 0$ such that

$$x \in D \text{ and } |x - c| < \delta \Longrightarrow |f(x) - f(c)| < \epsilon.$$

Since $x_n \to c$, there is $n_0 \in \mathbb{N}$ such that $|x_n - c| < \delta$ for all $n \geq n_0$. Hence $|f(x_n) - f(c)| < \epsilon$ for all $n \geq n_0$. Thus $f(x_n) \to f(c)$. This shows that $f$ is continuous at $c$.

Theorem: Let $D \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$ be a function. Then $f$ is uniformly continuous on $D$, as in definition (2.51), if and only if $f$ satisfies (2.50), i.e., for every $\epsilon > 0$, there is $\delta > 0$ such that

$$x, y \in D \text{ and } |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \epsilon. \tag{2.53}$$

Proof: Let $f$ be uniformly continuous on $D$. Suppose there is $\epsilon > 0$ such that for every $\delta > 0$, there are $x$ and $y$ in $D$ such that $|x - y| < \delta$, but $|f(x) - f(y)| \geq \epsilon$. Considering $\delta := 1/n$ for $n \in \mathbb{N}$, we obtain sequences $(x_n)$ and $(y_n)$ in $D$ such that $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \epsilon$ for all $n \in \mathbb{N}$. Then $x_n - y_n \to 0$, but $f(x_n) - f(y_n) \nrightarrow 0$. This contradicts the assumption that $f$ is uniformly continuous on $D$.

Conversely, assume that the uniform $\epsilon - \delta$ condition holds. Let $(x_n)$ and $(y_n)$ be any sequences in $D$ such that $x_n - y_n \to 0$. Let $\epsilon > 0$ be given. Then there is $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$, whenever $x, y \in D$ and $|x - y| < \delta$. Since $x_n - y_n \to 0$, we can find $n_0 \in \mathbb{N}$ such that $|x_n - y_n| < \delta$ for all $n \geq n_0$. But then $|f(x_n) - f(y_n)| < \epsilon$ for all $n \geq n_0$. Thus $f(x_n) - f(y_n) \to 0$. Hence $f$ is uniformly continuous on $D$.

Theorem: Show that uniformly continuous functions defined on the same domain form a vector space. That means, if $f, g : D \to \mathbb{R}$ are uniformly continuous functions, then $cf + dg$ is uniformly continuous, where $c, d \in \mathbb{R}$.

Proof: We show additivity and scalar multiplication (homogeneity) separately:

Additivity: Given $\epsilon > 0$ there are $\delta_1 > 0$ and $\delta_2 > 0$ such that for any $x, y \in D$, if $|x - y| < \delta_1$, then $|f(x) - f(y)| < \epsilon/2$ and if $|x - y| < \delta_2$, then $|g(x) - g(y)| < \epsilon/2$. Therefore, if $|x - y| < \min\{\delta_1, \delta_2\}$, then, from the triangle inequality,

$$|(f + g)(x) - (f + g)(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Homogeneity: By uniform continuity of $f$, we have that, given $\epsilon > 0$, there is a $\delta > 0$ such that, for any $x, y \in D$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon/|c|$, for $c \in \mathbb{R} \setminus \{0\}$. Therefore, if $|x - y| < \delta$,

$$|cf(x) - cf(y)| \leq |c||f(x) - f(y)| < |c|\epsilon/|c| = \epsilon.$$

Combining the two statements gives the result.

Theorem: Suppose that the functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are uniformly continuous and bounded. The product $fg : D \to \mathbb{R}$ is also uniformly continuous.

Proof: Because $f$ and $g$ are bounded, $\exists M_f > 0$ such that, $\forall u \in D$, $|f(u)| < M_f$; and $\exists M_g > 0$ such that $\forall u \in D$, $|g(u)| < M_g$. Set $M := \max\{M_f, M_g\}$. Because $f$ and $g$ are uniformly continuous on $D$, given $\epsilon > 0$, $\exists \delta > 0$ such that, if $u, v \in D$ and $|u - v| < \delta$, then

$$|f(u) - f(v)| < \frac{\epsilon}{2M} \quad \text{and} \quad |g(u) - g(v)| < \frac{\epsilon}{2M}. \tag{2.54}$$

Write

$$f(u)g(u) - f(v)g(v) = f(u)[g(u) - g(v)] + g(v)[f(u) - f(v)].$$

Taking the absolute value and applying the triangle inequality, we have, for $|u - v| < \delta$,

$$\begin{aligned} |f(u)[g(u) - g(v)] + g(v)[f(u) - f(v)]| &\le |f(u)[g(u) - g(v)]| + |g(v)[f(u) - f(v)]| \\ &\le |f(u)||g(u) - g(v)| + |g(v)||f(u) - f(v)| \\ &\le M|g(u) - g(v)| + M|f(u) - f(v)| \\ &< M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} = \epsilon, \end{aligned}$$

where we used the fact that $f$ and $g$ are bounded by $M$; and applying (2.54).

The next result is very important in analysis. Its proof invokes the use of subsequences and the Bolzano Weierstrass Theorem (3.58), the discussion of which we postpone until §3.5. Thus, the following proof can be skipped for now. Recall that, if $f$ is a continuous function on its domain $D$, we write $f \in \mathcal{C}^0(D)$.

Theorem: Let $D \subseteq \mathbb{R}$. Every uniformly continuous function on $D$ is continuous on $D$. Moreover, if $D$ is a closed and bounded set, and $f \in \mathcal{C}^0(D)$, then

$$f \textbf{ is uniformly continuous on } D. \tag{2.55}$$

Proof: (Ghorpade and Limaye, Prop. 3.20) Let $f : D \to \mathbb{R}$ be given. First assume that $f$ is uniformly continuous on $D$. If $c \in D$ and $(x_n)$ is any sequence in $D$ such that $x_n \to c$, then let $y_n := c$ for all $n \in \mathbb{N}$. Since $x_n - y_n \to 0$, we obtain $f(x_n) - f(c) = f(x_n) - f(y_n) \to 0$, that is, $f(x_n) \to f(c)$. Thus $f$ is continuous at $c$. Since this holds for every $c \in D$, $f$ is continuous on $D$.

Now assume that $D$ is a closed and bounded set and $f$ is continuous on $D$. Suppose $f$ is not uniformly continuous on $D$. Then there are sequences $(x_n)$ and $(y_n)$ in $D$ such that $x_n - y_n \to 0$, but $f(x_n) - f(y_n) \nrightarrow 0$. Consequently, there exist $\epsilon > 0$ and positive integers $n_1 < n_2 < \cdots$ such that $|f(x_{n_k}) - f(y_{n_k})| \ge \epsilon$ for all $k \in \mathbb{N}$. Since $D$ is a bounded set, the sequence $\{x_{n_k}\}$ is bounded. By the Bolzano-Weierstrass Theorem, it has a convergent subsequence, say $\{x_{n_{k_j}}\}$. Let us denote the sequences $\{x_{n_{k_j}}\}$ and $\{y_{n_{k_j}}\}$ by $(\tilde{x}_j)$ and $(\tilde{y}_j)$ for simplicity. Let $\tilde{x}_j \to c$. Then $c \in D$, since $D$ is a closed set. Because $x_n - y_n \to 0$, we see that $\tilde{x}_j - \tilde{y}_j \to 0$ and hence $\tilde{y}_j \to c$ as well. Since $f$ is continuous at $c$, we obtain $f(\tilde{x}_j) \to f(c)$ and $f(\tilde{y}_j) \to f(c)$. Thus

$$f(\tilde{x}_j) - f(\tilde{y}_j) \to f(c) - f(c) = 0.$$

But this is a contradiction, since $|f(\tilde{x}_j) - f(\tilde{y}_j)| \ge \epsilon$ for all $j \in \mathbb{N}$. Hence $f$ is uniformly continuous on $D$.

Another proof of (2.55), using the notion of topological compactness, is given later, in (3.78). The notions of uniform continuity and uniform convergence (the latter discussed in §2.6 below) play a major role in analysis. One example, of many, and which we will use, is in the construction of the Riemann integral in §2.5.1. We also require (2.55) for proving (6.24).

**Example 2.4** *We wish to demonstrate that if a function $f : \mathbb{R} \to \mathbb{R}$ is bounded and continuous, it is not necessarily uniformly continuous. We will require the Mean Value Theorem (MVT) (2.94), which states: If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then $\exists c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.*

*To show the claim, we use $f(x) = \sin(x^2)$. Let $x, y \in \mathbb{R}$. The MVT implies*

$$\exists k \in (x, y) \cup (y, x) \text{ such that } \left|\sin\left(x^2\right) - \sin\left(y^2\right)\right| = 2\,|k|\,\left|\cos\left(k^2\right)\right|\,|x - y|. \tag{2.56}$$

*Let $\epsilon > 0$. From (2.53), we need to show, $\forall x, y \in \mathbb{R}$,*

$$\exists \delta > 0 \text{ such that } |x - y| = d_X(x, y) < \delta \implies d_Y(f(x), f(y)) = \left|\sin\left(x^2\right) - \sin\left(y^2\right)\right| < \epsilon.$$

*Observe from (2.56) that, as $x, y \to \infty$, $k$ is not bounded, i.e., not less than $|x - y|$. Thus $\nexists \delta > 0$ that satisfies (2.53).* ■

Definition: A function $f : D \to \mathbb{R}$ is said to be *Lipschitz*, provided

$$\exists C \in \mathbb{R}_{>0} \text{ such that } \forall u, v \in D,\ |f(u) - f(v)| \le C|u - v|. \tag{2.57}$$

Theorem: A Lipschitz function is uniformly continuous.

> Proof: Suppose that $f$ is Lipschitz. From (2.57), there is a number $K > 0$ such that
>
> $$|f(x) - f(y)| \le K|x - y|, \qquad \text{for all } x, y \in E.$$
>
> Fix $\varepsilon > 0$, and let $\delta = \varepsilon/K$. If $x, y \in E$ satisfy $|x - y| < \delta$, then
>
> $$|f(x) - f(y)| \le K|x - y| < K\delta = \varepsilon.$$
>
> The result follows from (2.53).

**Example 2.5** *The function $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. Let $\varepsilon = 2$ and choose an arbitrary $\delta > 0$. Let $n_\delta$ be a natural number such that $1/n_\delta < \delta$. Further, let $x_\delta = n_\delta + 1/n_\delta$ and $y_\delta = n_\delta$. Then $|x_\delta - y_\delta| = 1/n_\delta < \delta$ while*

$$f\left(x_\delta\right) - f\left(y_\delta\right) = \left(n_\delta + 1/n_\delta\right)^2 - n_\delta^2 = 2 + 1/n_\delta^2 > \varepsilon.$$

*We conclude that $f$ is not uniformly continuous.*

*The function $f(x) = x^2$ is Lipschitz (and hence uniformly continuous) on any bounded interval $[a, b]$. For any $x, y \in [a, b]$ we obtain*

$$\begin{aligned}\left|x^2 - y^2\right| &= |(x + y)(x - y)| = |x + y||x - y| \\ &\le (|x| + |y|)|x - y| \le 2\max(|a|, |b|)|x - y|.\end{aligned}$$ ■

**Example 2.6** *Let $f : D \to \mathbb{R}$, $f(x) = \sqrt{x}$, with $D = [0,1]$.*

1. *Prove that $f$ is continuous.*
2. *Prove that $f$ is uniformly continuous.*
3. *Prove that $f$ is* not *Lipschitz.*
4. *Determine whether or not $f : [1, +\infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$, is uniformly continuous. Hint:*

$$|\sqrt{x} - \sqrt{y}| = |\sqrt{x} - \sqrt{y}| \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}.$$

*Solutions:*

1. *If $p > 0$, $|f(x) - f(p)| = |\sqrt{x} - \sqrt{p}| = |x - p|/(\sqrt{x} + \sqrt{p}) < \frac{1}{\sqrt{p}}|x - p|$. Let $\epsilon > 0$ be given. Set $\delta = \min\{p, \sqrt{p}\epsilon\}$. Then $|x - p| < \delta$ implies that $|f(x) - f(p)| < \epsilon$. Therefore $f$ is continuous at $p$. Note that $|x - p| < \delta \Leftrightarrow p - \delta < x < p + \delta$, and $x$ is restricted to $D = [0,1]$. This is why $\delta$ is taken to be $\delta = \min\{p, \sqrt{p}\epsilon\}$.*

   *If $p = 0$, $|f(x) - f(p)| = |\sqrt{x} - \sqrt{p}| = |x - p|/(\sqrt{x} + \sqrt{p})$. Set $\delta = \epsilon^2$.*

2. *From (2.55), a continuous function with domain a closed, bound interval is uniformly continuous.*

3. *One argument using the derivative is: Function $f$ is Lipschitz if $|f(x) - f(y)| \leq C|x - y|$, which implies $|\ \frac{f(x)-f(y)}{x-y}\ | \leq C$,. The left-hand side term is a difference quotient (or a growth of rate) of a function, or graphically the slope of the line joining $(x, f(x))$ and $(y, f(y))$. Thus $f$ is Lipschitz if all the secant lines are of bounded slope.*

   *A second argument not invoking derivatives is to consider sequence $x_n = 1/n$ for $n \in \mathbb{N}$. Observe*

$$\frac{\sqrt{1/n} - \sqrt{0}}{1/n - 0} = \frac{1/\sqrt{n}}{1/n} = \sqrt{n}.$$

   *This ratio can be made arbitrarily large as $n \to \infty$. Therefore, the square-root function fails to be Lipschitz.*

4. *The function is uniformly continuous on $[1, +\infty)$. Let $\epsilon > 0$, and take $\delta = \epsilon$. If $|x - y| < \delta$, then*

$$|\sqrt{x} - \sqrt{y}| = |\sqrt{x} - \sqrt{y}| \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{|x - y|}{2} < |x - y| < \delta = \epsilon. \qquad \blacksquare$$

We now gather some fundamental results, with proofs available in most all real analysis textbooks. Let $I = [a,b]$ be a closed, bounded interval. Let $f$ be a continuous function on $I$, i.e., $f \in \mathcal{C}^0[a,b]$. Then:

- The image of $f \in \mathcal{C}^0(I)$, $I = [a,b]$, forms a closed, bounded subset of $\mathbb{R}$, i.e.,

$$\forall x \in I, \exists m, M \in \mathbb{R} \textbf{ such that } m \leq f(x) \leq M. \tag{2.58}$$

- (*Extreme Value Theorem*) Function $f$ assumes minimum and maximum values on $I$, i.e.,[7]

$$f \in \mathcal{C}^0(I),\; I = [a,b] \quad \Longrightarrow \quad \exists x_0, x_1 \in I \textbf{ s.t. } \forall x \in I,\; f(x_0) \le f(x) \le f(x_1). \tag{2.59}$$

  The proof is most easily conducted using the concept of compactness, and so we relegate the proof to the end of §3.5.

- (*Intermediate Value Theorem*) Let $f : D \subset \mathbb{R} \to \mathbb{R}$, $I = [a,b] \subset D$, $f \in \mathcal{C}^0(I)$, and $\alpha = f(a)$ and $\beta = f(b)$.

$$\forall \gamma : \; \alpha < \gamma < \beta, \;\; \exists c \in (a,b) \textbf{ such that } f(c) = \gamma. \tag{2.60}$$

- As stated in (2.55) and proved there,

$$f \in \mathcal{C}^0(I),\; I = [a,b] \quad \Longrightarrow \quad f \textbf{ is uniformly continuous on } I. \tag{2.61}$$

These four facts together constitute what Pugh (2002, p. 39) argues could rightfully be called the *Fundamental Theorem of Continuous Functions.*

We conclude with the definition of a null sequence, and a basic, useful result.

Definition: A *null sequence* is any real-valued sequence $\{h_n\}_{n=0}^{\infty}$ that converges to 0 as $n \to \infty$. That is, for any $\epsilon > 0$, $\exists n_0 = n_0(\epsilon)$ such that $|h_n| < \epsilon$ for $n \ge n_0$. Examples of positive such sequences include $h_k = 1/k$ and $h_k = 1/2^k$.

Theorem: (Garling, Prop 3.2.8.) Let $\{\epsilon_n\}_{n=0}^{\infty}$ be a null sequence of positive numbers. Sequence $\{a_n\}_{n=0}^{\infty}$ converges to $\ell$ iff

$$\forall k, \exists n_k \text{ such that } |a_n - \ell| < \epsilon_k \text{ for } n \ge n_k. \tag{2.62}$$

> Proof:
> Necessary ($\Leftarrow$): As $\{\epsilon_n\}_{n=0}^{\infty}$ is a null sequence of positive numbers, for any given $\epsilon$, $\exists k \in \mathbb{N}$ such that $0 < \epsilon_k < \epsilon$. The condition then implies that, for $n > n_k$, $|a_n - \ell| < \epsilon_k < \epsilon$, this being the definition of sequence convergence.
>
> Sufficient ($\Rightarrow$): Assume sequence $\{a_n\}_{n=0}^{\infty}$ converge to $\ell$. Let $\epsilon_1 > 0$ be given. Then, as $\ell$ is the limit point of $\{a_n\}$, $\exists n_1 \in \mathbb{N}$ such that $|a_n - \ell| < \epsilon_1, \forall n \ge n_1$. Similarly, given $\epsilon_2$ such that $0 < \epsilon_2 < \epsilon_1$, $\exists n_2 \in \mathbb{N}$, $n_2 > n_1$, such that $|a_n - \ell| < \epsilon_2, \forall n \ge n_2$. Continuing, we obtain, as required, a strictly increasing sequence $\{n_k\} \in \mathbb{N}$ and strictly decreasing sequence $\{\epsilon_k\} \in \mathbb{R}_{>0}$ such that, for each $k \in \mathbb{N}$, $|a_n - \ell| < \epsilon_k, \forall n \ge n_k$.

[7] According to Petrovic, Advanced Calculus: Theory and Practice, 2nd ed., 2020, p. 99, this is known as "The Maximum Theorem", "The Extreme Value Theorem", and Weierstrass called it "The Principal Theorem" in his lectures in 1861. The result was originally proved by Bolzano, but his proof was not published until 1930. The first publication was by Cantor in 1870.

## 2.3 Differentiation

### 2.3.1 Definitions and Techniques

Let $f \in \mathcal{C}^0(I)$, where $I$ is an interval of nonzero length. If the *Newton quotient*

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2.63}$$

exists for $x \in I$, then $f$ is *differentiable* at $x$, the limit is the *derivative* of $f$ at $x$, and is denoted $f'(x)$ or $df/dx$. Similar to the notation for continuity, if $f$ is differentiable at each point in $I$, then $f$ is differentiable on $I$, and if $f$ is differentiable on its domain, then $f$ is differentiable. If $f$ is differentiable and $f'(x)$ is a continuous function of $x$, then $f$ is *continuously differentiable*, and is of *class* $\mathcal{C}^1$.

Observe that, for $h$ small,

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad f(x+h) \approx f(x) + hf'(x), \tag{2.64}$$

which, for constant $x$, is a linear function in $h$. By letting $h = y - x$, (2.63) can be equivalently written as

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x}, \tag{2.65}$$

which is sometimes more convenient to work with.

Lemma (Fundamental lemma of differentiation) This lemma makes the notion more precise that a differentiable function can be approximated at each point in (the interior of) its domain by a linear function whose slope is the derivative at that point. As in Protter and Morrey (1991, p. 85), let $f$ be differentiable at the point $x$. Then there exists a function $\eta$ defined on an interval about zero such that

$$f(x+h) - f(x) = \left[f'(x) + \eta(h)\right] \cdot h, \tag{2.66}$$

and $\eta$ is continuous at zero, with $\eta(0) = 0$. The proof follows by solving (2.66) for $\eta(h)$ and defining $\eta(0) = 0$, i.e.,

$$\eta(h) := \frac{1}{h}\left[f(x+h) - f(x)\right] - f'(x), \quad h \neq 0, \qquad \eta(0) := 0.$$

As $f$ is differentiable at $x$, $\lim_{h \to 0} \eta(h) = 0$, so that $\eta$ is continuous at zero.

**Example 2.7** *From the definition of limit,* $\lim_{h \to 0} 0/h = 0$*, so that the derivative of* $f(x) = k$ *for some constant* $k$ *is zero. For* $f(x) = x$*, it is easy to see from (2.63) that* $f'(x) = 1$*. For* $f(x) = x^2$*,*

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} (2x + h) = 2x.$$

*Now consider* $f(x) = x^n$ *for* $n \in \mathbb{N}$*. The binomial theorem implies*

$$f(x+h) = (x+h)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} h^i = x^n + nhx^{n-1} + \cdots + h^n,$$

*so that*

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \left(nx^{n-1} + \cdots + h^{n-1}\right) = nx^{n-1}. \tag{2.67}$$

*Now let* $f(x) = x^{-n}$ *for* $n \in \mathbb{N}$. *From* (1.10), *for any* $x \neq 0$ *and* $y \neq 0$,

$$f(y) - f(x) = y^{-n} - x^{-n} = \frac{x^n - y^n}{x^n y^n} = (y - x)\left[-\frac{y^{n-1} + y^{n-2}x + \cdots + x^{n-1}}{x^n y^n}\right],$$

*so that* (2.65) *implies*

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = -\lim_{y \to x}\left[\frac{y^{n-1} + y^{n-2}x + \cdots + x^{n-1}}{x^n y^n}\right] = -\frac{n x^{n-1}}{x^{2n}} = -n x^{-n-1}.$$

*Thus, for* $f(x) = x^n$ *with* $n \in \mathbb{Z}$, $f'(x) = n x^{n-1}$. ∎

Assume for functions $f$ and $g$ defined on $I$ that $f'(x)$ and $g'(x)$ exist on $I$. Then

$$\textbf{(sum rule)} \qquad (f+g)'(x) = f'(x) + g'(x), \tag{2.68}$$

$$\textbf{(product rule)} \qquad (fg)'(x) = f(x)g'(x) + g(x)f'(x), \tag{2.69}$$

$$\textbf{(quotient rule)} \qquad (f/g)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0, \tag{2.70}$$

$$\textbf{(chain rule)} \qquad (g \circ f)'(x) = g'(f(x))\, f'(x). \tag{2.71}$$

The chain rule is proven in all beginning real analysis books. It is simple to prove using the fundamental lemma of differentiation (2.66); see, e.g., Protter and Morrey (1991, p. 85), or Ghorpade and Limaye (2018, Proposition 4.10).

Remark 1: Simply, but usefully, from (2.68); and (2.69) with $f(x) = -1$,

$$(f - g)'(x) = f'(x) - g'(x). \tag{2.72}$$

Remark 2: The set of differential functions forms a vector space. That means, if $f$ and $g$ are differentiable functions on their domain $D$, and $a, b \in \mathbb{R}$, then functions $(af) : D \to \mathbb{R}$, with $(af)(x) := af(x)$, and $(f+g) : D \to \mathbb{R}$, with $(f+g)(x) := f(x)+g(x)$, are differentiable; these properties being called homogeneity and linearity, respectively.

Remark 3: With $y = f(x)$ and $z = g(y)$, the usual mnemonic for the chain rule is

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}.$$

**Example 2.8** *Result* (2.67) *for* $n \in \mathbb{N}$ *could also be established by using an induction argument: Let* $f(x) = x^n$ *and assume* $f'(x) = n x^{n-1}$. *It holds for* $n = 1$; *induction and the product rule imply for* $f(x) = x^{n+1} = x^n \cdot x$ *that* $f'(x) = x^n \cdot 1 + x \cdot n x^{n-1} = (n+1)\, x^n$. ∎

Theorem:

$$\textbf{If } f \textbf{ is differentiable at } a\textbf{, then } f \textbf{ is continuous at } a. \tag{2.73}$$

Proof: This is seen by taking limits of

$$f(x) = \frac{f(x) - f(a)}{x - a}(x - a) + f(a),$$

which, using (2.23) and (2.24), gives

$$\lim_{x \to a} f(x) = f'(a) \cdot 0 + f(a) = f(a),$$

and recalling the definition of continuity, e.g., (2.37).

The function $f(x) = |x|$ at $x = 0$ is the showcase example that continuity does not imply differentiability.

Proposition: Differentiability of $f$ does not imply that $f'$ is continuous.

> Proof: We just need a counterexample. A popular one takes $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Then
>
> $$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), \quad x \neq 0,$$
>
> and $\lim_{x \to 0} f'(x)$ does not exist. But, from the Newton quotient at $x = 0$,
>
> $$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} h \sin(1/h) = 0,$$
>
> so that $f'(0) = 0$. Thus, $f'(x)$ is not continuous.

What is true is that *uniform differentiability* implies uniform continuity of the derivative, as discussed next. I took this from Estep (2002, §32.4), and it is also stated as an exercise in Stoll (2021, p. 205, # 16).

Definition: A function $f$ is said to be uniformly differentiable on an interval $[a, b]$ if, $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$\left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| < \epsilon, \quad \forall x, y \in [a, b] \quad \text{with} \quad |x - y| < \delta.$$

Theorem: If $f$ is uniformly differentiable on $[a, b]$, then $f'(x)$ is uniformly continuous on $[a, b]$.

> Proof: If $f$ is uniformly differentiable, then for $x, y \in [a, b]$ and $\epsilon > 0$, we can find a $\delta > 0$ such that, for $|x - y| < \delta$,
>
> $$\begin{aligned} |f'(y) - f'(x)| &= \left| f'(y) - \frac{f(y) - f(x)}{y - x} + \frac{f(y) - f(x)}{y - x} - f'(x) \right| \\ &\leq \left| f'(y) - \frac{f(y) - f(x)}{y - x} \right| + \left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| < 2\epsilon. \end{aligned}$$
>
> Thus, $f'(x)$ is uniformly continuous on $[a, b]$.

**Example 2.9** *Function $f(x) = x^2$ is uniformly differentiable on any bounded interval $[a, b]$. The function $f(x) = 1/x$ is differentiable on $(0, 1)$, but is not uniformly differentiable on $(0, 1)$.* ■

Theorem (Fermat): Let $f$ be defined on an interval $[a, b]$ and suppose that it attains its greatest or its smallest value at a point $c \in (a, b)$.

$$\textbf{If } f \textbf{ is differentiable at } c\textbf{, then } f'(c) = 0. \tag{2.74}$$

Proof: We will assume that $f$ attains its greatest value at $c \in (a,b)$, i.e., that $f(x) \le f(c)$ for all $x \in [a,b]$. If $x < c$ then

$$\frac{f(x) - f(c)}{x - c} \ge 0,$$

and (2.47) implies that $f'(c) \ge 0$. On the other hand, if $x > c$ then

$$\frac{f(x) - f(c)}{x - c} \le 0,$$

so $f'(c) \le 0$. Combining the two, $f'(c) \ge 0$ and $f'(c) \le 0$, we obtain that $f'(c) = 0$.

Of great use is *l'Hôpital's rule*[8] for evaluating *indeterminate forms or ratios*:

Theorem (l'Hôpital's rule, 0/0 case): Let $f$ and $g$, and their first derivatives, be continuous functions on $(a,b)$. If $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$ and $\lim_{x\to a^+} f'(x)/g'(x) = L$, then

$$\lim_{x\to a^+} f(x)/g(x) = L. \tag{2.75}$$

Most students remember this very handy result, but few can intuitively justify it. Most real analysis textbooks give the rigorous proof, and also discuss and prove the $\infty/\infty$ case. We give a "heuristic justification" that is easy to remember.

Assume $f$ and $g$ are continuous at $a$, so that $f(a) = g(a) = 0$. Using (2.64) gives

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{h\to 0} \frac{f(a+h)}{g(a+h)} \approx \lim_{h\to 0} \frac{f(a) + hf'(a)}{g(a) + hg'(a)} = \frac{f'(a)}{g'(a)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

Another, related, quick way of seeing this part of the rule is the following. As in Stoll (p. 212, Exercise #2), suppose $f, g$ are differentiable on $(a,b)$, $x_0 \in (a,b)$, and $g'(x_0) \neq 0$. If $f(x_0) = g(x_0) = 0$, then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}.$$

This follows because, for $x \neq x_0$, write

$$\frac{f(x) - 0}{g(x) - 0} = \frac{f(x) - f(x_0)}{x - x_0} \frac{x - x_0}{g(x) - g(x_0)} = \frac{f(x) - f(x_0)}{x - x_0} \Bigg/ \frac{g(x) - g(x_0)}{x - x_0},$$

and take the limit, $x \to x_0$.

A different "rough proof" of l'Hôpital's rule is given by Pugh (2002, p. 143).

A similar result, also referred to as l'Hôpital's rule, holds for $x \to b^-$, and for $x \to \infty$; and also for the case when $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = \infty$.

**Example 2.10** *(Petrovic, Advanced Calculus: Theory and Practice, 2nd ed., 2020, Example 4.6.4) Determine*

$$\lim_{x\to 1} x^{\frac{1}{1-x}}.$$

[8]Named after Guillaume François Antoine Marquis de l'Hôpital (1661–1704), who was taught calculus by Johann Bernoulli (for a high price), and wrote the first calculus textbook (1696) based on Bernoulli's notes, in which the result appeared. Not surprisingly, l'Hôpital's rule was also known to Bernoulli (confirmed in Basel, 1922, with the discovery of certain written documents).

*This limit is of the form* $(1^{\infty})$*, so we cannot apply l'Hôpital's rule directly. Since* $x = e^{\ln x}$ *when* $x > 0$*, and since* $x \to 1$ *means that we can assume that* $x > 0$*, we have*

$$x^{\frac{1}{1-x}} = \exp\left(\ln x^{\frac{1}{1-x}}\right) = \exp\left(\frac{\ln x}{1-x}\right).$$

*When* $x \to 1$*, the exponent* $\ln x/(1-x)$ *is of the form* $\left(\frac{0}{0}\right)$*, so we can apply l'Hôpital's rule. Further,* $(1-x)' = -1 \neq 0$ *so*

$$\lim_{x\to 1} \frac{\ln x}{1-x} = \lim_{x\to 1} \frac{1/x}{-1} = -1,$$

*and we obtain that* $\lim_{x\to 1} x^{\frac{1}{1-x}} = e^{-1}$*.* ∎

Definition: Let $f$ be a strictly increasing continuous function on a closed interval $I$. Function $f$ on $I$ is said to be *invertible*, or is a bijection (injective and surjective; see page 6). The *inverse function* $g = f^{-1}$ is defined as the function such that $(g \circ f)(x) = x$ and $(f \circ g)(y) = y$. It is also continuous and strictly increasing.

For $f$ a strictly increasing continuous function on a closed interval $I$, (2.58) and the IVT (2.60) imply that the image of $f$ is also a closed interval. (See (3.75) for a more general result, namely, only continuity is required.) If $f$ is also differentiable in the interior of $I$ with $f'(x) > 0$, then a fundamental result is that

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{f'(g(y))}. \tag{2.76}$$

We prove a simpler version of this. It assumes existence of the derivative of $f^{-1}$.

Lemma: Let $X, Y \subset \mathbb{R}$, and let $f : X \to Y$ be an invertible function, with inverse $f^{-1} : Y \to X$. Suppose that $x_0 \in X$ and $y_0 \in Y$ are limit points of $X, Y$, respectively, such that $y_0 = f(x_0)$. This implies $x_0 = f^{-1}(y_0)$. If $f$ is differentiable at $x_0$, and $f^{-1}$ is differentiable at $y_0$, then

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}.$$

> Proof: (Tao, Analysis I, 4th ed., 2022, p. 226) First note that, if $f$ is the identity function, i.e., $f(x) = x$ for all $x \in X$, then $f$ is differentiable at $x_0$ and $f'(x_0) = 1$. From the chain rule (2.71),
>
> $$\left(f^{-1} \circ f\right)'(x_0) = \left(f^{-1}\right)'(y_0)\, f'(x_0).$$
>
> But $f^{-1} \circ f$ is the identity function on $X$, and hence $(f^{-1} \circ f)'(x_0) = 1$.

(Tao then gives the more general proof, which relaxes the requirement on $f^{-1}$ from differentiability to continuity.)

A useful application involving the arcsin and arctan functions is given below in Example 2.13.

We close this section with two definitions that are of occasional use. We will use them below in proving (2.150).

Definition: (As in Stoll, Def 5.1.2) Let $I \subset \mathbb{R}$ be an interval and let $f$ be a real-valued function with domain $I$. If $p \in I$ is such that $I \cap (p, \infty) \neq \emptyset$, then the right derivative of $f$ at $p$, denoted $f'_+(p)$, is defined as

$$f'_+(p) = \lim_{h \to 0^+} \frac{f(p+h) - f(p)}{h}, \tag{2.77}$$

provided the limit exists. Similarly, if $p \in I$ satisfies $(-\infty, p) \cap I \neq \emptyset$, then the left derivative of $f$ at $p$, denoted $f'_-(p)$, is given by

$$f'_-(p) = \lim_{h \to 0^-} \frac{f(p+h) - f(p)}{h}, \tag{2.78}$$

provided the limit exists.

NOTE: if $I = [a, b]$, the right derivative applies to $p \in [a, b)$, but not for $p = b$, because $I \cap (b, \infty) = \emptyset$. Similar for left derivative.

**Example 2.11** *A uniformly continuous function on* $[0, 1]$ *that is differentiable on* $(0, 1)$ *need not have* $f'$ *bounded on* $(0, 1)$*. For example, take* $f(x) = \sqrt{x}$*, and note the limit from the right at zero of* $f'$*, i.e.,* $f'(0+)$*.* ■

Theorem: Suppose $f$ is differentiable on an interval $I$. Then $f'$ is bounded on $I$ if and only if there exists a constant $M$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in I$.

> Proof:
>
> ($\Longrightarrow$) Take any $x, y \in I$. If $x = y$, then $0 \leq 0$ holds vacuously. Assume, w.l.o.g., $x < y$. Suppose that $f'$ is bounded on $I$, i.e., $\exists M > 0$ such that, $\forall c \in I, |f'(c)| \leq M$. Since $f$ is differentiable on $I$, $f$ is continuous on $[x, y]$ and differentiable on $(x, y)$. Thus, the MVT applies to $f$ (but not necessarily $f'$, which was not assumed continuous), so that $\exists x_0 \in (x, y)$ such that $f(y) - f(x) = f'(x_0)(y - x)$. As $|f'(x_0)| \leq M$ and $y - x > 0$,
>
> $$-M(y - x) \leq f(y) - f(x) = f'(x_0)(y - x) \leq M(y - x).$$
>
> That is, $|f(y) - f(x)| \leq M(y - x) = M|y - x|$.
>
> ($\Longleftarrow$) Suppose $\forall x, y \in I$, $|f(y) - f(x)| \leq M|y - x|$, where $M > 0$. Fix an arbitrary $x \in I$. Then
>
> $$\forall y \in I,\ y \neq x, \quad \left| \frac{f(y) - f(x)}{y - x} \right| \leq M, \quad \text{or} \quad -M \leq f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} \leq M.$$
>
> The limit exists because $f$ is differentiable on $I$. That is, $\forall x \in I$, $|f'(x)| \leq M$, i.e., $f'$ is bounded on $I$.

There is also the following related result.

Theorem: Suppose that $f$ has a bounded derivative on $(a, b)$. Then $f$ is uniformly continuous on $(a, b)$.

See Petrovic, exercise #4.4.27.

### 2.3.2 Trigonometric Functions

We turn now to some fundamental results on limits and derivatives for trigonometric functions. Recall the unit circle geometric representation of sine and cosine, and, from Pythagoras, the fundamental relation $\cos^2(x) + \sin^2(x) = 1$. Of great use are the following relations:

Theorem (Angle sum and difference identities): For $x, y \in \mathbb{R}$,

$$\sin(x + y) = \sin x \cos y + \cos x \sin y, \tag{2.79a}$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y. \tag{2.79b}$$

These can be demonstrated from a clever graphic, such as in Stillwell, Numbers and Geometry, 1998, §5.3, though I prefer the nice derivation from Kuttler, Calculus of One and Many Variables, p. 59), included here.

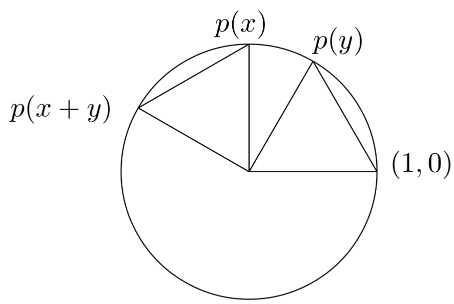



Figure 1: From Kuttler. Unit circle with two inscribed, equal triangles

Theorem: Let $x, y \in \mathbb{R}$. Then

$$\cos(x + y)\cos(x) + \sin(x + y)\sin(x) = \cos(y). \tag{2.80}$$

Proof: Recall that, for a real number $z$, there is a unique point $p(z)$ on the unit circle and the coordinates of this point are $\cos z$ and $\sin z$. Now it seems geometrically clear from Figure 1 that the length of the arc between $p(x + y)$ and $p(x)$ has the same length as the arc between $p(y)$ and $p(0)$.

Also from geometric reasoning the distance between the points $p(x+y)$ and $p(x)$ must be the same as the distance from $p(y)$ to $p(0)$. In fact, the two triangles have the same angles and the same sides. Writing this in terms of the definition of the trig functions and the distance formula,

$$(\cos(x + y) - \cos x)^2 + (\sin(x + y) - \sin x)^2 = (\cos y - 1)^2 + (\sin y - 0)^2.$$

Expanding, we get

$$\begin{aligned}\cos^2(x + y) + \cos^2 x - 2\cos(x + y)\cos x + \sin^2(x + y) + \sin^2 x - 2\sin(x + y)\sin x \\ = \cos^2 y - 2\cos y + 1 + \sin^2 y.\end{aligned}$$

Now using that $\cos^2 + \sin^2 = 1$,

$$2 - 2\cos(x + y)\cos(x) - 2\sin(x + y)\sin(x) = 2 - 2\cos(y),$$

which gives (2.80).

Continuing from Kuttler's presentation, we now prove (2.79).

Proof: The length of the unit circle is defined as $2\pi$. Thus, for example, $\sin\left(\frac{\pi}{2}\right) = 1$, $\cos\left(\frac{\pi}{2}\right) = 0$. Letting $x = \pi/2$, (2.80) implies (seen also from the unit circle)

$$\sin(y + \pi/2) = \cos y. \tag{2.81}$$

Now let $u = x + y$ and $v = x$. Then (2.80) implies $\cos u \cos v + \sin u \sin v = \cos(u - v)$. Also, from this and the basic relations

$$\cos(-x) = \cos(x) \qquad \text{and} \qquad \sin(-x) = -\sin(x), \tag{2.82}$$

we obtain

$$\begin{aligned} \cos(u+v) &= \cos(u - (-v)) = \cos u \cos(-v) + \sin u \sin(-v) \\ &= \cos u \cos v - \sin u \sin v. \end{aligned}$$

Thus, letting $v = \pi/2$ (and also graphically clear from the unit circle),

$$\cos(u + \pi/2) = -\sin u. \tag{2.83}$$

Then, from (2.81) and (2.83),

$$\begin{aligned} \sin(x+y) &= -\cos\left(x + \frac{\pi}{2} + y\right) \\ &= -\left[\cos\left(x + \frac{\pi}{2}\right)\cos y - \sin\left(x + \frac{\pi}{2}\right)\sin y\right] \\ &= \sin x \cos y + \sin y \cos x, \qquad \text{and} \\ \sin(x-y) &= \sin x \cos y - \cos x \sin y. \end{aligned}$$

Using (2.79b), $\cos(2x) = \cos(x + x) = \cos(x)\cos(x) - \sin(x)\sin(x) = \cos^2(x) - \sin^2(x)$. From this, we easily obtain two of the useful double-angle formulae,

$$\cos 2x = \cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x; \tag{2.84}$$

and

$$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x) = -1 + 2\cos^2 x. \tag{2.85}$$

Let $f(x) = \sin(x)$. Using (2.79a), the derivative of $f$ is

$$\begin{aligned} \frac{d\sin(x)}{dx} &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h\to 0} \frac{\sin x \ \cos h - \sin x}{h} + \lim_{h\to 0} \frac{\cos x \ \sin h}{h} \\ &= \sin(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h} \\ &= \cos(x), \end{aligned} \tag{2.86}$$

where

$$L_s := \lim_{h\to 0} \frac{\sin(h)}{h} = 1 \quad \text{and} \quad \lim_{h\to 0} \frac{\cos(h) - 1}{h} = 0. \tag{2.87}$$

Both limits in (2.87) need to be justified. If we *assume* that $L_s$ is not infinite, then the second limit in (2.87) is easy to prove: Write

$$\begin{aligned}\frac{\cos(h)-1}{h} &= \frac{h(\cos(h)+1)}{h(\cos(h)+1)}\frac{\cos(h)-1}{h} = \frac{h(\cos^2 h-1)}{h^2(\cos(h)+1)} \\ &= -\left(\frac{\sin h}{h}\right)^2 \frac{h}{\cos(h)+1}\end{aligned}$$

using $\cos^2(x)+\sin^2(x)=1$, so that, from (2.24) and because we assumed that $L_s \in \mathbb{R}$,

$$\lim_{h\to 0}\frac{\cos(h)-1}{h} = -\lim_{h\to 0}\left(\frac{\sin h}{h}\right)^2 \lim_{h\to 0}\frac{h}{\cos(h)+1} = 0.$$

Using (2.86), along with (2.81) and (2.83), i.e., $\cos(x) = \sin(x+\pi/2)$ and $\sin(x) = -\cos(x+\pi/2)$, the chain rule gives

$$\frac{d\cos x}{dx} = \frac{d\sin(x+\pi/2)}{dx} = \cos(x+\pi/2) = -\sin(x).$$

Students remember that derivative of sine and cosine involve, respectively, cosine and sine, but some forget the signs. To recall them, just think of the unit circle at angle $\theta = 0$ and the geometric definition of sine and cosine. A slight increase in $\theta$ increases the vertical coordinate (sine) and decreases the horizontal one (cosine).

The easiest way of proving the former limit in (2.87) is using (2.86); it follows trivially by using the derivative of $\sin x$, i.e.,

$$\lim_{h\to 0}\frac{\sin(h)}{h} = \lim_{h\to 0}\frac{\sin h - \sin 0}{h} = \left.\frac{d\sin x}{dx}\right|_{x=0} = \cos 0 = 1.$$

The limits in (2.87) also follow by applying l'Hôpital's rule: For the latter,

$$\lim_{h\to 0}\frac{\cos(h)-1}{h} = \frac{-\sin(h)}{1} = -\sin(0) = 0.$$

The circular logic between (2.86) and (2.87) is obviously not acceptable.[9] The properties of the sine and cosine functions can be correctly, elegantly and easily derived from an algebraic point of view by starting with functions $s$ and $c$ such that

$$s' = c,\ c' = -s,\ s(0) = 0 \text{ and } c(0) = 1 \tag{2.88}$$

(see e.g., Lang, 1997, §4.3). As definitions one takes

$$\cos(z) = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{(2k)!} \qquad \text{and} \qquad \sin(z) = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)!}, \tag{2.89}$$

which converge for all $z \in \mathbb{R}$; see Example 2.85 below for details. From (2.89), the properties of the trigonometric functions can be inferred, such as $\cos^2(x)+\sin^2(x) = 1$, (2.79), (2.81), (2.82), and (2.83). See e.g., Browder (1996, §3.6) or Hijab (1997, §3.5) for details.

[9] Of course, from a geometric point of view, it is essentially obvious that $\lim_{h\to 0} h^{-1}\sin(h) = 1$. Let $\theta$ be the angle in the first quadrant of the unit circle, measured in radians. Recall that $\theta$ then represents the length of the arc on the unit circle, of which the total length is $2\pi$. Then it seems apparent that, as $\theta$ decreases, the arc length coincides with $\sin(\theta)$.

From (2.82) and (2.79a),

$$\begin{aligned}\sin(x-y)+\sin(x+y) &= \sin x\cos y-\cos x\sin y \;+\; \sin x\cos y+\cos x\sin y\\ &= 2\sin x\cos y.\end{aligned}$$

Now let $b=x+y$ and $c=y-x$, so that $x=(b-c)/2$ and $y=(b+c)/2$. It follows that

$$\sin(b)-\sin(c)=2\sin\left(\frac{b-c}{2}\right)\cos\left(\frac{b+c}{2}\right), \tag{2.90}$$

which is one of the sum-to-product identities.

Finally, let $f(x)=\tan(x):=\sin(x)\,/\,\cos(x)$ so that

$$f'(x)=\frac{\cos(x)\cos(x)-\sin(x)(-\sin(x))}{\cos^2(x)}=1+\frac{\sin^2(x)}{\cos^2(x)}=1+\tan^2(x) \tag{2.91}$$

from the quotient rule.

**Example 2.12** *To find the derivative of* $y=\arcsin x$, *note that* $\sin y=x$, *so that* $y$ *can be considered an acute angle in a right triangle with a sine ratio of* $x/1$*; see Figure 2.*

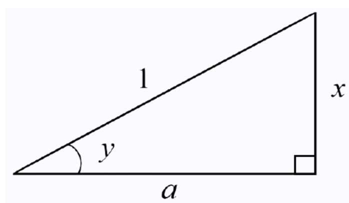



Figure 2: $x=\sin y$ and $\cos y=a$

*Differentiating* $\sin y=x$ *with respect to* $x$ *and using the chain rule and* (2.86) *gives*

$$\cos y\cdot\frac{dy}{dx}=1,\quad \text{or}\quad \frac{dy}{dx}=\frac{1}{\cos y}.$$

*Note from Figure 2 that* $\cos y=a$. *From Pythagoras,* $a^2+x^2=1^2$ *or* $a=\sqrt{1-x^2}$, *so that*

$$\frac{d}{dx}(\arcsin x)=\frac{1}{\sqrt{1-x^2}}. \qquad \blacksquare$$

**Example 2.13** *Let* $f(x)=\sin(x)$ *for* $-\pi/2<x<\pi/2$, *with derivative* $f'(x)=\cos x$ *from (2.86). From (2.76) and relation* $\cos^2(x)+\sin^2(x)=1$, *the inverse function* $g(y)=\arcsin(y)$ *has derivative*

$$g'(y)=\frac{1}{\cos(\arcsin(y))}=\frac{1}{\sqrt{1-\sin^2(\arcsin(y))}}=\frac{1}{\sqrt{1-y^2}},$$

*which agrees with Example 2.12. Similarly, let* $f(x)=\tan(x)$, *for* $-\pi/2<x<\pi/2$ *with inverse function* $g(y)=\arctan(y)$, *so that, from* (2.91),

$$g'(y)=\frac{1}{1+\tan^2(\arctan(y))}=\frac{1}{1+y^2}. \tag{2.92}$$

*Now let* $z$ *be a constant. Using (2.92) and the chain rule gives*

$$\frac{d}{dx}\arctan(z-x)=-\frac{1}{1+(z-x)^2}, \tag{2.93}$$

*which we will use below in Example 2.58.* $\blacksquare$

### 2.3.3 Mean Value Theorem and Function Extreme Points

Theorem (Mean Value Theorem, MVT): Let $f$ be a continuous function on its domain $[a, b]$, $b > a$, and differentiable on $(a, b)$. Then $\exists \xi \in (a, b)$ such that $f(b) - f(a) = f'(\xi)(b - a)$. The MVT is perhaps more easily remembered as

$$\frac{f(b) - f(a)}{b - a} = f'(\xi), \quad b - a \neq 0. \tag{2.94}$$

The proof is given below, after we prove Rolle's theorem. Many common and important calculus results (see the list below) hinge on this result, or a generalization of it, and we will also see it used extensively in the multivariate setting. The MVT becomes intuitive from Figure 3, for a differentiable (and, thus, continuous) function, and such that $f$ is continuous (from the right, and the left, respectively) also at endpoints $a$ and $b$, as stated in the theorem.

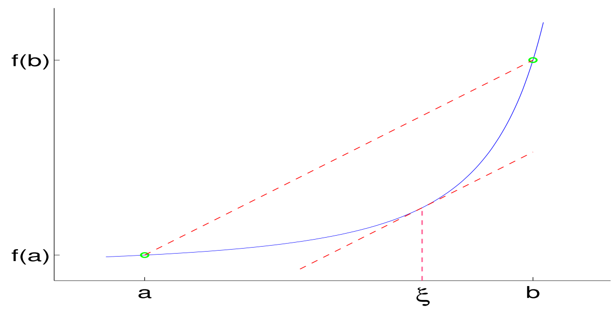



Figure 3: The mean value theorem of the differential calculus

Still, without a proof, a convincing argument and clever graphic are not adequate. Pugh (Real Mathematical Analysis, 2nd ed., 2015, p. 4) says it quite well, and is the only analysis book I have ever seen that discusses this in such detail. Here is an excerpt:

> When is a mathematical statement accepted as true? Generally, mathematicians would answer "Only when it has a proof inside a familiar mathematical framework." A picture may be vital in getting you to believe a statement. An analogy with something you know to be true may help you understand it. An authoritative teacher may force you to parrot it. A formal proof, however, is the ultimate and only reason to accept a mathematical statement as true.
>
> There has been a tendency in recent years to take the notion of proof down from its pedestal. Critics point out that standards of rigor change from century to century. New gray areas appear all the time. Is a proof by computer an acceptable proof? Is a proof that is spread over many journals and thousands of pages, that is too long for any one person to master, a proof? And of course, venerable Euclid is full of flaws, some filled in by Hilbert, others possibly still lurking.
>
> Clearly it is worth examining closely and critically the most basic notion of mathematics, that of proof. On the other hand, it is important to bear in mind that all distinctions and niceties about what precisely constitutes a proof are mere quibbles compared to the enormous gap between any generally accepted version of a proof and the notion of a convincing argument. Compare Euclid, with all his flaws to the most eminent of the ancient exponents of the convincing argument – Aristotle. Much of Aristotle's reasoning was brilliant, and he certainly convinced most thoughtful people for over a thousand years. In some cases his analyses were exactly right, but in others, such as heavy objects falling faster than light ones, they turned out to be totally wrong. In contrast, there is not to my knowledge a single theorem stated in Euclid's Elements that in the course of two thousand years turned out to be false. That is quite an astonishing record, and an extraordinary validation of proof over convincing argument.

Theorem (Rolle): Suppose that $f$ is a function defined and continuous on an interval $[a,b]$, that it is differentiable in $(a,b)$, and that $f(a)=f(b)$. Then

$$\exists c \in (a,b) \text{ such that } f'(c)=0. \tag{2.95}$$

The equivalent contrapositive will be used below:

$$\nexists c \in (a,b) \text{ such that } f'(c)=0 \;\Rightarrow\; f(a) \neq f(b). \tag{2.96}$$

> Proof: We start with the EVT (2.59), which guarantees that $f$ attains its largest value $M$ and its smallest value $m$ on $[a,b]$. There are two possibilities: either $M=m$ or $M>m$. In the former case, the inequality $m \leq f(x) \leq M$ implies that $f$ is constant on $[a,b]$, so $f'(x)=0$ for all $x \in (a,b)$ and we can take for $c$ any point in $(a,b)$. If $M>m$, the assumption that $f(a)=f(b)$ shows that at least one of $M$ and $m$ is attained at a point $c \in (a,b)$. By Fermat's Theorem (2.74), $f'(c)=0$.

Proof of the MVT:

> The function
>
> $$F(x)=f(x)-\frac{f(b)-f(a)}{b-a}x$$
>
> satisfies $F(a)=F(b)$. Since linear functions are differentiable (and, hence, continuous), $F$ satisfies all the hypotheses of Rolle's Theorem (2.95). It follows that there exists $c \in (a,b)$ such that $F'(c)=0$. Clearly,
>
> $$F'(x)=f'(x)-\frac{f(b)-f(a)}{b-a}$$
>
> so
>
> $$0=F'(c)=f'(c)-\frac{f(b)-f(a)}{b-a}.$$

**Example 2.14** *As in Pons, Thm 5.4.5, let $f$ and $g$ be functions, differentiable on $(0,\infty)$ and continuous on $[0,\infty)$. Prove: If $f'(x) \leq g'(x)$ for every $x \in (0,\infty)$ and $f(0)=g(0)$, then $f(x) \leq g(x)$ for every $x \in [0,\infty)$.*

*Proof: Let $h=g-f$, so $h'(x)=g'(x)-f'(x) \geq 0$ for all $x \in (0,\infty)$. Fix $x_0 \in (0,\infty)$ and apply the MVT to $h \in (0,x_0)$, showing $\exists c \in (0,x_0)$ such that*

$$h'(c)=\frac{h\left(x_0\right)-h(0)}{x_0-0}.$$

*The quotient and the denominator are nonnegative; thus it must be the case that $h\left(x_0\right)-h(0) \geq 0$. Substituting for $f$ and $g$,*

$$0 \leq h\left(x_0\right)-h(0)=g\left(x_0\right)-f\left(x_0\right)-(g(0)-f(0))=g\left(x_0\right)-f\left(x_0\right),$$

*implying $f\left(x_0\right) \leq g\left(x_0\right)$.* ■

We now collect some further useful results.

- If $\forall x \in I$, $f'(x)=0$, then the MVT (2.94) implies that, $\forall x,y \in I$, $f(y)=f(x)$, i.e.,

$$\boldsymbol{f \text{ is constant on } I.} \tag{2.97}$$

- Let $I$ be the open interval $(a,b)$. If $f$ is differentiable on $I$ and, $\forall x \in I$, $|f'(x)| \leq M$, then the MVT implies that $|f(y) - f(x)| \leq M|y-x|$ for all $x, y \in I$. This is referred to as the *(global) Lipschitz condition.*

- The MVT is mainly used for proving other results, including the fundamental theorem of calculus (see §2.5.2), the validity of interchanging derivative and integral (§6.3), and the fundamental optimization results, proven below, in (2.105) and (2.106).

- If (i) $f'(c) > 0$ for some point $c \in I$, and (ii) $f'$ is continuous at $c$, then, from (2.46), $\exists\, \delta > 0$ such that, $\forall x \in (c-\delta, c+\delta)$, $f'(x) > 0$, i.e., $f$ is increasing on that interval. See (2.100) below for proof. Condition (ii) cannot be dropped: See, e.g., Stoll, 2021, the remark on p. 199.

- The MVT can be generalized to the *Cauchy* or *Ratio Mean Value Theorem*, as stated and proved below. It is used, for example, to rigorously prove l'Hôpital's rule (2.75). Given the prominence of the MVT, Stoll (2001, p. 204) argues that it could justifiably called the *Fundamental Theorem of Differential Calculus.*

Theorem (Cauchy Mean Value Theorem): If $f$ and $g$ are continuous functions on $[a,b]$ and differentiable on $I = (a,b)$, then $\exists\, c \in I$ such that $[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$ or, easier to remember, if $g(b) - g(a) \neq 0$,

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \tag{2.98}$$

Notice that this reduces to the usual mean value theorem when $g(x) = x$.

> Proof: Let $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$. Then, from (2.38) and (2.39), $h$ is continuous on $[a,b]$; and, from (2.68) and (2.69), differentiable on $(a,b)$ with
>
> $$h(a) = f(b)g(a) - f(a)g(b) = h(b).$$
>
> Thus by Rolle's theorem, there exists $c \in (a,b)$ such that $h'(c) = 0$, which gives the result.
>
> If $g'(x) \neq 0$ for all $x \in (a,b)$, then (2.96) implies $g(a) \neq g(b)$, so that (2.98) can be written as
>
> $$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Remark: Recall the intermediate value theorem (IVT) in (2.60). Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be differentiable on $I$. If $f'$ is continuous on $I$, then the IVT applied to $f'$ implies that, for $a, b \in I$ with $a < b$, $\alpha = f'(a)$, $\beta = f'(b)$ and a value $\gamma \in \mathbb{R}$ with either $\alpha < \gamma < \beta$ or $\alpha > \gamma > \beta$, $\exists\, c \in (a,b)$ such that $f'(c) = \gamma$. More intriguing is the fact that this still holds *even if $f'$ is not continuous*, a result attributed to Jean Gaston Darboux (1842–1917); see Stoll (2001, p. 184), Browder (1996, Thm. 4.25) or Pugh (2002, p. 144). It is referred to as the *Intermediate Value Theorem for Derivatives.*

**Remark**: The need occasionally arises to construct simple graphics like Figure 3, and it is often expedient to use the plotting and graphics generation capabilities of Matlab or other such software. In this case, the graph was constructed using the function $f(x) = 1/(1-x)^2$ with endpoints $a = 0.6$ and $b = 0.9$.

This is also a good excuse to illustrate Matlab's symbolic toolbox (which uses the Maple computing engine). The top third of the code in Listing 2 uses some basic commands from the symbolic toolbox to compute $\xi$ based on our choice of $f$, $a$ and $b$. The rest of the code constructs Figure 3.

While Matlab supports interactive graphics editing, use of the native graphics commands ("batch code") in Listing 2 is not only faster the first time around (once you are familiar with them of course), but ensures that the picture can be identically reproduced. ■

We now turn to the most basic concepts of function minimization / maximization. The first theorem is the same as Fermat's theorem given above in (2.74), which we needed for proving Rolle.

Theorem: Let $I$ be a neighborhood of $x_0$ and suppose that the function $f : I \to \mathbb{R}$ is differentiable at $x_0$. If the point $x_0$ is either a maximizer or a minimizer of the function $f : I \to \mathbb{R}$, then

$$f'(x_0) = 0. \tag{2.99}$$

Proof: Observe that, by the definition of a derivative,

$$\lim_{x\to x_0, x<x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x\to x_0, x>x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).$$

First suppose that $x_0$ is a maximizer. Then

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0 \quad \text{for } x \text{ in } I \text{ with } x < x_0,$$

and hence, from (2.26),

$$f'(x_0) = \lim_{x\to x_0, x<x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

On the other hand,

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0 \quad \text{for } x \text{ in } I \text{ with } x > x_0,$$

and hence

$$f'(x_0) = \lim_{x\to x_0, x>x_0} \frac{f(x) - f(x_0)}{x - x_0} \leq 0.$$

Thus, $f'(x_0) = 0$.

In the case where $x_0$ is a minimizer, the same proof applies, with inequalities reversed.

Theorem: Let $I$ be an open interval and the function $f : I \to \mathbb{R}$ be differentiable. Suppose that $f'(x) > 0$ for all $x$ in $I$. Then

$$f : I \to \mathbb{R} \textbf{ is strictly increasing.} \tag{2.100}$$

```
function meanvaluetheorem
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%          Use the symbolic toolbox to compute xi          %%%%%
 syms x xi real  % declare x and xi to be real symbolic variables  %
 f=1/(1-x)^2     % our function                                    %
 a=0.6; b=0.9;   % use these two end points                        %
 fa=subs(f,'x',a); fb=subs(f,'x',b); % evaluate f at a and b       %
 ratio=(fb-fa)/(b-a)  % slope of the line                          %
 df=diff(f)           % first derivative of f                      %
 xi = solve(df-ratio) % find x such that f'(x) = ratio             %
 xi=eval(xi(1))  % there is only one real solution                 %
 subs(df,'x',xi) % just check if equals ratio                      %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Plot function and the slope line
xx=0.57:0.002:0.908; ff=1./(1-xx).^2; h=plot(xx,ff)
hold on
  h=plot([a ; b],[fa ; fb],'go'); set(h,'linewidth',28)
hold off
set(h,'LineWidth',1.5), bot=-6; axis([0.53 0.96 bot 125])
set(gca,'fontsize',21,'Box','off', ...
        'YTick',[fa fb], 'YtickLabel',{'f(a)' ; 'f(b)'}, ...
        'XTick',[a b],   'XTickLabel',{'a' ; 'b'})
h=line([a b],[fa fb]);
set(h,'linestyle','--','color',[1 0 0],'linewidth',0.8)

% plot line y-y0 = m(x-x0) where m is slope and goes through (x0,y0)
x0=xi; y0=subs(f,'x',x0);
xa=a+0.4*(b-a);        xb=b-0.0*(b-a);
ya = y0+ratio*(xa-x0); yb = y0+ratio*(xb-x0);
h=line([xa xb],[ya yb]);
set(h,'linestyle','--','color',[1 0 0],'linewidth',0.8)

% vertical line at xi with label at xi
h=line([xi xi],[bot y0]);
set(h,'linestyle','--','color',[1 0.4 0.6],'linewidth',1.2)
text(xi-0.005,-13,'\xi','fontsize',24)
% Text command (but not the XTickLabel) supports use of
%  LaTeX-like text strings
```

**Program Listing 2:** Computes $\xi$ in the mean value theorem, and creates Figure 3

Proof: Let $u$ and $v$ be points in $I$ with $u < v$. Then we can apply the Mean Value Theorem to the restriction of $f$ to the closed bounded interval $[u, v]$ and choose a point $x_0$ in the open interval $(u, v)$ at which

$$f'(x_0) = \frac{f(v) - f(u)}{v - u}.$$

Since $f'(x_0) > 0$ and $v - u > 0$, it follows that $f(u) < f(v)$.

By replacing $f : I \to \mathbb{R}$ with $-f : I \to \mathbb{R}$, the above implies that if $f : I \to \mathbb{R}$ has a negative derivative at each point $x$ in $I$, then $f : I \to \mathbb{R}$ is strictly decreasing.

Definition: A point $x_0$ in the domain of a function $f : D \to \mathbb{R}$ is said to be a local maximizer for $f$ provided that there is some $\delta > 0$ such that

$$f(x) \leq f(x_0) \quad \text{for all } x \text{ in } D \text{ such that } |x - x_0| < \delta.$$

We call $x_0$ a local minimizer for $f$ provided that there is some $\delta > 0$ such that

$$f(x) \geq f(x_0) \quad \text{for all } x \text{ in } D \text{ such that } |x - x_0| < \delta.$$

The above result (2.99) asserts that, if $I$ is a neighborhood of $x_0$ and $f : I \to \mathbb{R}$ is differentiable at $x_0$, then for $x_0$ to be either a local minimizer or a local maximizer for $f$, it is necessary that

$$f'(x_0) = 0.$$

However, knowing that $f'(x_0) = 0$ does not guarantee that $x_0$ is either a local maximizer or a local minimizer. For instance, if $f(x) = x^3$ for all $x$, then $f'(0) = 0$, but the point 0 is neither a local maximizer nor a local minimizer for the function $f$. In order to establish criteria that are sufficient for the existence of local maximizers and local minimizers, it is necessary to introduce higher derivatives.

The second derivative of $f$, if it exists, is the derivative of $f'$, and denoted $f''$ or $f^{(2)}$, and likewise for higher order derivatives. If $f^{(r)}$ exists, then $f$ is said to be $r$th *order differentiable*, and if $f^{(r)}$ exists for all $r \in \mathbb{N}$, then $f$ is *infinitely differentiable*, or *smooth* (see, e.g., Pugh, 2002, p. 147). Let $f^{(0)} \equiv f$. As differentiability implies continuity, it follows that, if $f$ is $r$th order differentiable, then $f^{(r-1)}$ is continuous, and that smooth functions and all their derivatives are continuous. If $f$ is $r$th order differentiable and $f^{(r)}$ is continuous, then $f$ is *continuously* $r$th *order differentiable*, and $f$ is of *class* $\mathcal{C}^r$. An infinitely differentiable function is of *class* $\mathcal{C}^\infty$.

For a differentiable function $f : I \to \mathbb{R}$ that has as its domain an open interval $I$, we say that $f : I \to \mathbb{R}$ has one derivative if $f : I \to \mathbb{R}$ is differentiable and define $f^{(1)}(x) = f'(x)$ for all $x$ in $I$. If the function $f' : I \to \mathbb{R}$ itself has a derivative, we say that $f : I \to \mathbb{R}$ has two derivatives, or has a second derivative, and denote the derivative of $f' : I \to \mathbb{R}$ by $f'' : I \to \mathbb{R}$ or by $f^{(2)} : I \to \mathbb{R}$. Now let $k$ be a natural number for which we have defined what it means for $f : I \to \mathbb{R}$ to have $k$ derivatives and have defined $f^{(k)} : I \to \mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to have $k+1$ derivatives if $f^{(k)} : I \to \mathbb{R}$ is differentiable, and we define $f^{(k+1)} : I \to \mathbb{R}$ to be the derivative of $f^{(k)} : I \to \mathbb{R}$. In this context, is it useful to denote $f(x)$ by $f^{(0)}(x)$.

In general, if a function has $k$ derivatives, it does not necessarily have $k+1$ derivatives. For instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|x$ for all $x$ is differentiable but does not have a second derivative.

The goal now is to determine what conditions on the second derivative of function $f$ are required in order to conclude that $f'(x_0) = 0 \Rightarrow x_0$ is a local minimizer or maximizer of $f$. The conditions are given below in (2.105) and (2.106). To prove them, we first require a preliminary result.

Lemma: For an open interval $I \subset \mathbb{R}$, let $g : I \to \mathbb{R}$ be differentiable, and let $x_0 \in I$. If $g'(x_0) > 0$, then

$$\exists \delta > 0 \text{ such that } 0 < |x - x_0| < \delta \quad \Rightarrow \quad [g(x) - g(x_0)] / [x - x_0] > 0. \tag{2.101}$$

Proof: By definition of the derivative, and the assumption $g'(x_0) > 0$,

$$g'(x_0) = \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} > 0.$$

If $g'$ is continuous at $x_0$, then (2.101) follows from (2.46) applied to $g'$. Now consider the case without the continuity assumption. Define $h : I \to \mathbb{R}$ as

$$h(x) = \begin{cases} [g(x) - g(x_0)] / [x - x_0], & \text{if } x \neq x_0, \\ g'(x_0), & \text{if } x = x_0. \end{cases}$$

As $g$ is differentiable, $g$ is continuous from (2.73). From (2.38) and (2.40), $h$ is continuous for $x \neq x_0$. As $\lim_{x \to x_0} h(x) = h(x_0)$, (2.37) implies $h$ is continuous also at $x = x_0$, and thus $h : I \to \mathbb{R}$ is continuous. Result (2.101) now follows from (2.46) applied to $h$.

Before proceeding, we work a bit further with this lemma. It is equivalent to the remark in Stoll (2021, p. 199):

> It needs to be emphasized that if the derivative of a function $f$ is positive at a point $c$, then this does not imply that $f$ is increasing on an interval containing $c$; it could be non-monotone on any interval containing $c$. If $f'(c) > 0$, the only conclusion that can be reached is: $\exists \delta > 0$ such that
>
> $$\forall x \in (c - \delta, c),\ f(x) < f(c) \text{ and } \forall x \in (c, c + \delta),\ f(x) > f(c). \tag{2.102}$$
>
> This does not mean that $f$ is increasing on $(c - \delta, c + \delta)$.

While perhaps a bit tricky to visualize because $f'$ is not continuous, imagine, for example, a differentiable (and thus continuous) function that is (pathologically) oscillatory for $x \in (c - \delta, c + \delta)$ but such that (2.102) is satisfied, i.e., all its values for $x \in (c - \delta, c)$ lie below $f(c)$, and all its values are $x \in (c, c + \delta)$ lie above $f(c)$. Here is the proof of (2.102):

Proof: By hypothesis, $f'(c)$ exists and $f'(c) > 0$. Let $\epsilon = f'(c) > 0$. Then by existence of the derivative, $\exists \delta > 0$ such that

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \epsilon, \quad \forall x \in I \text{ with } 0 < d(x, c) < \delta. \tag{2.103}$$

This implies that

$$\underbrace{f'(c) - \epsilon}_{=0} < \frac{f(x) - f(c)}{x - c}, \quad \forall x \in I \text{ with } 0 < d(x, c) < \delta. \tag{2.104}$$

Note that, whenever $x > c$ (with $0 < x - c < \delta$), the denominator is positive, so that $f(x) > f(c)$. Similarly, whenever $c > x$ (with $0 < c - x < \delta$), we must have $f(c) > f(x)$. We have therefore proven the existence of a $\delta$ as required.

Theorem: Let $I$ be an open interval containing the point $x_0$ and suppose that the function $f : I \to \mathbb{R}$ has a second derivative. Suppose that $f'(x_0) = 0$. Then,

$$\textbf{If } f''(x_0) > 0\textbf{, then } x_0 \textbf{ is a local minimizer of } f. \tag{2.105}$$

$$\textbf{If } f''(x_0) < 0\textbf{, then } x_0 \textbf{ is a local maximizer of } f. \tag{2.106}$$

Proof: First suppose that $f''(x_0) > 0$ (and not necessarily continuous). Since

$$f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} > 0,$$

it follows from Lemma (2.101) that there is a $\delta > 0$ such that the open interval $(x_0 - \delta, x_0 + \delta)$ is contained in $I$ and

$$\frac{f'(x) - f'(x_0)}{x - x_0} > 0 \quad \text{if } x \text{ belongs to } (x_0 - \delta, x_0 + \delta). \tag{2.107}$$

But $f'(x_0) = 0$, so (2.107) amounts to the assertion that

$$f'(x) > 0 \text{ if } x_0 < x < x_0 + \delta \quad \text{and} \quad f'(x) < 0 \text{ if } x_0 - \delta < x < x_0. \tag{2.108}$$

From the first inequality in (2.108), the MVT implies $\exists \xi \in (x_0, x)$ such that

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(\xi) > 0 \quad \Rightarrow \quad f(x) > f(x_0);$$

while from the second inequality in (2.108), the MVT implies $\exists \xi \in (x, x_0)$ such that

$$\frac{f(x_0) - f(x)}{x_0 - x} = f'(\xi) < 0 \quad \Rightarrow \quad f(x) > f(x_0).$$

Thus, for $0 < |x - x_0| < \delta$, $f(x) > f(x_0)$, which is (2.105).

A similar argument applies for $f''(x_0) < 0$ to prove (2.106).

The preceding theorem provides no information about $f(x_0)$ as a local extreme point if both $f'(x_0) = 0$ and $f''(x_0) = 0$. As we see from examining functions of the form $f(x) = cx^n$ for all $x$ at $x_0 = 0$, if $f'(x_0) = 0$ and $f''(x_0) = 0$, then $x_0$ may be a local maximizer, a local minimizer, or neither.

### 2.3.4 Exponential and Logarithm

> For the sake of brevity, we will always represent this number 2.718281828459... by the letter e. (Leonhard Euler)

The *exponential function* arises ubiquitously in mathematics, and so it is worth spending some time understanding it. As in Lang (1997, §4.1, §4.2), let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$\text{(i) } f'(x) = f(x) \quad \text{and} \quad \text{(ii) } f(0) = 1. \tag{2.109}$$

From the product rule (2.69), the chain rule (2.71), and using (i),

$$[f(x)\, f(-x)]' = -f(x)\, f'(-x) + f(-x)\, f'(x) = -f(x)\, f(-x) + f(-x)\, f(x) = 0,$$

so that, from (2.97), $f(x)\, f(-x)$ is constant, and from (ii), equals 1. Thus, $f(x) \neq 0$ and $f(-x) = 1/f(x)$. As $f$ is differentiable, $f$ is continuous. From (ii) and the contrapositive of the IVT (2.60), $\forall x$, $f(x) > 0$. Thus, from (i), $\forall x$, $f'(x) > 0$, i.e.,

$$f \text{ is strictly increasing.} \tag{2.110}$$

Further, as $f'' = f' = f$, $f$ is strictly convex (see §2.4).

Theorem:

$$\textbf{A function } f \textbf{ satisfying (i) and (ii) in (2.109) is unique.} \tag{2.111}$$

> Proof: Suppose $g$ is any function such that $g' = g$. From (2.70), differentiating $g/f$ (and $\forall x$, $f(x) \neq 0$) yields 0. Hence, from (2.97), $g/f = K$ for some constant $K$, and thus $g = Kf$. If $g(0) = 1$, then $g(0) = Kf(0)$ so that $K = 1$ and $g = f$.

Theorem:

$$f(x+y) = f(x)\, f(y) \quad \text{and} \quad f(nx) = [f(x)]^n, \quad n \in \mathbb{N}. \tag{2.112}$$

> Proof: For the first result in (2.112), fix a number $a$, and consider the function $g(x) = f(a+x)$. Then $g'(x) = f'(a+x) = f(a+x) = g(x)$. From the previous uniqueness proof, $g' = g$ implies $g(x) = Kf(x)$ for some constant $K$. Letting $x = 0$ shows that $K = g(0) = f(a)$. Hence, $f(a+x) = f(a)f(x)$ for all $x$, as contended. For the second result in (2.112), this is true when $n = 1$, and assuming it for $n$, we have
>
> $$f((n+1)a) = f(na+a) = f(na)f(a) = f(a)^n f(a) = f(a)^{n+1},$$
>
> by induction. The second result in (2.112) also holds for $n \in \mathbb{R}$.

Function $f$ is the exponential, written $\exp(\cdot)$. Defining $e = f(1)$, and using that the second result in (2.112) also holds for $n \in \mathbb{R}$, we can write

$$\exp(x) = f(x) = f(1 \cdot x) = [f(1)]^x = e^x. \tag{2.113}$$

As shown above, $f(x)$ is strictly increasing; and, as $f(0) = 1$, we have that $f(1) = e > 1$.

It follows from (1.35) and the fact that $e > 1$ that

$$\forall k \in \mathbb{N}, \quad \lim_{n\to\infty} e^n/n^k = \infty, \quad n \in \mathbb{N}. \tag{2.114}$$

Now replace $n$ by $x$, for $x \in \mathbb{R}_{>0}$. Use of the quotient rule (2.70) gives

$$\frac{d}{dx}\left(\frac{e^x}{x^k}\right) = \frac{x^k e^x - e^x k x^{k-1}}{x^{2k}} = \frac{e^x}{x^k}(1 - k/x),$$

which is positive for $k < x$, i.e., for $x$ large enough, $e^x/x^k$ is increasing. This and the limit result for $n \in \mathbb{N}$ implies that

$$\lim_{x\to\infty} \frac{e^x}{x^k} = \infty, \quad \text{for all } k \in \mathbb{N}, \tag{2.115}$$

a result we will use below in (2.135). Recall the discussion just above (2.76): As $f(x)$ is strictly increasing, the inverse function $g$ exists; and as $f(x) > 0$, $g(y)$ is defined for $y > 0$. From (ii), $g(1) = 0$. From (2.76) and (i) in (2.109),

$$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{f(g(y))} = \frac{1}{y}. \tag{2.116}$$

For $a > 0$, the weighted sum and chain rules for differentiation, (2.72) and (2.71), yield

$$[g(ax) - g(x)]' = ag'(ax) - g'(x) = \frac{a}{ax} - \frac{1}{x} = 0,$$

and (2.97) then implies $g(ax) - g(x) = c$ or $g(ax) = c + g(x)$. Letting $x = 1$ gives $g(a) = c + g(1) = c$, which then implies

$$g(ax) = g(a) + g(x). \tag{2.117}$$

By induction, $g(x^n) = ng(x)$. Function $g$ is the *natural logarithm*, denoted $\log(y)$, $\log y$ or, from the French *logarithm natural*, $\ln y$. Thus,

$$\ln 1 = 0, \tag{2.118}$$

$$\ln x^n = n \ln x, \quad n \in \mathbb{N}, \ x > 0, \tag{2.119}$$

and, from (2.116),

$$\frac{d}{dx} \ln x = \frac{1}{x}, \quad x > 0. \tag{2.120}$$

As $\ln 1 = 0$, write $0 = \ln 1 = \ln(x/x) = \ln x + \ln(1/x)$ from (2.117), so that $\ln x^{-1} = -\ln x$. The last two results generalize to (see Stoll, 2001, p. 234)

$$\ln(x^p) = p \cdot \ln(x), \quad p \in \mathbb{R}, \ x \in \mathbb{R}_{>0}. \tag{2.121}$$

The reader can have a peak at Example 2.47 below regarding (2.119) and (2.121), where the log function is defined in terms of an integral. Based on their properties, the exponential and logarithmic functions are also used in the following way:

$$\textbf{For } r \in \mathbb{R} \textbf{ and } x \in \mathbb{R}_{>0}, \ x^r \ \underline{\textbf{is defined by}} \ x^r := \exp(r \ln x)\textbf{.} \tag{2.122}$$

**Example 2.15** *To evaluate $\lim_{x \to 0^+} x^x$, use l'Hôpital's rule and (2.120) to see that*

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-x^{-2}} = -\lim_{x \to 0^+} x = 0.$$

*Then, by (2.37) and the continuity of the exponential function,*

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} \exp(\ln x^x) = \lim_{x \to 0^+} \exp(x \ln x) = \exp\left(\lim_{x \to 0^+} x \ln x\right) = \exp 0 = 1. \quad \blacksquare$$

**Example 2.16** *We will make important use of the following two basic limit results. From the continuity of the exponential function and use of l'Hôpital's rule,*

$$\lim_{k \to \infty} k^{1/k} = \lim_{k \to \infty} \exp\left(\ln k^{1/k}\right) = \lim_{k \to \infty} \exp\left(\frac{\ln k}{k}\right) = \exp \lim_{k \to \infty} \frac{\ln k}{k} = \exp \lim_{k \to \infty} (1/k) = 1.$$

*Similarly, and again using the continuity of the exponential function, for any* $a \in \mathbb{R}_{>0}$,

$$\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} a^{1/n} = \lim_{n\to\infty} \exp\left(\frac{\ln a}{n}\right) = \exp \lim_{n\to\infty}\left(\frac{\ln a}{n}\right) = \exp(0) = 1. \tag{2.123}$$

*We now give a proof of* (2.123) *using much less sophisticated machinery. As in Petrovic (Example 2.9.1), first let* $a \geq 1$. *Recall Bernoulli's inequality (which, for* $x \geq 0$, *is just the first term in the binomial theorem expansion* (1.34)*; or can be proven by induction): For* $x > -1$ *and* $n \in \mathbb{N}$, $(1+x)^n \geq 1 + nx$. *With* $x := \sqrt[n]{a} - 1 \geq 0$,

$$a = (1+x)^n \geq 1 + nx = 1 + n(\sqrt[n]{a} - 1), \quad \text{or} \quad 0 \leq \sqrt[n]{a} - 1 \leq \frac{a-1}{n}.$$

*Taking the limit as* $n \to \infty$ *and use of the Squeeze Theorem* (2.9) *implies* $\lim a_n = 1$.

*Now consider the case for which* $0 < a < 1$. *Let* $b = 1/a > 1$. *From the previous result,* $1 = \lim \sqrt[n]{b} = \lim b^{1/n}$. *From the limits of ratios result* (2.25),

$$\lim \sqrt[n]{a} = \lim \frac{1}{\sqrt[n]{b}} = \frac{\lim 1}{\lim \sqrt[n]{b}} = \frac{1}{1} = 1.$$

*(Enter a positive number in your calculator, repeatedly press the* $\sqrt{\ }$ *key, and see what happens: Either the key will break, or a 1 will result).* ■

**Example 2.17** *Let* $f(x) = x^r$, *for* $r \in \mathbb{R}$ *and* $x \in \mathbb{R}_{>0}$. *From (2.122) and the chain rule,*

$$f'(x) = \exp(r \ln x)\frac{r}{x} = x^r \frac{r}{x} = r x^{r-1},$$

*which extends the results in Example 2.7 in a natural way.* ■

**Example 2.18** *Consider the case when the variable is not the base, but the exponent:*

$$\textbf{\textit{For }} t \in \mathbb{R}_{>0} \textbf{\textit{ and }} f(x) = t^x, \ f'(x) = t^x \ln t. \tag{2.124}$$

*From* (2.122), $f(x) = \exp(x \ln t)$. *Then* $f'(x) = \exp(x \ln t)(\ln t) = t^x \ln t$ *(chain rule).* ■

This next example gives an application of (2.124). We need the following definition, which we take from §2.6.8, where further detail will be found. Let $\{f_n(x)\}$ be a *sequence of functions* with the same domain, say $D$. The function $f$ is the *pointwise limit* of sequence $\{f_n\}$, or $\{f_n\}$ *converges pointwise* to $f$, if, $\forall x \in D$, $\lim_{n\to\infty} f_n(x) = f(x)$. That is, $\forall x \in D$ and for every given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$, $\forall n > N$.

**Example 2.19** *(Stade, Fourier Analysis, p. 157) For domain* $D = [0, 2\pi]$, $N \in \mathbb{N}$, *let* $f_N : D \to \mathbb{R}$ *be the function defined by*

$$f_N(x) = N\left(\frac{x}{2\pi}\right)^N \sqrt{2\pi - x}. \tag{2.125}$$

*Also let* $f(x) = 0$ *for all* $x \in [0, 2\pi]$. *Show that the* $f_N$*'s converge pointwise to* $f$ *but do not converge to* $f$ *in norm.*

*Solution: Let's first take care of the cases* $x = 0$ *and* $x = 2\pi$, *which are easy:* $f_N(0) = f_N(2\pi) = 0 \to 0 = f(0) = f(2\pi)$ *as* $N \to \infty$, *as required.*

*Next, for any fixed element $x$ of $(0, 2\pi)$, use l'Hôpital's rule and (2.124) as follows:*

$$\begin{aligned}
\lim_{N\to\infty} f_N(x) &= \lim_{N\to\infty} N\left(\frac{x}{2\pi}\right)^N \sqrt{2\pi - x} = \sqrt{2\pi - x}\lim_{N\to\infty}\frac{N}{(2\pi/x)^N} \\
&= \sqrt{2\pi - x}\lim_{N\to\infty}\frac{1}{(2\pi/x)^N \ln(2\pi/x)} \\
&= \frac{\sqrt{2\pi - x}}{\ln(2\pi/x)}\lim_{N\to\infty}\left(\frac{x}{2\pi}\right)^N = 0.
\end{aligned}$$

*(The limit on the right is zero because $0 < x < 2\pi$.) So the $f_N$'s converge pointwise to $f$ on $[0, 2\pi]$, as required. Pointwise convergence is defined in (2.274) below.* ■

**Example 2.20** *For $x > 0$ and $p \in \mathbb{R} \setminus \{0\}$, the chain rule and (2.120) imply*

$$\frac{d}{dx}(\ln x)^p = \frac{p(\ln x)^{p-1}}{x}, \tag{2.126}$$

*so that, dividing both sides by p, integrating both sides (and using the fundamental theorem of calculus; see §2.5.2 below),*

$$\frac{(\ln x)^p}{p} = \int \frac{dx}{x(\ln x)^{1-p}}, \tag{2.127}$$

*which we require below in Example 2.68. Also, from (2.120) and the chain rule,*

$$\frac{d}{dx}\ln(\ln x) = \frac{1}{\ln x}\frac{d}{dx}\ln x = \frac{1}{x\ln x}, \tag{2.128}$$

*also required in Example 2.68.* ■

**Example 2.21** *For $y > 0$, $k \in \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ given by $f(p) = y^{kp}$,*

$$f'(p) = \frac{d}{dp}y^{kp} = \frac{d}{dp}\exp(kp\ln y) = \exp(kp\ln y)\,k\ln y = y^{kp}k\ln y. \tag{2.129}$$

*With $k = -1$, $x > 1$, and $y = \ln x$, (2.129) implies*

$$\frac{d}{dp}\left((\ln x)^{-p}\right) = -(\ln x)^{-p}\ln(\ln x).$$

*Also, for $y > 0$ and $k = -1$, (2.129) implies*

$$\frac{d}{dp}y^{1-p} = y\frac{d}{dp}y^{-p} = -y^{1-p}\ln y, \tag{2.130}$$

*which we will use in the next example.* ■

**Example 2.22** *In microeconomics, a* utility function*, $U(\cdot)$, is a preference ordering for different goods of choice ("bundles" of goods and services, amount of money, etc.) For example, if bundle $A$ is preferable to bundle $B$, then $U(A) > U(B)$. Let $U : A \to \mathbb{R}$, $A \subset \mathbb{R}_{>0}$, be a continuous and twice differentiable utility function giving a preference ordering for overall wealth, $W$. Not surprisingly, one assumes that $U'(W) > 0$, i.e., people prefer more wealth to less, but also that $U''(W) < 0$, i.e., the more wealth you have, the less additional utility you reap upon obtaining a fixed increase in wealth. (In this case, $U$ is a concave* *function and the person is said to be* risk-averse.*) A popular choice of $U$ is $U(W;\gamma) = W^{1-\gamma}/(1-\gamma)$ for a fixed parameter $\gamma \in \mathbb{R}_{>0} \setminus \{1\}$ and $W > 0$. (Indeed, an easy calculation verifies that $U'(W) > 0$ and $U''(W) < 0$). Interest centers on the limit of $U$ as $\gamma \to 1$. In this case, $\lim_{\gamma\to 1} W^{1-\gamma} = 1$ and $\lim_{\gamma\to 1}(1-\gamma) = 0$ so that l'Hôpital's rule is not applicable. However, as utility is a relative measure, we can let $U(W;\gamma) = (W^{1-\gamma}-1)/(1-\gamma)$ instead. Then, from (2.130), $(d/d\gamma)\, W^{1-\gamma} = -W^{1-\gamma}\ln W$, so that*

$$\lim_{\gamma\to 1} U(W;\gamma) = \lim_{\gamma\to 1}\frac{W^{1-\gamma}-1}{1-\gamma} = \lim_{\gamma\to 1}\frac{(d/d\gamma)\left(W^{1-\gamma}-1\right)}{(d/d\gamma)(1-\gamma)} = \lim_{\gamma\to 1} W^{1-\gamma}\ln W = \ln W. \qquad \blacksquare$$

**Example 2.23** *A useful fact is that $\ln(1+x) < x$, for all $x \in \mathbb{R}_{>0}$, easily seen as follows. With $f(x) = \ln(1+x)$ and $g(x) = x$, note that $f$ and $g$ are continuous and differentiable, with $f(0) = g(0) = 0$, but their slopes are such that $f'(x) = (1+x)^{-1} < 1 = g'(x)$, so that, from Example 2.14, $f(x) < g(x)$ for all $x \in \mathbb{R}_{>0}$.*

*We can also prove that $\ln(1+x) < x$ for $x > 0$ (and more) using the MVT. As in Stoll (2021, Example 5.2.7), we wish to prove that*

$$\frac{x}{1+x} \le \ln(1+x) \le x \quad \text{for all} \quad x > -1. \tag{2.131}$$

*Let $f(x) = \ln(1+x)$, $x \in (-1,\infty)$. Then $f(0) = 0$. If $x > 0$, then by the MVT, $\exists c \in (0,x)$ such that*

$$\ln(1+x) = f(x) - f(0) = f'(c)x. \tag{2.132}$$

*But $f'(c) = (1+c)^{-1}$ and $(1+x)^{-1} < (1+c)^{-1} < 1$ for all $c \in (0,x)$. Therefore*

$$\frac{x}{1+x} < f'(c)x < x, \tag{2.133}$$

*and, as a consequence of (2.132) and (2.133), and adding the $x = 0$ case,*

$$\frac{x}{1+x} \le \ln(1+x) \le x \quad \text{for all} \quad x \ge 0.$$

*Now suppose $-1 < x < 0$. Observe, as $f(0) = 0$,*

$$\ln(1+x) = f(x) - f(0) = \frac{-[f(0)-f(x)]}{0-x}(0-x) = x\frac{f(0)-f(x)}{0-x},$$

*and, again by the MVT, $\exists c \in (x,0)$ such that*

$$\frac{f(0)-f(x)}{0-x} = f'(c) = \frac{1}{1+c},$$

*i.e., multiplying this by $x$,*

$$\ln(1+x) = f(x) - f(0) = \frac{x}{1+c}. \tag{2.134}$$

*But as $x < c < 0$, we have $1 < (1+c)^{-1} < (1+x)^{-1}$, and as $x$ is negative,*

$$\frac{x}{1+c} > \frac{x}{1+x}.$$

*From (2.134) and that $c < 0$, we have $1 + c < 1$, so that (as $x < 0$) $\ln(1+x) < x$. Thus,*

$$\frac{x}{1+x} < \frac{x}{1+c} = \ln(1+x) < x.$$

*Hence, the desired inequality holds for all $x > -1$, with equality if and only if $x = 0$.* $\blacksquare$

**Example 2.24** *For any $k \in \mathbb{N}$ and $z \in \mathbb{R}$, letting $x = e^z$ shows that*

$$\lim_{x\to\infty} \frac{(\ln x)^k}{x} = \lim_{z\to\infty} \frac{z^k}{e^z} = 0 \tag{2.135}$$

*from (2.115). Also, for some $p > 0$, with $z = x^p$, (2.135) implies*

$$\lim_{x\to\infty} \frac{\ln x}{x^p} = p^{-1} \lim_{z\to\infty} \frac{(\ln z)}{z} = 0, \tag{2.136}$$

*a result required below in Example 2.67.* ■

Most students will be familiar with a (common and correct) different definition of the exponential function. Here is the connection. As the derivative of $\ln x$ at $x = 1$ is $1/x = 1$, the Newton quotient (2.63), that $\ln x^p = p \ln x$ from (2.121), and the continuity of the log function combined with continuity result (2.37) imply that

$$1 = \lim_{h\to 0} \frac{\ln(1+h) - \ln 1}{h} = \lim_{h\to 0} \frac{\ln(1+h)}{h} = \lim_{h\to 0} \left(\ln(1+h)^{1/h}\right) = \ln\left(\lim_{h\to 0}(1+h)^{1/h}\right).$$

Taking the inverse function (exponential) and recalling (2.113) gives

$$\exp(1) = e = \lim_{h\to 0}(1+h)^{1/h} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

We now prove the other direction. First observe that, from the chain rule and (2.120),

$$\frac{d}{dn}\ln(1+\lambda/n) = \frac{1}{1+\lambda/n}(-\lambda n^{-2}) = -\frac{\lambda}{n\lambda + n^2}.$$

To evaluate $\lim_{n\to\infty}(1+\lambda/n)^n$, take the log of this, use the continuity of the log function, and l'Hôpital's rule to get

$$\begin{aligned}
\ln \lim_{n\to\infty}(1+\lambda/n)^n &= \lim_{n\to\infty} \ln(1+\lambda/n)^n = \lim_{n\to\infty} n\ln(1+\lambda/n) \\
&= \lim_{n\to\infty} \frac{\frac{d}{dn}\ln(1+\lambda/n)}{\frac{d}{dn}n^{-1}} = \lim_{n\to\infty} \frac{-\frac{\lambda}{n\lambda+n^2}}{-\frac{1}{n^2}} = \lambda \lim_{n\to\infty}\left(\frac{n}{n+\lambda}\right) = \lambda,
\end{aligned}$$

i.e.,

$$\lim_{n\to\infty}(1+\lambda/n)^n = e^\lambda. \tag{2.137}$$

## 2.4 Convexity

Here we investigate some basic relations between convexity, continuity, and derivatives. Convexity is an extremely important property in optimization. There are several books dedicated to convexity and optimization. This section is based primarily on Ghorpade and Limaye.

Most students will have learned the following: Given a twice differentiable function $f : \mathbb{R} \to \mathbb{R}$, $f$ is convex if $f''(x) \geq 0$ for all $x \in \mathbb{R}$. Likewise, $f$ is concave if $f''(x) \leq 0$ for all $x \in \mathbb{R}$. Figure 4 shows an illustration of convex and concave functions. For example, $f(x) = ax^2 + bx + c$ is convex if $a \geq 0$, and is concave if $a \leq 0$. These definitions are too specific, requiring $f$ to be twice differentiable.

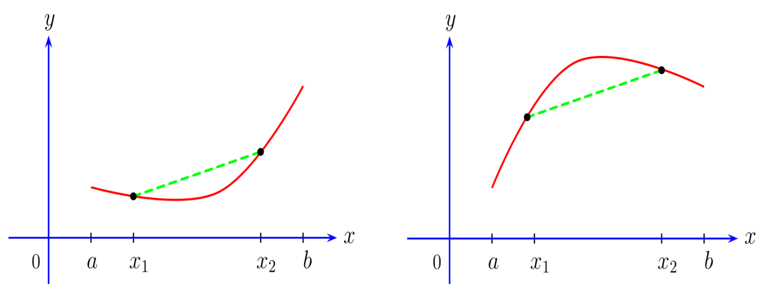



Figure 4: From Ghorpade and Limaye, page 25

We now develop the more general definition, ultimately given below in (2.141), and some basic results.

Geometrically, a function is *convex* if the line segment joining any two points on its graph lies on or above the graph. A function is *concave* if any such line segment lies on or below the graph. Another geometrically visible fact is that, for a convex function, each tangent line of the function lies entirely below the graph of the function. More specifically, Let $f : (a, b) \to \mathbb{R}$ be a convex function. Then, for every point $c \in (a, b)$, one can prove there exists a line $L$ in $\mathbb{R}^2$ with the following properties:

(a) $L$ passes through the point $(c, f(c))$. (b) The graph of $f$ lies entirely above $L$.

Any line satisfying the above is referred to as a tangent line for $f$ at $c$. Note that $f$ does not need to be differentiable. If not, then the slope of a tangent line may not be uniquely determined. As an example, consider $f : [0, 1] \to \mathbb{R}$, with $f(x) = x/2$ for $x \in [0, 1/2]$; and $f(x) = x - 1/2$, for $x \in (1/2, 1]$. Another canonical example is $f(x) = |x|$.

Theorem:

$$\text{Every convex function is continuous.} \tag{2.138}$$

> Proof: As in https://e.math.cornell.edu/people/belk/measuretheory/Inequalities.pdf: Let $f : (a, b) \to \mathbb{R}$ be a convex function, and let $c \in (a, b)$. Let $L$ be a linear function whose graph is a tangent line for $f$ at $c$, and let $P$ be a piecewise linear function consisting of two chords to the graph of $f$ meeting at $c$. See Figure 5. Then $L \leq f \leq P$ in a neighborhood of $c$, and $L(c) = f(c) = P(c)$. As $L$ and $P$ are continuous at $c$, it follows from the Squeeze Theorem and the sequential definition of continuity that $f$ is also continuous at $c$.

Analytically, for $x_1 < x < x_2$, we think in terms of the slope of the line from $x_1$ to $x$, compared to the slope of the line from $x_1$ to $x_2$. For convex, the latter should be larger than the former. This gives rise to the following.

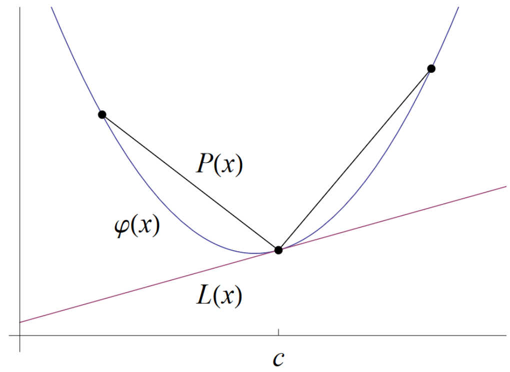



Figure 5: Convex function $f$ is continuous at each point in an open interval of its domain. Taken from https://e.math.cornell.edu/people/belk/measuretheory/Inequalities.pdf.

Definition: Let $D \subseteq \mathbb{R}$ be such that $D$ contains an interval $I$, and let $f : D \to \mathbb{R}$ be a function. We say that $f$ is convex on $I$ if

$$x_1, x_2, x \in I, x_1 < x < x_2 \implies f(x) - f(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \tag{2.139}$$

and $f$ is concave on $I$ if

$$x_1, x_2, x \in I, x_1 < x < x_2 \implies f(x) - f(x_1) \geq \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1). \tag{2.140}$$

An alternative way, and the one more commonly seen in the literature, to formulate the definitions of convexity and concavity is as follows.

Proposition: Function $f$ is convex on $I$ if (and only if)

$$\forall x_1, x_2 \in I, \quad x_1 < x_2, \quad \forall t \in (0,1), \quad f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2).$$

> Proof: First note that, for all $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$, the points $x$ between $x_1$ and $x_2$ are of the form $(1-t)x_1 + tx_2$ for some $t \in (0,1)$; in fact, $t$ and $x$ determine each other uniquely, since
>
> $$x = (1-t)x_1 + tx_2 \iff t = \frac{x - x_1}{x_2 - x_1}.$$
>
> Substituting this into the previous definition gives the result.

In the previous result, the roles of $t$ and $1-t$ can be readily reversed, and with this in view, one need not assume that $x_1 < x_2$. Thus, we arrive at our final definition.

Definition: Function $f$ is convex on $I$ if (and only if)

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) \qquad \text{for all } x_1, x_2 \in I \text{ and } t \in (0,1). \tag{2.141}$$

Similarly, $f$ is concave on $I$ if (and only if)

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2) \qquad \text{for all } x_1, x_2 \in I \text{ and } t \in (0,1). \tag{2.142}$$

Theorem: A function $f : (a,b) \to \mathbb{R}$ is convex if and only if it is continuous on $(a,b)$ and satisfies

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}, \quad \forall x_1, x_2 \in (a,b). \tag{2.143}$$

See Ghorpade and Limaye, p. 102, #3.34 for this result. We will in fact prove one direction of (2.143) next, in (2.144), and for a more general linear combination of $x_i$. The result is well-known, and very important; it is called Jensen's inequality, given in (2.153) below.

Theorem: Let $f$ be convex on $(a,b)$ as in (2.141). Then

$$f \textbf{ is continuous on } (a,b). \tag{2.144}$$

Proof: This is proven without appealing to geometric arguments as above; so purely analytic. The result also holds for $f$ concave. As in Ghorpade and Limaye (Prop 3.15), let $I$ be an open interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be convex on $I$ or concave on $I$.

First, suppose $f$ is convex. Let $c \in I$. Then there is $r > 0$ such that $[c-r, c+r] \subseteq I$. Let $M := \max\{f(c-r), f(c+r)\}$. For each $x \in [c-r, c+r]$, there is $t \in [0,1]$ such that $x = (1-t)(c-r) + t(c+r)$, and, hence, from (2.141),

$$f(x) \le (1-t)f(c-r) + tf(c+r) \le (1-t)M + tM = M. \tag{2.145}$$

Given any $\epsilon > 0$ with $\epsilon \le 1$, and $x \in \mathbb{R}$, we claim that

$$|x-c| \le r\epsilon \Longrightarrow x \in I \text{ and } |f(x) - f(c)| \le \epsilon(M - f(c)).$$

Suppose $|x-c| \le r\epsilon$. Then $x \in [c-r, c+r]$, since $\epsilon \le 1$, and so $x \in I$. Define

$$y := c + \frac{x-c}{\epsilon} \quad \text{and} \quad z := c - \frac{x-c}{\epsilon}.$$

Then $|y-c| = |z-c| = |x-c|/\epsilon \le r$, and so $y, z \in [c-r, c+r]$. Moreover,

$$x = (1-\epsilon)c + \epsilon y \quad \text{and} \quad c = \frac{1}{1+\epsilon}x + \frac{\epsilon}{1+\epsilon}z.$$

Since $f$ is convex and $0 < \epsilon \le 1$, we see that

$$f(x) \le (1-\epsilon)f(c) + \epsilon f(y), \quad \text{that is,} \quad f(x) - f(c) \le \epsilon(f(y) - f(c)). \tag{2.146}$$

Recall $y \in [c-r, c+r]$ and, for each $y \in [c-r, c+r]$, (2.145) implies $f(y) \le M$. Thus, (2.146) implies that $f(x) - f(c) \le \epsilon(M - f(c))$. Also, as $f$ is convex and $x, y, z \in [c-r, c+r]$,

$$f(c) \le \frac{1}{1+\epsilon}f(x) + \frac{\epsilon}{1+\epsilon}f(z), \quad \text{that is,} \quad (1+\epsilon)f(c) \le f(x) + \epsilon f(z).$$

The last inequality implies that $f(c) - f(x) \le \epsilon(f(z) - f(c)) \le \epsilon(M - f(c))$. It follows that $|f(x) - f(c)| \le \epsilon(M - f(c))$, and thus the claim is established. The result of continuity of $f$ at $c$ follows from the $\delta$-$\epsilon$ definition of continuity. If $f$ is concave, it suffices to apply the result just proved to $-f$.

Theorem: Let $I$ be an interval containing more than one point, and let $f : I \to \mathbb{R}$ be a differentiable function. Then

$$\textbf{(i) } f' \textbf{ is monotonically increasing on } I \iff f \textbf{ is convex on } I. \tag{2.147}$$

Similarly,
(ii) $f'$ is monotonically decreasing on $I \iff f$ is concave on $I$.

(iii) $f'$ is strictly increasing on $I \iff f$ is strictly convex on $I$.
(iv) $f'$ is strictly decreasing on $I \iff f$ is strictly concave on $I$.

Proof of (2.147): As in Ghorpade and Limaye (Prop 4.33). First, assume that $f'$ is monotonically increasing on $I$. Let $x_1, x_2, x \in I$ be such that $x_1 < x < x_2$. By the MVT, there are $c_1 \in (x_1, x)$ and $c_2 \in (x, x_2)$ satisfying

$$f(x) - f(x_1) = f'(c_1)(x - x_1) \quad \text{ and } \quad f(x_2) - f(x) = f'(c_2)(x_2 - x).$$

Now $c_1 < c_2$ and $f'$ is monotonically increasing on $I$, and so

$$\frac{f(x) - f(x_1)}{x - x_1} = f'(c_1) \leq f'(c_2) = \frac{f(x_2) - f(x)}{x_2 - x}.$$

Collecting only the terms involving $f(x)$ on the left side, we obtain

$$f(x)\left(\frac{1}{x - x_1} + \frac{1}{x_2 - x}\right) \leq \frac{f(x_1)}{x - x_1} + \frac{f(x_2)}{x_2 - x}.$$

Multiplying throughout by $(x - x_1)(x_2 - x)/(x_2 - x_1)$, we see that

$$f(x) \leq \frac{f(x_1)(x_2 - x) + f(x_2)(x - x_1)}{x_2 - x_1} = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1),$$

where the last equality follows by writing $x_2 - x = (x_2 - x_1) - (x - x_1)$. Thus, recalling (2.139), $f$ is convex on $I$.

Conversely, assume that $f$ is convex on $I$. Let $x_1, x_2, x \in I$ be such that $x_1 < x < x_2$. Then

$$\begin{aligned} f(x) &\leq f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}[(x_2 - x_1) - (x_2 - x)] \\ &= f(x_1) + [f(x_2) - f(x_1)] - \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_2 - x) = f(x_2) - \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_2 - x). \end{aligned}$$

As a consequence, the slopes of chords are increasing, that is,

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}.$$

Taking limits as $x \to x_1^+$ and $x \to x_2^-$, we obtain

$$f'(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'(x_2)$$

Thus, $f'$ is monotonically increasing on $I$.

Theorem: Suppose $f''(x)$ exists for all $x \in (a, b)$. Then

$$f \text{ convex on } (a, b) \iff \forall x \in (a, b),\ f''(x) \geq 0. \tag{2.148}$$

Proof: Just apply (2.147) and the slight variant of (2.100), namely: Suppose $f : I \to \mathbb{R}$ is differentiable on the interval $I$. If $f'(x) \geq 0$ for all $x \in I$, then $f$ is monotone increasing on $I$.

Theorem: (Ghorpade and Limaye, p. 40, exercise #1.63): Let $I$ be an interval containing more than one point and let $f : I \to \mathbb{R}$ be a function. Define $\phi(x_1, x_2) := (f(x_1) - f(x_2)) / (x_1 - x_2)$ for $x_1, x_2 \in I$ with $x_1 \neq x_2$. Then $f$ is convex on $I$ if and only if $\phi$ is a monotonically increasing function of $x_1$, that is,

$$\forall x_1, x_2 \in I,\ \ x_1 < x_2,\ \ \forall x \in I \setminus \{x_1, x_2\}, \quad \phi(x_1, x) \leq \phi(x_2, x). \tag{2.149}$$

Proof: Ghorpade and Limaye do not provide the proof. Here is one.

**Montonically increasing if convex**: Suppose that $f$ is convex. Assume for contradiction that $\phi(x_1, x)$ is not monotonically increasing in $x_1$. That is, we can find $x_1, x_2 \in I$ with $x_1 < x_2$ and $x \in I \setminus \{x_1, x_2\}$ such that $\phi(x_1, x) > \phi(x_2, x)$. Assume $x_1 < x < x_2$.[10] Note that, since $x_1 < x < x_2$, $\exists t \in (0,1)$ such that $tx_1 + (1-t)x_2 = x$. Then

$$\begin{aligned}
\frac{f(x_1) - f(tx_1 + (1-t)x_2)}{x_1 - (tx_1 + (1-t)x_2)} &> \frac{f(x_2) - f(tx_1 + (1-t)x_2)}{x_2 - (tx_1 + (1-t)x_2)} \\
\Leftrightarrow \frac{f(x_1) - f(tx_1 + (1-t)x_2)}{(1-t)(x_1 - x_2)} &> \frac{f(x_2) - f(tx_1 + (1-t)x_2)}{-t(x_1 - x_2)} \\
\Leftrightarrow -t\left[f(x_1) - f(tx_1 + (1-t)x_2)\right] &> (1-t)\left[f(x_2) - f(tx_1 + (1-t)x_2)\right] \\
\Leftrightarrow f(tx_1 + (1-t)x_2) &> (1-t)f(x_2) + tf(x_1)
\end{aligned}$$

Note that going from the second to third row we multiply by $-t(1-t)(x_1 - x_2) > 0$, therefore the inequalities do not switch. The last line contradicts that $f$ is convex. We therefore conclude that $\phi(x_1, x)$ is monotonically increasing in $x_1$.

**Convex if montonically increasing**: Conversely, suppose that $\phi(x_1, x) \leq \phi(x_2, x)$ for any $x_1, x_2 \in I$ and $x \in I \setminus \{x_1, x_2\}$. Now assume for contradiction that $f$ is not convex, i.e., $\exists t \in (0,1)$ and $x_1, x_2 \in I$ (with $x_1 \neq x_2$)[11] such that $f(tx_1 + (1-t)x_2) > tf(x_1) + (1-t)f(x_2)$. Assume w.l.o.g. that $x_1 < x_2$. Let $x = tx_1 + (1-t)x_2$, then clearly $x \in I$. Then

$$\begin{aligned}
f(tx_1 + (1-t)x_2) &> tf(x_1) + (1-t)f(x_2) \\
\Leftrightarrow f(tx_1 + (1-t)x_2) - f(x_2) &> t(f(x_1) - f(x_2)) \\
\Leftrightarrow \frac{f(tx_1 + (1-t)x_2) - f(x_2)}{tx_1 + (1-t)x_2 - x_2} &< \frac{t(f(x_1) - f(x_2))}{tx_1 + (1-t)x_2 - x_2} \\
\Leftrightarrow \frac{f(tx_1 + (1-t)x_2) - f(x_2)}{tx_1 + (1-t)x_2 - x_2} &< \frac{f(x_1) - f(x_2)}{x_1 - x_2} \\
\Leftrightarrow \phi(x, x_2) &< \phi(x_1, x_2)
\end{aligned}$$

The inequality flips in the third line when we divide both sides by $(tx_1 + (1-t)x_2 - x_2) < 0$. The last line contradicts our assumption that $\phi(x_1, x)$ is monotonically increasing in $x_1$, because $x_1 < x$ but $\phi(x, x_2) < \phi(x_1, x_2)$. Therefore, we conclude that $f$ is convex.

Theorem: If $f$ is convex on $(a, b)$, then

$$f'_+(p) \textbf{ and } f'_-(p) \textbf{ exist for every } p \in (a, b). \tag{2.150}$$

(This appears in Stoll, 2021, p. 221, Misc. Exercise #3, without solution.)

> Proof: The case of interest is when $f$ is not differentiable at some $p \in (a, b)$. From (2.138) and (2.144), we know $f$ is continuous at $p$, i.e., $f(p) = f_+(p) = f_-(p)$. From (2.77) and (2.78), we need to show the existence of
>
> $$f'_+(p) = \lim_{h \to 0^+} \frac{f(p+h) - f(p)}{h} \quad \text{and} \quad f'_-(p) = \lim_{h \to 0^-} \frac{f(p+h) - f(p)}{h}.$$
>
> Recall that a limit of function $f : D \to \mathbb{R}$ as $x \to c$ exists, denoted $f(x) \to \ell$ as $x \to c$, or $\lim_{x \to c} f(x) = \ell$, if there exists $\ell \in \mathbb{R}$ such that, for any sequence $\{x_n\} \in D \setminus \{c\}$ with $x_n \to c$, $f(x_n) \to \ell$. Consider $f'_+(p)$ and let $x_n = p + 1/n$, for $n \in \{n_0, n_0 + 1, \ldots\}$, where $n_0$ is the smallest value of $n \in \mathbb{N}$ such that $p + 1/n < b$ (which we know exists, by invoking the well-ordering principle and the Archimedean Property). Let $h, k \in \mathbb{N}$ such that $n_k > n_h \geq n_0$, so $p < x_{n_k} < x_{n_h}$. From (2.139),
>
> $$\frac{f(x_{n_k}) - f(p)}{x_{n_k} - p} \leq \frac{f(x_{n_h}) - f(p)}{x_{n_h} - p}, \tag{2.151}$$
>
> The result now follows from (2.149). The proof for $f'_-(p)$ is similar, or possibly could be elicited from that of $f'_+(p)$ and some clever “symmetry” argument, defining some function $g$ in terms of $f$.

As an example that a convex function on $(a, b)$ need not be differentiable on $(a, b)$, consider the following. For any $a > 0$, let $I = [-a, a]$, and $f : I \to \mathbb{R}$ defined by $f(x) = |x|$. Function $f$ is clearly convex, but not differentiable at the interior point $0 \in (-a, a)$. Similarly, $-f$ is concave on $I$, but not differentiable at 0.

Theorem (Young’s inequality): Let $a, b \in \mathbb{R}_{\geq 0}$ and $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.152}$$

> Proof: (Nair, Lemma 5.2.3) Function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) = e^x, \quad x \in \mathbb{R}$ is convex, i.e., for every $x, y \in \mathbb{R}$ and $0 < \lambda < 1$, $\varphi(\lambda x + (1 - \lambda)y) \leq \lambda\varphi(x) + (1 - \lambda)\varphi(y)$. Taking $\lambda = 1/p$ we have $1 - \lambda = 1/q$ and
>
> $$e^{x/p + y/q} \leq \frac{e^x}{p} + \frac{e^y}{q}.$$
>
> Now, taking $x > 0$ and $y > 0$ such that $a = e^{x/p}$ and $b = e^{y/q}$, that is, $x = \ln(a^p)$ and $y = \ln(b^q)$, we obtain (2.152).

Theorem (Jensen’s inequality, Finite Version): Let $\varphi : (a, b) \to \mathbb{R}$ be a convex function, where $-\infty \leq a < b \leq \infty$, and let $x_1, \ldots, x_n \in (a, b)$. Then

$$\varphi(\lambda_1 x_1 + \cdots + \lambda_n x_n) \leq \lambda_1 \varphi(x_1) + \cdots + \lambda_n \varphi(x_n) \tag{2.153}$$

for any $\lambda_1, \ldots, \lambda_n \in [0, 1]$ satisfying $\lambda_1 + \cdots + \lambda_n = 1$.

NOTE: A more general version that subsumes this case is Jensen’s inequality for the Lebesgue integral. An excellent presentation can be found in M. Thamban Nair’s *Measure and Integration: A First Course*, Thm 5.2.5. Nair subsequently also shows that Young’s inequality is a special case of Jensen.

Proof: (https://e.math.cornell.edu/people/belk/measuretheory/Inequalities.pdf) Let $c = \lambda_1 x_1 + \cdots + \lambda_n x_n$, and let $L$ be a linear function whose graph is a tangent line for $\varphi$ at $c$. Since $\lambda_1 + \cdots + \lambda_n = 1$, we know that $L(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \lambda_1 L(x_1) + \cdots + \lambda_n L(x_n)$. As $L \leq \varphi$ and $L(c) = \varphi(c)$, we conclude that

$$\begin{aligned}\varphi(c) = L(c) &= L(\lambda_1 x_1 + \cdots + \lambda_n x_n) \\ &= \lambda_1 L(x_1) + \cdots + \lambda_n L(x_n) \leq \lambda_1 \varphi(x_1) + \cdots + \lambda_n \varphi(x_n).\end{aligned}$$

Theorem (AM-GM Inequality): Let $n \in \mathbb{N}$ and let $a_1, \ldots, a_n$ be nonnegative real numbers. Then the arithmetic mean of $a_1, \ldots, a_n$ is greater than or equal to their geometric mean, that is,

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}. \tag{2.154}$$

Moreover, equality holds if and only if $a_1 = \cdots = a_n$.

Proof: As in Ghorpade and Limaye (Prop 1.11) If some $a_i = 0$, then the result is obvious. Assume $a_i > 0$. Let $g = (a_1 \cdots a_n)^{1/n}$ and $b_i = a_i/g$ for $i = 1, \ldots, n$. Then $b_1, \ldots, b_n$ are positive and $b_1 \cdots b_n = 1$. We shall now show, using induction on $n$, that $b_1 + \cdots + b_n \geq n$. This is clear if $n = 1$ or if each of $b_1, \ldots, b_n$ equals 1. Suppose $n > 1$ and not every $b_i$ equals 1 . Then $b_1 \cdots b_n = 1$ implies that among $b_1, \ldots, b_n$ there is a number $< 1$ as well as a number $> 1$. Relabeling $b_1, \ldots, b_n$ if necessary, we may assume that $b_1 < 1$ and $b_n > 1$. Let $c_1 = b_1 b_n$. Then $c_1 b_2 \cdots b_{n-1} = 1$, and hence by the induction hypothesis $c_1 + b_2 + \cdots + b_{n-1} \geq n - 1$. Now observe that

$$\begin{aligned}b_1 + \cdots + b_n &= (c_1 + b_2 + \cdots + b_{n-1}) + b_1 + b_n - c_1 \\ &\geq (n-1) + b_1 + b_n - b_1 b_n = n + (1 - b_1)(b_n - 1) > n,\end{aligned}$$

because $b_1 < 1$ and $b_n > 1$. This proves that $b_1 + \cdots + b_n \geq n$, and moreover, the inequality is strict unless $b_1 = \cdots = b_n = 1$. Substituting $b_i = a_i/g$, we obtain the desired result.

Theorem (AM-GM Inequality, Unequal Weights): Let $x_1, \ldots, x_n > 0$, and let $\lambda_1, \ldots, \lambda_n \in [0, 1]$ so that $\lambda_1 + \cdots + \lambda_n = 1$. Then

$$x_1^{\lambda_1} \cdots x_n^{\lambda_n} \leq \lambda_1 x_1 + \cdots + \lambda_n x_n. \tag{2.155}$$

Proof: This theorem is equivalent to the convexity of the exponential function. Specifically, from (2.153),

$$\exp\{\lambda_1 t_1 + \cdots \lambda_n t_n\} \leq \lambda_1 e^{t_1} + \cdots + \lambda_n e^{t_n}, \quad \forall t_1, \ldots, t_n \in \mathbb{R}.$$

Substituting $x_i = e^{t_i}$ gives the desired result.

## 2.5 Integration

> If we evolved a race of Isaac Newtons, that would not be progress. For the price Newton had to pay for being a supreme intellect was that he was incapable of friendship, love, fatherhood, and many other desirable things. As a man he was a failure; as a monster he was superb.
>
> (Aldous Huxley)

Every schoolchild knows the formula for the area of a rectangle. Under certain conditions, the area under a curve can be approximated by summing the areas of adjacent rectangles with heights coinciding with the function under study and ever-decreasing widths. Related concepts go back at least to Archimedes. This idea was of course known to Gottfried Leibniz (1646–1716) and Isaac Newton (1642–1727), though they viewed the integral as an antiderivative (see below) and used it as such. Augustin Louis Cauchy (1789–1857) is credited with using limits of sums as in the modern approach of integration, which led him to prove the fundamental theorem of calculus. Building on the work of Cauchy, Georg Bernhard Riemann (1826–1866) entertained working with discontinuous functions, and ultimately developed the modern definition of what is now called the Riemann integral in 1853, along with necessary and sufficient conditions for its existence. Contributions to its development were also made by Jean Gaston Darboux (1842–1917), while Thomas–Jean Stieltjes (1856–1894) pursued what is now referred to as the Riemann–Stieltjes integral.[12]

### 2.5.1 Definitions, Existence, and Properties

> The simplest schoolboy is now familiar with facts for which Archimedes would have sacrificed his life. (Ernest Renan)

To make precise the aforementioned notion of summing the area of rectangles, some notation is required. Let $A = [a, b]$, $a < b$, be a bounded interval in $\mathbb{R}$. A *partition* of $A$ is a finite set $\pi = \{x_k\}_{k=0}^{n}$ such that $a = x_0 < x_1 < \cdots < x_n = b$, and its *mesh* (sometimes called the norm, or size), is given by $\mu(\pi) = \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}$.

Let $\pi_1$ and $\pi_2$ be partitions of $I$.

$$\textbf{If } \pi_1 \subset \pi_2\textbf{, then } \pi_2 \textbf{ is a } \textbf{\textit{refinement}} \textbf{ of } \pi_1. \tag{2.156}$$

A *selection* associated to a partition $\pi = \{x_k\}_{k=0}^{n}$ is any set $\{\xi_k\}_{k=1}^{n}$ such that $x_{k-1} \leq \xi_k \leq x_k$ for $k = 1, \ldots, n$.

[12] See Stoll (2001, Ch. 6) and Browder (1996, p. 121) for some historical commentary, and Hawkins (1970) for a detailed account of the development of the Riemann and Lebesgue integrals.
The Riemann integral was fundamentally superseded and generalized by the work of Henri Léon Lebesgue (1875–1941) in 1902, as well as Émile Borel (1871–1956) and Constantin Carathéodory (1873–1950), giving rise to the Lebesgue integral. While it is considerably more complicated than the Riemann integral, it has important properties not shared by the latter, to the extent that the Riemann integral is considered by some to be just a historical relic. Somewhat unexpectedly, in the 1950's, Ralph Henstock and Jaroslav Kurzweil independently proposed an integral formulation that generalizes the Riemann integral, but in a more direct and much simpler fashion, without the need for notions of measurable sets and functions, or $\sigma$-algebras. It is usually referred to as the gauge integral, or some combination of the pioneers names. It not only nests the Lebesgue integral, but also the improper Riemann and Riemann–Stieltjes integrals. There are several textbooks that discuss it, such as those by or with Charles W. Swartz, and by or with Robert G. Bartle. See the web page by Eric Schechter, `http://www.math.vanderbilt.edu/~schectex/ccc/gauge/` and the references therein for more information.

Now let $f : D \to \mathbb{R}$ with $A \subset D \subset \mathbb{R}$, $\pi = \{x_k\}_{k=0}^{n}$ be a partition of $A$, and $\sigma = \{\xi_k\}_{k=1}^{n}$ a selection associated to $\pi$. The *Riemann sum* for function $f$, with partition $\pi$ and selection $\sigma$, is given by

$$S(f, \pi, \sigma) = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}). \tag{2.157}$$

Observe how $S$ is just a sum of areas of rectangles with heights dictated by $f$, $\pi$ and $\sigma$. If the Riemann sum converges to a real number as the level of refinement increases, then $f$ is *integrable.*

Definition: Function $f$ is said to be *(Riemann) integrable* over $A = [a, b]$ if there is a number $I \in \mathbb{R}$ such that: $\forall \epsilon > 0$, there exists a partition $\pi_0$ of $A$ such that, for every refinement $\pi$ of $\pi_0$, and every selection $\sigma$ associated to $\pi$, we have $|S(f, \pi, \sigma) - I| < \epsilon$. If $f$ is Riemann integrable over $[a, b]$, then we write $f \in \mathcal{R}[a, b]$.

The number $I$ is called the *integral* of $f$ over $[a, b]$, and denoted by $\int_a^b f$ or $\int_a^b f(x)\,dx$. Observe how, in the latter notation, $x$ is a "dummy variable", in that it could be replaced by any other letter (besides, of course, $f$, $a$ or $b$), and also how it mirrors the notation in (2.157), i.e., the term $\sum_{k=1}^{n}$ is replaced by $\int_a^b$, the term $f(\xi_k)$ is replaced by $f(x)$ and the difference $(x_k - x_{k-1})$ by $dx$. Indeed, the integral symbol $\int$ is an elongated letter S, for summation, introduced by Leibniz, and the word integral in this context was first used by Jakob Bernoulli.

For $f$ to be integrable, it is necessary (but not sufficient) that $f$ be bounded on $A = [a, b]$. To see this, observe that, if $f$ were *not* bounded on $A$, then, for every given partition $\pi = \{x_k\}_{k=0}^{n}$, $\exists k \in \{1, \ldots, n\}$ and an $x \in [x_{k-1}, x_k]$ such that $|f(x)|$ is arbitrarily large. Thus, by varying the element $\xi_k$ of the selection $\sigma$ associated to $\pi$, the Riemann sum $S(f, \pi, \sigma)$ can be made arbitrarily large, and there can be no value $I$ such that $|S(f, \pi, \sigma) - I| < \epsilon$.

**Example 2.25** *Let $f(x) = x$. Then the graph of $f$ from $0$ to $b > 0$ forms a triangle with area $b^2/2$. For the equally–spaced partition $\pi_n = \{x_k\}_{k=0}^{n}$, $n \in \mathbb{N}$, with $x_k = kb/n$ and selection $\sigma = \{\xi_k\}_{k=1}^{n}$ with $\xi_k = x_k = kb/n$, the Riemann sum is*

$$S(f, \pi, \sigma) = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} \frac{kb}{n}\left(\frac{kb}{n} - \frac{(k-1)\,b}{n}\right),$$

*which simplifies to*

$$S(f, \pi, \sigma) = \left(\frac{b}{n}\right)^2 \sum_{k=1}^{n} k = \left(\frac{b}{n}\right)^2 \left(\frac{n(n+1)}{2}\right) = \frac{b^2}{2}\frac{(n+1)}{n}. \tag{2.158}$$

*This overestimates the area of the triangle because $f$ is increasing and we took the selection $\xi_k = x_k$; likewise, choosing $\xi_k = x_{k-1}$ would underestimate it with $S = \frac{b^2}{2}\frac{(n-1)}{n}$; and, because of the linearity of $f$, choosing the midpoint $\xi_k = (x_{k-1} + x_k)/2$ gives exactly $b^2/2$. From the boundedness of $f$ on $[a, b]$, the choice of selection will have vanishing significance as $n$ grows, so that, from (2.158), as $n \to \infty$, $S(f, \pi, \sigma) \to b^2/2 = I$. (Of course, to strictly abide by the definition, the partitions would have to be chosen as successive refinements, which is clearly possible.)*[13] ■

[13] The more general case with $f(x) = x^p$ for $x \geq 0$ and $p \neq -1$ is particularly straightforward when using a wise choice of non-equally-spaced partition, as was first shown by Pierre De Fermat before the fundamental theorem of calculus was known to him; see Browder (1996, pp. 102, 121) and Stahl (1999, p. 16) for details.

Let $\pi = \{x_k\}_{k=0}^n$ be a partition of $f$. The *upper (Darboux) sum* of $f$ for $\pi$ is defined as $\overline{S}(f,\pi) = \sup_\sigma \{S(f,\pi,\sigma)\}$, i.e., the supremum of $S(f,\pi,\sigma)$ over all possible $\sigma$ associated to $\pi$. Likewise, the *lower (Darboux) sum* is $\underline{S}(f,\pi) = \inf_\sigma \{S(f,\pi,\sigma)\}$, and $\underline{S}(f,\pi) \leq \overline{S}(f,\pi)$. By defining

$$m_k = \inf\{f(t) : t \in [x_{k-1}, x_k]\} \quad \text{and} \quad M_k = \sup\{f(t) : t \in [x_{k-1}, x_k]\}, \tag{2.159}$$

we can write $\underline{S}(f,\pi) = \sum_{k=1}^n m_k (x_k - x_{k-1})$ and $\overline{S}(f,\pi) = \sum_{k=1}^n M_k (x_k - x_{k-1})$. Also, if $m \leq f(x) \leq M$ for $x \in [a,b]$, then $m(b-a) \leq \underline{S}(f,\pi) \leq \overline{S}(f,\pi) \leq M(b-a)$. It should be intuitively clear that, if $\pi$ and $\pi'$ are partitions of $[a,b]$ such that $\pi \subset \pi'$, then

$$\underline{S}(f,\pi) \leq \underline{S}(f,\pi') \leq \overline{S}(f,\pi') \leq \overline{S}(f,\pi). \tag{2.160}$$

Also, for *any* two partitions $\pi_1$ and $\pi_2$ of $[a,b]$, let $\pi_3 = \pi_1 \cup \pi_2$ be their *common refinement*, so that, from (2.160), $\underline{S}(f,\pi_1) \leq \underline{S}(f,\pi_3) \leq \overline{S}(f,\pi_3) \leq \overline{S}(f,\pi_2)$, i.e., the lower sum of any partition is less than or equal to the upper sum of any (other) partition. This fact is useful for proving the intuitively plausible result, due to Riemann, but going back to Archimedes (see the Wikipedia entry Method of Exhaustion), and thus sometimes referred to as the Archimedes-Riemann Theorem:

Theorem: If $f$ is a bounded function on $[a,b]$, then

$$\int_a^b f \textbf{ exists iff } \forall\epsilon > 0, \exists\pi \textbf{ of } [a,b] \textbf{ s.t. } \overline{S}(f,\pi) - \underline{S}(f,\pi) < \epsilon. \tag{2.161}$$

Proofs can be found in most real analysis books, e.g., Stoll and Fitzpatrick. This, in turn, is used for proving the following important results.

Theorem: If $f$ is a bounded function on $[a,b]$, then

$$\textbf{If } f \textbf{ is monotone on } [a,b]\textbf{, then } \int_a^b f \textbf{ exists.} \tag{2.162}$$

> Proof: Let $f$ be a (bounded and) monotone increasing function on $[a,b]$. Let $\pi = \{x_k\}_{k=0}^n$ be a partition of $[a,b]$ with $x_k = a + (k/n)(b-a)$. Then (2.159) implies that $m_k = f(x_{k-1})$ and $M_k = f(x_k)$, and, as $x_k - x_{k-1} = (b-a)/n$,
>
> $$\begin{aligned} \overline{S}(f,\pi) - \underline{S}(f,\pi) &= \sum_{k=1}^n [f(x_k) - f(x_{k-1})](x_k - x_{k-1}) \\ &= \frac{b-a}{n} \sum_{k=1}^n [f(x_k) - f(x_{k-1})] = \frac{b-a}{n}(f(b) - f(a)). \end{aligned}$$
>
> As $f$ is bounded and increasing, $0 \leq f(b) - f(a) < \infty$, and $n$ can be chosen such that the rhs is less than any $\epsilon > 0$. Thus, by (2.161), $\int_a^b f$ exists.

Theorem: A continuous function on a closed and bounded interval is integrable:

$$\textbf{If } f \in \mathcal{C}^0[a,b]\textbf{, then } \int_a^b f \textbf{ exists.} \tag{2.163}$$

Proof: Recall from (2.55) that continuity of $f$ on a closed, bounded interval $I$ implies that $f$ is uniformly continuous on $I$. Thus, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|f(x) - f(y)| < \epsilon/(b-a)$ when $|x - y| < \delta$. Let $\pi = \{x_k\}_{k=0}^n$ be a partition of $[a, b]$ with mesh $\mu(\pi) < \delta$. Then, for any values $s, t \in [x_{k-1}, x_k]$, $|s - t| < \delta$ and $|f(s) - f(t)| < \epsilon/(b-a)$. In particular, from (2.159), $M_k - m_k \leq \epsilon/(b-a)$ (the strict inequality is replaced with $\leq$ because of the nature of inf and sup) and

$$\overline{S}(f, \pi) - \underline{S}(f, \pi) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \leq \frac{\epsilon}{b-a} \sum_{k=1}^{n} (x_k - x_{k-1}) = \epsilon.$$

Thus, by (2.161), $\int_a^b f$ exists.

Theorem (Riemann-Lebesgue): If $f$ is a bounded function on $[a, b]$ whose set of discontinuities has measure zero, then

$$\int_a^b f \textbf{ exists}. \tag{2.164}$$

See Browder (1996, p. 104) for a short, easy proof when there exists a measure zero cover $C$ for the set of discontinuity points, and $C$ consists of a *finite* set of disjoint open intervals. This restriction to a finite set of intervals can be lifted, and is referred to as *Lebesgue's Theorem*, given by him in 1902. See e.g., Stoll (2001, §6.7) or Pugh (2002, pp. 165-7) for detailed proofs, and Terrell, §12.5 for the proof for the multivariate Riemann integral, the theorem of which we state in §6.2.3.

Theorem: Let $f$ and $\phi$ be functions such that $f \in \mathcal{R}[a, b]$, with $m \leq f(x) \leq M$ for all $x \in [a, b]$, and $\phi \in \mathcal{C}^0[m, M]$. Then

$$\phi \circ f \in \mathcal{R}[a, b]. \tag{2.165}$$

See, e.g., Browder (1996, pp. 106-7) or Stoll (2001, pp. 217-8) for elementary proofs, and Stoll (2001, p. 220) or Pugh (2002, p. 168) for the extremely short proof using Lebesgue's theorem.

Valuable special cases include $\phi(y) = |y|$ and $\phi(y) = y^2$, i.e., if $f$ is integrable on $I = [a, b]$, then so are $|f|$ and $f^2$ on $I$. It is *not* necessarily true that the composition of two integrable functions is integrable.

We now state some important properties. (Proofs can be found in all real analysis textbooks.) With $f, g \in \mathcal{R}[a, b]$ and $I = [a, b]$, we have *monotonicity*, i.e.,

$$\text{if } f(x) \leq g(x) \text{ for all } x \in I, \text{ then } \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx. \tag{2.166}$$

For any constants $k_1, k_2 \in \mathbb{R}$, we have *linearity*, or *additivity* and *homogeneity*:

$$\int_a^b (k_1 f + k_2 g) = k_1 \int_a^b f + k_2 \int_a^b g, \tag{2.167}$$

with obvious extension to the sum of $n \in \mathbb{N}$ such integrals. These can be used to prove (the also intuitively obvious)

$$\left| \int_a^b f \right| \leq \int_a^b |f|. \tag{2.168}$$

If $f, g \in \mathcal{R}[a, b]$, then

$$fg \in \mathcal{R}[a, b] \tag{2.169}$$

(see, e.g., Ghorpade and Limaye, Prop 6.16 (iii) for proof), and

$$\left( \int_a^b fg \right)^2 \leq \left( \int_a^b f^2 \right) \left( \int_a^b g^2 \right), \tag{2.170}$$

known as the *Schwarz*, *Cauchy–Schwarz*, or *Bunyakovsky–Schwarz* inequality.[14]

Let $a < c < b$ and let $f$ be a function on $[a, b]$. Then, $\int_a^b f$ exists iff $\int_a^c f$ and $\int_c^b f$ exist, in which case

$$\int_a^b f = \int_a^c f + \int_c^b f, \tag{2.171}$$

which is referred to as *domain additivity*, or *additivity over partitions*. Also, as definitions,

$$\int_a^a f := 0 \quad \text{and} \quad \int_b^a := - \int_a^b f. \tag{2.172}$$

The motivation for the former is obvious; for the latter definition, one reason is so that (2.171) holds even for $c < a < b$. That is,

$$\int_a^b f = \int_a^c f + \int_c^b f = - \int_c^a f + \int_c^b f = \int_c^b f - \int_c^a f,$$

which corresponds to our intuitive notion of working with pieces of areas.

The *Mean Value Theorem for Integrals* states: Let $f, p \in \mathcal{C}^0(I)$ for $I = [a, b]$, with $p$ nonnegative. Then $\exists c \in I$ such that

$$\int_a^b f(x)p(x)\, dx = f(c) \int_a^b p(x)\, dx. \tag{2.173}$$

A popular and useful form of the theorem just takes $p(x) \equiv 1$, so that $\int_a^b f = f(c)(b - a)$. To prove (2.173), use (2.59) to let

$$m = \min \{f(t) : t \in I\} \quad \text{and} \quad M = \max \{f(t) : t \in I\}. \tag{2.174}$$

As $p(t) \geq 0$, $mp(t) \leq f(t)\, p(t) \leq Mp(t)$ for $t \in I$, so that

$$m \int_a^b p(t)\ dt \leq \int_a^b f(t)\, p(t)\ dt \leq M \int_a^b p(t)\ dt,$$

or, assuming $\int_a^b p(t)\, dt > 0$, $m \leq \gamma \leq M$ where $\gamma = \int_a^b f(t)\, p(t)\, dt \ / \ \int_a^b p(t)\, dt$. From (2.174) and the IVT (2.60), $\exists c \in I$ such that $f(c) = \gamma$, implying (2.173).

The *Bonnet Mean Value Theorem* states that, if $f, g \in \mathcal{C}^0(I)$ and $f$ is positive and decreasing, then $\exists\, c \in I$ such that

$$\int_a^b f(x)g(x)\, dx = f(a) \int_a^c g(x)\, dx. \tag{2.175}$$

It is credited to Pierre Ossian Bonnet (1819–1892), who discovered it in 1849. Lang (1997, p. 107) provides an outline of the proof. There is a similar statement to (2.175) for $f$ positive and increasing. Oddly, most real analysis books do not include this result. A related result, termed the first mean value theorem for integrals, is given in Ghorpade and Limaye (2018, p. 231).

[14] Bunyakovsky was first, having published the result in 1859, while Schwarz found it in 1885 (see Browder, 1996, p. 121). The finite-sum analog of (2.170) is (1.22).

### 2.5.2 Fundamental Theorem of Calculus

Definition: Let $f : I \to \mathbb{R}$. The function $F : I \to \mathbb{R}$ is called an *antiderivative*, or a *primitive* of $f$ if, $\forall x \in I$, $F'(x) = f(x)$.

The fundamental theorem of calculus, or, in short, FTC, is the link between the differential and integral calculus, of which there are two forms, say FTC (i) and FTC (ii).

Theorem (FTC i): For $f \in \mathcal{R}[a,b]$ with $F$ a primitive of $f$,

$$\int_a^b f = F(b) - F(a). \tag{2.176}$$

> Proof: Let $\pi = \{x_k\}_{k=0}^n$ be a partition of $I = [a,b]$. From the definition, $F'(t) = f(t)$ for all $t \in I$. Applying the MVT to $F$ implies that $\exists \xi_k \in (x_{k-1},\ x_k)$ such that
>
> $$F(x_k) - F(x_{k-1}) = F'(\xi_k)(x_k - x_{k-1}) = f(\xi_k)(x_k - x_{k-1}).$$
>
> The set of $\xi_k$, $k = 1, \ldots n$, forms a selection, $\sigma = \{\xi_k\}_{k=1}^n$, associated to $\pi$, so that
>
> $$S(f, \pi, \sigma) = \sum_{k=1}^n f(\xi_k)(x_k - x_{k-1}) = \sum_{k=1}^n (F(x_k) - F(x_{k-1})) = F(b) - F(a).$$
>
> This holds for any partition $\pi$, so that $\int_a^b f(t)\, dt = F(b) - F(a)$.

Observe from (2.176), (2.72), and that, $\forall x \in I$, $F'(x) = f(x)$, that

$$\frac{d}{db}\int_a^b f = \frac{d}{db}F(b) - \frac{d}{db}F(a) = f(b). \tag{2.177}$$

**Example 2.26** *Let $I = [a,b]$, $f : I \to \mathbb{R}$ be a continuous (and thus integrable) function on $I$, and $x \in I$. Differentiating the relation $\int_a^b f = \int_a^x f + \int_x^b f$ w.r.t. $x$ and using (2.177) gives*

$$0 = \frac{d}{dx}\left(\int_a^x f\right) + \frac{d}{dx}\left(\int_x^b f\right) = f(x) + \frac{d}{dx}\left(\int_x^b f\right),$$

*which implies that*

$$\frac{d}{dx}\left(\int_x^b f(t)\, dt\right) = -f(x), \quad x \in I. \qquad \blacksquare$$

Theorem (FTC ii, a, b): For $f \in \mathcal{R}[a,b]$, define $F(x) = \int_a^x f$, $x \in I = [a,b]$. Then (a):

$$F \in \mathcal{C}^0[a,b], \tag{2.178}$$

and, if $f$ is continuous at $x \in I$, then (b):

$$F'(x) = f(x). \tag{2.179}$$

Proof: For part (a), i.e., (2.178), we demonstrate the $\epsilon - \delta$ formulation of continuity, as given in (2.52). That is, for any given $\epsilon > 0$, and any given $x \in I$, $\exists \delta > 0$ such that

$$y \in I \text{ and } |y - x| < \delta \Longrightarrow |F(y) - F(x)| < \epsilon. \tag{2.180}$$

Fix an $\epsilon > 0$ and $x \in I$, and let $M = \sup\{|f(t)| : t \in I\} \geq 0$. Then, for any $h \in \mathbb{R}$ with $y = x + h \in I$ (in particular, $h$ can be negative, so $x = b$ is allowed), $|y - x| = |h|$, and

$$\begin{aligned} |F(x+h) - F(x)| &= \left| \int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt \right| \\ &= \left| \int_x^{x+h} f(t)\, dt \right| \leq \int_x^{x+h} |f(t)|\, dt \leq M\,|h|\,. \end{aligned} \tag{2.181}$$

By choosing $\delta = \epsilon / M$ and taking $y$ such that $|h| = |y - x| < \delta$, (2.181) becomes

$$|F(x+h) - F(x)| \leq M\,|h| = M|y - x| < M\delta = M(\epsilon/M) = \epsilon,$$

which is (2.180). Thus, $F$ is continuous at $x$. As $x$ was chosen to be any value in $I$, $F$ is continuous on $I$.

To prove (2.179), note that, from (2.52), if $f$ is continuous at $x \in I$, then, for any $\epsilon > 0$, $\exists \delta > 0$ such that, for $t \in I$ with $|t - x| < \delta$, $|f(t) - f(x)| < \epsilon$.

First consider the case of establishing the result for $x \in I \setminus \{b\} = [a, b)$. We can always find an $h > 0$ such that $x + h \in [a, b)$. Choose $h$ such that $0 < h < \delta$. Then $\int_x^{x+h} dt = h$, $h^{-1} \int_x^{x+h} f(y)\, dt = f(y)$, and, using inequality (2.168),

$$\begin{aligned} \left| \frac{F(x+h) - F(x)}{h} - f(x) \right| &= \left| \frac{1}{h} \int_x^{x+h} f(t)\, dt - f(x) \right| = \left| \frac{1}{h} \int_x^{x+h} [f(t) - f(x)]\, dt \right| \\ &\leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\, dt < \frac{\epsilon}{h} \int_x^{x+h} dt = \epsilon, \end{aligned}$$

showing that, for $x \in [a, b)$, $F'(x) = f(x)$.

We could consider $x \in I \setminus \{a\} = (a, b]$ and $h < 0$ such that $x + h \in (a, b]$. Instead, we develop the general proof valid for $x \in I$ and $h \in \mathbb{R} \setminus \{0\}$ (negative or positive), and can always be chosen such that $x + h \in I$. Let $|h| < \delta$. Then, noting that $\int_x^{x+h} dt = h$,

$$\begin{aligned} \left| \frac{F(x+h) - F(x)}{h} - f(x) \right| &= \left| \frac{1}{h} \int_x^{x+h} f(t)\, dt - f(x) \right| = \frac{1}{|h|} \left| \int_x^{x+h} [f(t) - f(x)]\, dt \right| \\ &\leq \frac{1}{|h|} \left| \int_x^{x+h} |f(t) - f(x)|\, dt \right| < \frac{\epsilon}{|h|} \left| \int_x^{x+h} dt \right| = \frac{\epsilon}{|h|}\,|h|\,. \end{aligned}$$

Thus, $\forall x \in [a, b]$, $F'(x) = f(x)$. It is also easy to show that (2.179) implies (2.176) when $f$ is continuous; see Browder (1996, p. 112) or Priestley (1997, Thm. 5.5).

An informal, graphical proof of (2.179) is of great value for remembering and understanding the result, as well as for convincing others. The left panel of Figure 6 shows a plot of part of a continuous function, $f(x) = x^3$, with vertical lines indicating $x = 3$ and $x = 3.1$. This is "magnified" in the middle panel, with a vertical line at $x = 3.01$, but keeping the same scaling on the $y$–axis to emphasize the approximate linearity of the function over a relatively small range of $x$–values. The rate of change, via the Newton quotient, of the area $A(t)$ under the curve from $x = 1$ to $x = t = 3$ is

$$\frac{A(t+h) - A(t)}{h} = \frac{\int_1^{3+h} f - \int_1^3 f}{h} = \frac{1}{h} \int_3^{3+h} f \approx \frac{h\, f(t)}{h},$$

because, as $h \to 0$ the region under study approaches a rectangle with base $h$ and height $f(t)$; see the right panel of Figure 6.

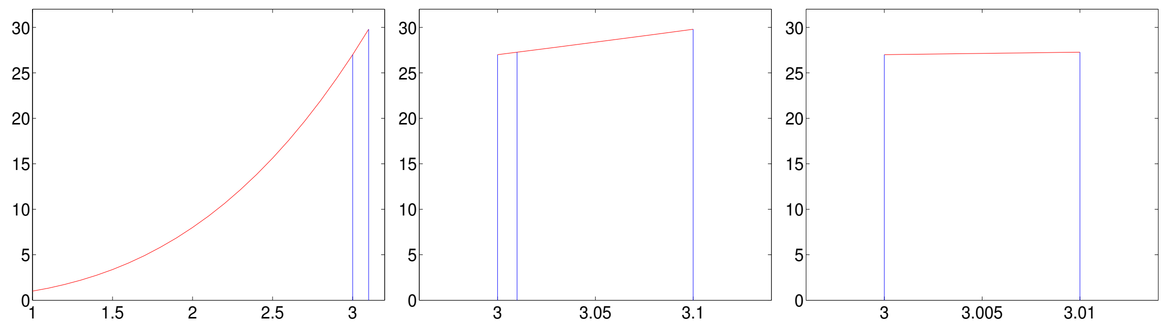



Figure 6: Graphical illustration of the FTC (2.179).

Theorem:

$$\textbf{Any two primitives of } f \textbf{ differ by a constant.} \tag{2.182}$$

> Proof: As in Hijab (1997, p. 103), let $F$ and $G$ be primitives of $f$ on $(a, b)$, so that $H = F - G$ is a primitive of zero, i.e., $\forall x \in (a, b)$, $H'(x) = (F(x) - G(x))' = 0$. The MVT (2.94) implies that $\exists c \in (a, b)$ such that, for $a < x < y < b$,
>
> $$H(x) - H(y) = H'(c)(x - y) = 0,$$
>
> i.e., $H$ is a constant. Thus, any two primitives of $f$ differ by a constant.

Before proceeding to the next example, we need the concept of an indefinite integral. First recall FTC (i), i.e., for $f \in \mathcal{R}[a, b]$ with $F$ a primitive of $f$, $\int_a^b f = F(b) - F(a)$.

Definition: If an integrable function $f : [a, b] \to \mathbb{R}$ has a primitive $F$, then $F$ is called an *indefinite integral of* $f$, and it is denoted by $\int f(x)dx$. This notation is ambiguous, because, from (2.182), a primitive of $f$ is unique only up to an additive constant. For this reason, one writes

$$\int f(x)\, dx = F(x) + C, \tag{2.183}$$

where $C$ denotes an arbitrary constant. Notice that, in this case, $\int_a^b f(x)dx = F(b) - F(a)$, where the right side is independent of the choice of an indefinite integral. The right side of the above equality is sometimes denoted by $[F(x)]_a^b$ or $F(x)|_a^b$. With this in mind, the Riemann integral of $f : [a, b] \to \mathbb{R}$ is sometimes referred to as the *definite integral of* $f$ *over* $[a, b]$. As a simple example, for $f(x) = x$, an antiderivative of $f$ is $F(x) = x^2/2$, so (2.183) implies $\int f(x)dx = F(x) + C$, where $C$ is an arbitrary number. We have $\int_0^1 f = F(x)|_a^b = 1^2/2 + C - (0^2/2 + C) = 1/2$.

**Example 2.27** *(Bóna and Shabanov, Concepts in Calculus II, 2012, p. 32) To compute $\int \cos^4 x\, dx$, use (2.85) to get that $\cos^2 x = (1+\cos 2x)/2$, and so*

$$\cos^4 x = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{1}{4} + \frac{\cos 2x}{2} + \frac{\cos^2 2x}{4}.$$

*Applying (2.85) again, with $2x$ replacing $x$, we get that $\cos^2 2x = (1+\cos 4x)/2$, so*

$$\cos^4 x = \frac{3}{8} + \frac{\cos 2x}{2} + \frac{\cos 4x}{8}.$$

*From the FTC (i) (2.176) but via (2.183); and (2.86),*

$$\int \cos^4 x\, dx = \int \left(\frac{3}{8} + \frac{\cos 2x}{2} + \frac{\cos 4x}{8}\right) dx = \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C. \quad \blacksquare$$

A vastly useful technique for resolving integrals is the *change of variables.*

Theorem (Integration by Substitution): Let $I = [a, b]$ for $a < b$, and let $f : I \to \mathbb{R}$ be continuous. Let $\phi : [\alpha, \beta] \to \mathbb{R}$ be such that (i) $\phi([\alpha, \beta]) = [a, b] = I$; (ii) $\phi$ is differentiable; and (iii) $\phi'$ is integrable on $[\alpha, \beta]$. Then $(f \circ \phi)\phi' : [\alpha, \beta] \to \mathbb{R}$ is integrable and

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx = \int_{\alpha}^{\beta} f(\phi(t))\phi'(t)dt. \tag{2.184}$$

Proof: From (2.163), $f$ is integrable. Define $F : I \to \mathbb{R}$ by $F(x) = \int_a^x f(u)du$. From FTC (ii, b) in (2.179), $F$ is differentiable, and, $\forall x \in I$, $F'(x) = f(x)$. Next define $H : [\alpha, \beta] \to \mathbb{R}$ by $H = F \circ \phi$. As $F$ and $\phi$ are differentiable, the chain rule (2.71) implies

$$\forall t \in [\alpha, \beta],\quad H'(t) = F'(\phi(t))\phi'(t) = f(\phi(t))\phi'(t), \quad \text{i.e.,} \quad H' = (f \circ \phi)\phi'.$$

Since $\phi$ is differentiable, it is continuous, and since $f$ is also continuous, from (2.48), the composite $f \circ \phi$ is continuous and hence, from (2.163), integrable. As $\phi'$ is integrable, (2.169) implies that $H' = (f \circ \phi)\phi'$ is integrable. Hence, from FTC (i) in (2.176),

$$\int_{\alpha}^{\beta} H'(t)dt = H(\beta) - H(\alpha) = \int_a^{\phi(\beta)} f(x)dx - \int_a^{\phi(\alpha)} f(x)dx = \int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx,$$

where the last equality follows from domain additivity (2.171). This proves (2.184).

**Example 2.28** *(Sasane, p. 224)*[15] *Consider the integral $\int_0^1 t\sqrt{1-t^2}\, dt$. Let $u = \phi(t) = 1 - t^2$, $t \in [0, 1]$, and take $f(u) = \sqrt{u}$, $u \in [0, 1]$, so that $du = -2tdt$, $tdt = -du/2$, $t = 0 \Rightarrow u = 1$, and $t = 1 \Rightarrow u = 0$. Thus, from (2.184),*

$$\begin{aligned}\int_0^1 t\sqrt{1-t^2}\, dt &= \int_1^0 \sqrt{u}\left(-\frac{1}{2}\right) du = \frac{1}{2}\int_0^1 \sqrt{u}\, du \\ &= \frac{1}{2} \cdot \frac{1}{1+\frac{1}{2}} u^{1+\frac{1}{2}}\Big|_0^1 = \frac{1}{2} \cdot \frac{2}{3} \cdot \left(1^{3/2} - 0^{3/2}\right) = \frac{1}{3}. \quad \blacksquare\end{aligned}$$

[15] The How and Why of One Variable Calculus, 2015

**Example 2.29** *(Sasane, p. 224) Consider the integral* $\int_0^{\pi/2}(\sin t)^5 \cos t\, dt$. *Let* $u = \phi(t) = \sin t$, $t \in \left[0, \frac{\pi}{2}\right]$, *and* $f(u) = u^5$, $u \in [0, 1]$, $du = \cos t dt$, $t = 0 \Rightarrow u = 0$, $t = \frac{\pi}{2} \Rightarrow u = 1$. *Thus*

$$\int_0^{\pi/2} (\sin t)^5 \cos t\, dt = \int_0^1 u^5 du = \frac{1}{6}u^6\bigg|_0^1 = \frac{1}{6}. \qquad \blacksquare$$

**Example 2.30** *(Sasane, p. 225) Consider the integral* $\int_2^5 \frac{1}{t\log t}\, dt$. *Let* $u = \phi(t) = \log t$, $t \in [2, 5]$, *and* $f(u) = 1/u$ *for* $u \in [\log 2, \log 5]$, $du = dt/t$, $t = 2 \Rightarrow u = \log 2$, $t = 5 \Rightarrow u = \log 5$. *Thus, from* (2.120) *and FTC (i) in* (2.176),

$$\int_2^5 \frac{1}{t\log t}\, dt = \int_{\log 2}^{\log 5} \frac{1}{u} du = \log u|_{\log 2}^{\log 5} = \log(\log 5) - \log(\log 2). \qquad \blacksquare$$

**Example 2.31** *(Bóna and Shabanov, p. 33) To compute* $\int \sin^3 x\, dx$, *write*

$$\sin^3 x = \sin x \cdot \sin^2 x = \sin x \cdot \left(1 - \cos^2 x\right) = \sin x - \sin x \cos^2 x,$$

*so that, with* $u = \cos x$, $du/dx = -\sin x$, *and*

$$\int -\sin x \cos^2 x\, dx = \int u^2 du = \frac{u^3}{3} + C = \frac{\cos^3 x}{3} + C.$$

*As* $\int \sin x\, dx = -\cos x$, *we get*

$$\int \sin^3 x\, dx = -\cos x + \frac{\cos^3 x}{3} + C. \qquad \blacksquare$$

Many more examples similar to Examples 2.30 and 2.31 can be found at https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)/07%3A_Techniques_of_Integration/7.02%3A_Trigonometric_Integrals

**Example 2.32** *(Bóna and Shabanov, p. 37) For the integral* $\int \left(1 + x^2\right)^{-2} dx$, *substitute* $x = \tan y$, *so* $y = \tan^{-1}(x)$, *and from* (2.92), $dy = dx/\left(1 + x^2\right)$. *This yields*

$$\begin{aligned}\int \frac{dx}{\left(1 + x^2\right)^2} &= \int \frac{dy}{1 + x^2} = \int \frac{dy}{1 + \tan^2 y} \quad (\tan = \sin / \cos)\\ &= \int \cos^2 y\, dy = \frac{y}{2} + \frac{\sin 2y}{4} = \frac{y}{2} + \frac{\sin y \cos y}{2} + C,\end{aligned}$$

*where* $\int \cos^2 y\, dy$ *is resolved as shown in Example 2.27, and then having used* (2.79a). *Now noting that*

$$\frac{x}{x^2 + 1} = \frac{\tan y}{1 + \tan^2 y} = \tan y \cos^2 y = \sin y \cos y,$$

*we have*

$$\int f := \int \frac{dx}{\left(1 + x^2\right)^2} = \frac{y}{2} + \frac{\sin y \cos y}{2} = \frac{1}{2} \cdot \tan^{-1}(x) + \frac{1}{2} \cdot \frac{x}{x^2 + 1} + C.$$

Indeed, differentiating the rhs (call it $F$) gives

$$\begin{aligned} F' &= \frac{1}{2}\frac{1}{1+x^2} + \frac{1}{2}\left(\frac{(x^2+1) - x\,(2x)}{(1+x^2)^2}\right) = \frac{1}{2x^2+2} + \frac{1-x^2}{4x^2+2x^4+2} \\ &= \left(\frac{1}{2x^2+2}\right)\left(\frac{4x^2+2x^4+2}{4x^2+2x^4+2}\right) + \left(\frac{1-x^2}{4x^2+2x^4+2}\right)\left(\frac{2x^2+2}{2x^2+2}\right) \\ &= \frac{4x^2+2x^4+2}{(2x^2+2)\,(4x^2+2x^4+2)} + \frac{2-2x^4}{(2x^2+2)\,(4x^2+2x^4+2)} \\ &= \frac{1}{(2x^2+x^4+1)} = \frac{1}{(1+x^2)^2} = f. \qquad \blacksquare \end{aligned}$$

**Example 2.33** *(Sasane, p. 280) Consider the circle given by $x^2+y^2=r^2$, where $r>0$. The area of the circular disk enclosed by the circle is the area of the region between the graphs of the functions $f^+(x) := \sqrt{r^2-x^2}$ and $f_-(x) := -\sqrt{r^2-x^2}$. Thus the area of the disk is*

$$\text{Area}(R) = \int_{-r}^{r}\left(\sqrt{r^2-x^2} - \left(-\sqrt{r^2-x^2}\right)\right)dx = 2\int_{-r}^{r}\sqrt{r^2-x^2}\,dx = 4\int_0^r \sqrt{r^2-x^2}\,dx,$$

*where the last equality follows because $x \mapsto \sqrt{r^2-x^2}$ is an even function. We now use the substitution $x = r\cos\theta$, so that $dx = -r\sin\theta d\theta$, and when $x=0$, we have $\theta = \pi/2$, while if $x=r$ then we have $\theta=0$. So we obtain, using (2.84), i.e., $\cos 2x = 1-2\sin^2 x$,*

$$\begin{aligned} \text{Area}(R) = 4\int_0^r \sqrt{r^2-x^2}\,dx &= 4\int_{\pi/2}^{0}\sqrt{r^2-r^2(\cos\theta)^2}\cdot(-r\sin\theta)d\theta \\ &= 4r^2\int_0^{\pi/2}(\sin\theta)^2 d\theta = 2r^2\int_0^{\pi/2}(1-\cos(2\theta))d\theta \\ &= 2r^2\left(\frac{\pi}{2} - \left.\frac{\sin(2\theta)}{2}\right|_0^{\pi/2}\right) = 2r^2\left(\frac{\pi}{2}-0\right) = \pi r^2, \end{aligned} \tag{2.185}$$

*i.e., $\pi r^2$ is the area of a circular disk of radius $r$.* $\blacksquare$

**Example 2.34** *(Ghorpade and Limaye, 2018, Prop 8.2) Let $a, b$ be positive real numbers. We wish to show: The area of the region enclosed by an ellipse given by $(x^2/a^2)+(y^2/b^2)=1$ is equal to $\pi ab$. As an important special case, setting $b=a$, we see that the area of a disk of radius $a$ is equal to $\pi a^2$, as in (2.185).*

*Proof: The area enclosed by the given ellipse is four times the area between the curves given by $y = b\sqrt{a^2-x^2}/a$, $y=0$ and between the lines given by $x=0$, $x=a$. Hence (with explanations following) it is equal to*

$$4\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx = \frac{4b}{a}\cdot a^2\int_0^{\pi/2}\cos^2\theta\,d\theta = 4ab\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}d\theta = \pi ab, \tag{2.186}$$

*where the first equality is obtained from substituting $x = a\sin\theta$, $\theta = \arcsin(x/a)$, $dx = a\cos\theta\,d\theta$, so that*

$$\int_0^a \sqrt{a^2-x^2}\,dx = \int_0^{\pi/2}\sqrt{a^2-a^2\sin^2\theta}\,a\cos\theta\,d\theta = a^2\int_0^{\pi/2}\cos^2\theta\,d\theta;$$

*the second equality is from (2.85); and the third follows from FTC (i) in (2.176), and*

$$\int_0^{\pi/2} \cos(2\theta) d\theta = \frac{1}{2} \int_0^{\pi} \cos(u) du = \frac{1}{2}[\sin \pi - \sin 0] = 0,$$

*having used substitution $u = 2\theta$ and (2.184).* ■

For computing the volume of a three-dimensional object, we need the concept of slices. We use the presentation from Ghorpade and Limaye, 2018, p. 302.

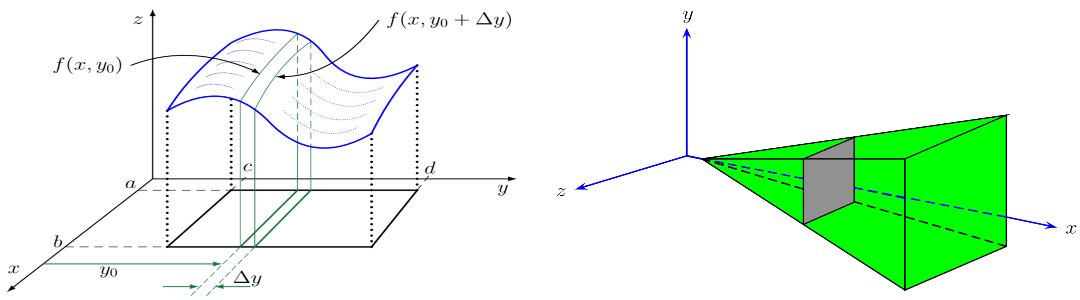



Figure 7: Adding slices to determine volume. Left is from Miklavcic, An Illustrative Guide to Multivariable and Vector Calculus (2020, p. 185); right is from Ghorpade and Limaye, 2018, p. 302.

> Let $D$ be a bounded subset of $\mathbb{R}^3 := \{(x, y, z) : x, y, z \in \mathbb{R}\}$ lying between two parallel planes and let $L$ denote a line perpendicular to these planes. A cross-section of $D$ by a plane is called a *slice* of $D$. See Figure 7. Let us assume that we are able to determine the "area" of a slice of $D$ by any plane perpendicular to $L$.
>
> For the sake of concreteness, let the line $L$ be the $x$-axis and assume that $D$ lies between the planes given by $x = a$ and $x = b$, where $a, b \in \mathbb{R}$ with $a < b$. For $s \in [a, b]$, let $A(s)$ denote the area of the slice $\{(x, y, z) \in D : x = s\}$ obtained by intersecting $D$ with the plane given by $x = s$. If $\{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$, then the solid $D$ gets divided into $n$ subsolids
>
> $$\{(x, y, z) \in D : x_{i-1} \leq x \leq x_i\}, \quad i = 1, \ldots, n.$$
>
> Let us choose $s_i \in [x_{i-1}, x_i]$ and replace the $i$th subsolid by a rectangular slab having volume equal to $A(s_i)(x_i - x_{i-1})$ for $i = 1, \ldots, n$. Then it is natural to consider
>
> $$\sum_{i=1}^{n} A(s_i)(x_i - x_{i-1})$$
>
> as an approximation of the desired volume of $D$. We therefore define the volume of $D$ to be
>
> $$\mathrm{Vol}(D) := \int_a^b A(x) dx, \tag{2.187}$$
>
> provided the "area function" $A : [a, b] \to \mathbb{R}$ is integrable.

**Example 2.35** *(Bóna and Shabanov, p. 11) Let $S$ be the right circular cone whose symmetry axis is the $y$ axis, whose apex is at $y = h$, and whose base is a circle in the plane $y = 0$ with its center at the origin and with radius $r$. See Figure 8. Find $V(S)$, the volume of $S$.*

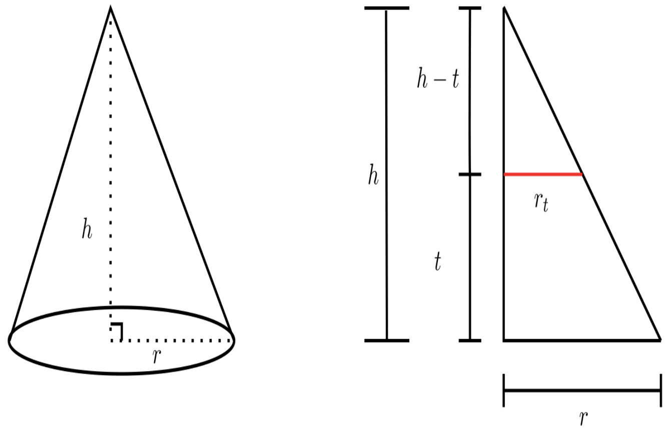



Figure 8: Right circular cone, and similar triangles

*Solution: The cone $S$ is between the planes $y = 0$ and $y = h$. The intersection of the plane $y = t$ and $S$ is a circle. The radius $r_t$ of this circle, by similar triangles, satisfies*

$$\frac{r_t}{r} = \frac{h-t}{h},$$

*showing that $r_t = r(h-t)/h$. From (2.185), let $B(t) = r^2(h-t)^2\pi/h^2$ be the area of the circular region, e.g., $\pi r_t^2$. Thus, from (2.187),*

$$V(S) = \int_0^h B(t)dt = \frac{r^2\pi}{h^2}\int_0^h \left(h^2 - 2ht + t^2\right)dt = \frac{r^2\pi}{h^2}\left[h^2t - ht^2 + \frac{t^3}{3}\right]_0^h = \frac{1}{3}hr^2\pi. \qquad \blacksquare$$

**Example 2.36** *(Ghorpade and Limaye, 2018, Prop 8.5) The volume of a solid enclosed by an ellipsoid given by $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$, where $a, b, c > 0$, is equal to $4\pi abc/3$. Letting $b = a$ and $c = a$, the volume of the spherical ball of radius $a$ is equal to $4\pi a^3/3$.*

*Proof: The given ellipsoid lies between the planes given by $x = -a$ and $x = a$. Also, for $s \in (-a, a)$, the area $A(s)$ of its slice*

$$\left\{(s, y, z) \in \mathbb{R}^3 : \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 - \frac{s^2}{a^2}\right\}$$

*by the plane given by $x = s$ is the area enclosed by the ellipse*

$$\frac{y^2}{b^2\left(1-(s^2/a^2)\right)} + \frac{z^2}{c^2\left(1-(s^2/a^2)\right)} = 1,$$

*and, hence, from (2.186),*

$$A(s) = \pi\left(b\sqrt{1-(s^2/a^2)}\right)\left(c\sqrt{1-(s^2/a^2)}\right) = \pi bc\left(1 - \frac{s^2}{a^2}\right).$$

*Thus the volume enclosed by the ellipsoid is equal to*

$$\int_{-a}^{a} A(x)dx = \pi bc\int_{-a}^{a}\left(1 - \frac{x^2}{a^2}\right)dx = \pi bc\left(2a - \frac{2a^3}{3a^2}\right) = \frac{4}{3}\pi abc. \qquad \blacksquare$$

**Example 2.37** *(Petrovic, Example 5.1.3) We wish to evaluate $\int \frac{dx}{x-a}$. If $x > a$, then $\ln(x-a)$ has derivative $f(x) = 1/(x-a)$; and from (2.183), $F(x) = \int f(x)dx$ is an indefinite integral of $f$, because $F'(x) = f(x)$ for all $x > a$. Thus,*

$$\int \frac{dx}{x-a} = \ln(x-a) + C, \text{ if } x > a. \tag{2.188}$$

*On the other hand, if $x < a$, then $\ln(x-a)$ is not defined. However, $\ln(a-x)$ is defined and differentiable, and its derivative is also $1/(x-a)$, so*

$$\int \frac{dx}{x-a} = \ln(a-x) + C, \text{ if } x < a. \tag{2.189}$$

*These two formulae are usually combined to yield*

$$\int \frac{dx}{x-a} = \ln|x-a| + C. \qquad \blacksquare$$

Augmenting the previous example a bit, consider the definite integral $I = \int_2^3 (x-1)^{-1} dx$, so $a = 1$, which, from (2.188) and the discussion just after (2.183), resolves to $\ln(x-1)|_2^3 = \ln 2$, recalling (2.118). Integral $I$ can also be resolved with the substitution $u = x-1$, $du = dx$, so $I = \int_1^2 u^{-1}du = \ln 2$. For the case with $x < a$, let $a = 1$ and $I = \int_{-3}^{-2} (x-1)^{-1} dx = \ln(1-x)|_{-3}^{-2} = \ln 3 - \ln 4$, from (2.189). Alternatively, with $u = x - 1$ and $du = dx$, $I = \int_{-4}^{-3} u^{-1}du$, which appears not to work, because we would get logs of negative numbers. But, from a plot of $1/u$, we know the area exists (and is negative). Let $v = -u$, $dv = -du$, so that $I = \int_4^3 (-v)^{-1}(-dv) = \int_4^3 v^{-1}dv = \ln 3 - \ln 4$.

**Example 2.38** *To compute $\int \frac{\sqrt{x}}{\sqrt{x}+1}dx$, use the substitution $\sqrt{x} = y$, so $dy/dx = \frac{1}{2\sqrt{x}} = \frac{1}{2y}$. Note that*

$$y - 1 + \frac{1}{y+1} = \frac{(y-1)(y+1)}{(y+1)} + \frac{1}{y+1} = \frac{y^2-1+1}{(y+1)} = \frac{y^2}{y+1}.$$

*Thus,*

$$\begin{aligned}\int \frac{\sqrt{x}}{\sqrt{x}+1}dx &= \int \frac{y}{y+1}2ydy = \int \frac{2y^2}{y+1}dy = \int \left[2(y-1) + \frac{2}{y+1}\right]dy \\ &= y^2 - 2y + 2\ln(y+1) = x - 2\sqrt{x} + 2\ln(\sqrt{x}+1) + C,\end{aligned}$$

*having used the results in Example 2.37.* $\blacksquare$

**Example 2.39** *Observe that $F(x) = e^{kx}/k + C$ is a primitive of $f(x) = e^{kx}$ for $k \in \mathbb{R} \setminus \{0\}$ and any constant $C \in \mathbb{R}$, because, via the chain rule, $dF(x)/dx = f(x)$. Thus, from (2.176),*

$$\int_a^b f = F(b) - F(a) = k^{-1}(e^{kb} - e^{ka}). \tag{2.190}$$

*See (2.204) for an example that is associated with the exponential distribution.* $\blacksquare$

**Example 2.40** *Let $I(x) = \int_0^{x^2} e^{-t}dt = 1 - e^{-x^2}$, so that $I'(x) = \frac{d}{dx}\left(1 - e^{-x^2}\right) = 2xe^{-x^2}$. Alternatively, let $G(y) = \int_0^y e^{-t}dt$, so that $I(x) = G(x^2) = G(f(x))$, where $f(x) = x^2$. From (2.177), $G'(y) = e^{-y}$, and from the chain rule,*

$$I'(x) = G'(f(x))\, f'(x) = e^{-x^2} \cdot 2x,$$

*as before, but without having to actually evaluate $I(x)$.* $\blacksquare$

**Example 2.41** *Recall* (2.173), *the Mean Value Theorem for Integrals: For* $f, p \in \mathcal{C}^0(I)$, $I = [a, b]$, *with* $p$ *nonnegative,* $\exists c \in I$ *such that* $\int_a^b f(x)p(x)\,dx = f(c)\int_a^b p(x)\,dx$. *The FTC allows an easy proof of this. As* $f$ *is integrable from (2.163), let* $F(x) = \int_a^x f$. *From* (2.178), $F$ *is continuous; and from (2.179),* $F'(x) = f(x)$ *for all* $x \in I$, *i.e.,* $F$ *is differentiable on* $I$. *The Mean Value Theorem (2.94) thus implies* $\exists c \in I$ *such that*

$$\frac{F(b) - F(a)}{b - a} = F'(c),$$

*i.e.,* $\int_a^b f(x)dx = F(b) - F(a) = F'(c)(b - a) = f(c)(b - a)$. ■

The simple technique of *integration by parts* can be invaluable in many situations.

Theorem: Let $f, g \in \mathcal{R}[a, b]$ with primitives $F$ and $G$, respectively. Then, in the definite integral case,

$$\begin{aligned}\int_a^b F(t)\,g(t)\,dt &= F(b)\,G(b) - F(a)\,G(a) - \int_a^b f(t)\,G(t)\,dt \\ &= FG\Big|_a^b - \int_a^b f(t)\,G(t)\,dt,\end{aligned} \tag{2.191}$$

while for the indefinite integral,

$$\int F(t)\,g(t)\,dt = FG - \int f(t)\,G(t)\,dt. \tag{2.192}$$

Proof: Use the product rule to get $(FG)' = F'G + FG' = fG + Fg$. Integrating both sides of this and using FTC (2.176) for the lhs gives, from the linearity property of the Riemann integral (2.167),

$$F(b)\,G(b) - F(a)\,G(a) = \int_a^b \left[f(t)\,G(t) + F(t)\,g(t)\right]\,dt,$$

and

$$FG = \int \left[f(t)\,G(t) + F(t)\,g(t)\right]\,dt.$$

These are equivalent to (2.191) and (2.192), respectively.

Throughout, we will use the popular notation

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du \qquad \text{or} \qquad \int u\,dv = uv - \int v\,du, \tag{2.193}$$

where $uv|_a^b := u(b)\,v(b) - u(a)\,v(a)$.

**Example 2.42** *Let* $f(x) = x\exp(-x)$ *and* $I = \int f$. *Using the latter equation in* (2.193) *with* $u = x$ *and* $dv = \exp(-x)dx$, *we obtain*

$$I = uv - \int v du = -x\exp(-x) + \int \exp(-x)dx = -x\exp(-x) - \exp(-x).$$

*Differentiating the rhs gives* $(-x)(-\exp(-x)) + (-1)\exp(-x) + \exp(-x) = f(x)$. ■

**Example 2.43** *To compute $\int \ln x\, dx$, let $u = \ln x$, $du = dx/x$, $dv = dx$, $v = x$, so that*

$$\int \ln x\, dx = x \ln x - \int x(1/x) dx = x \ln x - x.$$

*In the definite integral case, let $f(t) = \ln t$ for $t \in (0, 1]$. For $x \in (0, 1]$,*

$$\int_x^1 f(t) dt = (t \ln t - t)|_x^1 = x - 1 - x \ln x.$$

*We need to appeal to (2.202) for improper integrals. Since $x \ln x \to 0$ as $x \to 0^+$, we see that $\int_{0<t\leq 1} \ln t dt$ is convergent and its value is $-1$.* ■

**Example 2.44** *Applying integration by parts to $\int_0^1 x^r (\ln x)^r dx$ for $r \in \mathbb{N}$ with $u = (\ln x)^r$ and $dv = x^r dx$ (so that $v = x^{r+1}/(r+1)$ and $du = r(\ln x)^{r-1} x^{-1} dx$) gives*

$$\int_0^1 x^r (\ln x)^r dx = (\ln x)^r \frac{x^{r+1}}{r+1}\bigg|_0^1 - \int_0^1 \frac{x^{r+1}}{r+1} \frac{r}{x} (\ln x)^{r-1} dx = -\frac{r}{r+1} \int_0^1 x^r (\ln x)^{r-1} dx.$$

*Repeating this "in a feast of integration by parts" (Havil, 2003, p. 44) leads to*

$$\int_0^1 x^r (\ln x)^r dx = (-1)^r \frac{r!}{(r+1)^{r+1}}, \quad r \in \mathbb{N}, \tag{2.194}$$

*which is used in Example 2.89 below.* ■

**Example 2.45** *(Bóna and Shabanov, p. 30) To compute $\int e^x \cos x\, dx$, set $u = \cos x$ and $dv = e^x\, dx$. Then $du = -\sin x\, dx$, $v = e^x$, and*

$$\int e^x \cos x\, dx = e^x \cos x + \int e^x \sin x\, dx. \tag{2.195}$$

*So we could solve our problem if we could compute the integral $\int e^x \sin x\, dx$. We can do that by applying the technique of integration by parts again, obtaining*

$$\int e^x \sin x\, dx = e^x \sin x - \int e^x \cos x\, dx. \tag{2.196}$$

*Finally, note that (2.195) and (2.196) is a system of equations with unknowns $\int e^x \cos x\, dx$ and $\int e^x \sin x\, dx$. By adding these two equations, we get*

$$\int e^x \cos x\, dx = e^x(\cos x + \sin x) - \int e^x \cos x\, dx$$

*or*

$$\int e^x \cos x\, dx = \frac{e^x}{2}(\cos x + \sin x).$$

*Note that substituting the obtained expression for $\int e^x \cos x\, dx$ into (2.196), we get a formula for $\int e^x \sin x\, dx$, namely,*

$$\int e^x \sin x\, dx = \frac{e^x}{2}(\sin x - \cos x). \qquad ■$$

**Example 2.46** *(Petrovic, Example 5.1.6) To compute $\int \frac{dx}{x\ln x}$, use $u = \ln x$, so $du = dx/x$, and, recalling Example 2.37,*

$$\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C. \qquad \blacksquare$$

**Example 2.47** *Recall the development of the log function in §2.3.4, and, notably, results* (2.119) *and (2.121). The natural logarithm can be equivalently defined (and often is in some presentations)*

$$\ln x = \int_1^x t^{-1} dt, \quad x > 0. \tag{2.197}$$

*Our task in this example is to confirm that use of this definition indeed results in all the same properties of the log function that we previously established in §2.3.4. For instance, we need to confirm $\ln 1 = 0$ and $\ln(xy) = \ln x + \ln y$. The former follows directly from the first equation in* (2.172)*. For the latter, let $u = t/x$, so that $t = xu$ and $dt = xdu$. Then*

$$\int_x^{xy} t^{-1} dt = \int_1^y \frac{1}{u} du.$$

*Thus, using domain additivity* (2.171)*,*

$$\ln(xy) = \int_1^{xy} t^{-1}\, dt = \int_1^x t^{-1}\, dt + \int_x^{xy} t^{-1}\, dt = \int_1^x t^{-1}\, dt + \int_1^y t^{-1}\, dt = \ln x + \ln y.$$

*Similarly, $\ln(x/y) = \ln x - \ln y$.*

*Now let $y \in \mathbb{Z}$. If we use the definition of the log from §2.3.4* and its various properties *there, we obtain*

$$\frac{d}{dx} \ln x^y = \frac{1}{x^y} \frac{dx^y}{dx} = \frac{1}{x^y} y x^{y-1} = \frac{y}{x}$$

*and*

$$\frac{d}{dx} y \ln x = \frac{y}{x},$$

*so that $\ln x^y$ and $y \ln x$ have the same first derivative, and thus differ by a constant, $C = \ln x^y - y\ln x$. With $y = 0$, $C = \ln 1 = 0$ from* (2.118)*, arriving at what we already know from* (2.119) *and* (2.121)*, namely $\ln x^y = y \ln x$.*

*What we wish to do is repeat this exercise, but using the integral definition, $\ln x = \int_1^x t^{-1} dt$. From the integral representation, $\ln x^y = \int_1^{x^y} t^{-1} dt$. Thus, the FTC and the chain rule imply that*

$$\frac{d}{dx} \ln x^y = \frac{1}{x^y} \frac{dx^y}{dx} = \frac{1}{x^y} y x^{y-1} = \frac{y}{x}.$$

*Likewise,*

$$\frac{d}{dx} y \ln x = y \frac{d}{dx} \int_1^x t^{-1} dt = \frac{y}{x},$$

*so that, as before, $\ln x^y$ and $y \ln x$ have the same first derivative, and thus differ by a constant, $C = \ln x^y - y \ln x$. With $y = 0$, and using the first definition in* (2.172)*, $C = \ln 1 - 0 = 0$, so that $\ln x^y = y \ln x$. This is easily extended to $y \in \mathbb{Q}$ (see, e.g., Protter and Morrey, 1991, p. 118). The extension to $y \in \mathbb{R}$ follows by* defining *$x^r = \exp(r \ln x)$, as was given in §2.3.4.* $\blacksquare$

**Example 2.48** *Let $b, c \in \mathbb{R}$ and $T \in \mathbb{R}_{>0}$. Recall (2.82), i.e.,* $\cos(-x) = \cos(x)$ *and* $\sin(-x) = -\sin(x)$. *Letting $u = -t$,*

$$\begin{aligned} A &= \int_{-T}^{T} t^{-1} \sin(bt)\sin(ct)dt \\ &= \int_{T}^{-T} (-u^{-1})\sin(-bu)\sin(-cu)(-1)du = -\int_{-T}^{T} u^{-1}\sin(bu)\sin(cu)du = -A, \end{aligned}$$

*so that $A = 0$. Similarly,*

$$\begin{aligned} B &= \int_{0}^{T} t^{-1} \sin(bt)\cos(ct)dt \\ &= \int_{0}^{-T} (-u^{-1})\sin(-bu)\cos(-cu)(-1)du = \int_{-T}^{0} u^{-1}\sin(bu)\cos(cu)du, \end{aligned}$$

*so that*

$$\int_{-T}^{T} t^{-1}\sin(bt)\cos(ct)dt = 2\int_{0}^{T} t^{-1}\sin(bt)\cos(ct)dt.$$

*These results are useful when working with characteristic functions of random variables.* ■

Another useful integration technique when working with ratios of polynomials is *partial fraction decomposition*. Nice presentations can be found in, e.g., Petrovic, Advanced Calculus: Theory and Practice, 2nd ed., 2020; and Bóna and Shabanov, Concepts in Calculus II, 2012. We give one such example, from the latter book, to illustrate the idea; and the reader is encouraged to inspect the two aforementioned books, and others, for more detail and further examples.

**Example 2.49** *(Bóna and Shabanov, p. 41) To compute* $\int \frac{1}{x^2+3x+2}\,dx$, *note that* $x^2+3x+2 = (x+1)(x+2)$. *Using that observation, we are looking for real numbers $A$ and $B$ such that*

$$\frac{1}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2} \tag{2.198}$$

*as functions, that is, such that (2.198) holds for all real numbers $x$. Multiplying both sides by $x^2+3x+2$, we get*

$$1 = A(x+2) + B(x+1). \tag{2.199}$$

*If (2.199) holds for all real numbers $x$, it must hold for $x = -1$ and $x = -2$ as well. However, if $x = -1$, then (2.199) reduces to $1 = A$, and if $x = -2$, then (2.199) reduces to $1 = -B$. So we conclude that $A = 1$ and $B = -1$ are the numbers we wanted to find. It is now easy to compute the requested integral as follows:*

$$\int \frac{dx}{x^2+3x+2} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2} = \ln(x+1) - \ln(x+2) + C,$$

*having used the results in Example 2.37.* ■

**Example 2.50** *We will require the trigonometric secant function, given by* $\sec\theta = 1/\cos\theta$. *We wish to show the related integrals*

$$\int \sec y \; dy = \ln|\sec y + \tan y| \quad \text{and} \quad \int \left(x^2-1\right)^{-1/2} dx = \ln\left|x + \sqrt{x^2-1}\right|.$$

*Before proceeding, we use symbolic computing (Maple in this case) to check (apparent) equivalence of power series expansions of the first integral. Indeed, the two expansions appear equal, namely, and having used termwise integration,*

$$\begin{aligned} \sec y &= 1+\frac{1}{2}y^2+\frac{5}{24}y^4+\frac{61}{720}y^6+\frac{277}{8064}y^8+\cdots \\ \int \sec y dy &= y+\frac{1}{6}y^3+\frac{1}{24}y^5+\frac{61}{5040}y^7+\frac{277}{72\,576}y^9+\cdots = \ln\left(\sec y+\tan y\right). \end{aligned}$$

*Now note the simple identity*

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{1-\cos^2\theta}{\cos^2\theta} = \sec^2\theta - 1, \tag{2.200}$$

*which implies, for* $y\in[0,\pi/2)$, *that* $\tan\theta=\sqrt{\sec^2\theta-1}$, *so that the substitution* $x=\sec y$ *could be useful. Note from the quotient rule for derivatives that*

$$\frac{dx}{dy}\sec y = \frac{\sin y}{\cos^2 y} = \tan y\cdot\sec y, \quad i.e., \quad dx=\tan y\cdot\sec y\ dy.$$

*Using the (for now, and subsequently proven) result for the first integral above; the suggested substitution* $x=\sec y$*; and that, from (2.200),* $\sqrt{x^2-1}=\sqrt{\sec^2 y-1}=\tan y$*, we get*

$$\int\frac{dx}{\sqrt{x^2-1}} = \int\frac{\tan y\cdot\sec y}{\tan y}dy = \int \sec y dy = \ln|\sec y+\tan y| \ = \ln\left|x+\sqrt{x^2-1}\right|.$$

*As a "numerical confirmation", indeed,*

$$\int_1^3 \left(x^2-1\right)^{-1/2} dx \approx 1.7627 \approx \ln\left|x+\sqrt{x^2-1}\right|\Big|_1^3.$$

*What remains is to prove that* $\int \sec y\ dy=\ln|\sec y+\tan y|$*. It turns out that this integral has quite some history, and for which there is a highly informative and detailed Wikipedia web page, Integral_of_the_secant_function, from which we obtain the following derivation.* ■

**Example 2.51** *Here we resolve the interesting integral in Example 2.50, using, as said, the Wikipedia entry. It turns out that there are several equivalent expressions for the integral, the common first three of which are (and leaving off the "constant of integration" $C$)*

$$\int \sec\theta d\theta = \frac{1}{2}\ln\frac{1+\sin\theta}{1-\sin\theta} = \ln|\sec\theta+\tan\theta| = \ln\left|\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right|.$$

*We first prove that these are equivalent, because*

$$\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = |\sec\theta+\tan\theta| = \left|\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right|. \tag{2.201}$$

*Proof of (2.201): We can separately apply the so-called tangent half-angle substitution* $t=\tan\frac{1}{2}\theta$ *to each of the three forms, and show them equivalent to the same expression in terms of* $t$*. Under this substitution,* $\cos\theta=\left(1-t^2\right)/\left(1+t^2\right)$ *and* $\sin\theta=2t/\left(1+t^2\right)$*. First,*

$$\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sqrt{\frac{1+\frac{2t}{1+t^2}}{1-\frac{2t}{1+t^2}}} = \sqrt{\frac{1+t^2+2t}{1+t^2-2t}} = \sqrt{\frac{(1+t)^2}{(1-t)^2}} = \left|\frac{1+t}{1-t}\right|.$$

*Second,*

$$|\sec\theta+\tan\theta| = \left|\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}\right| = \left|\frac{1+t^2}{1-t^2}+\frac{2t}{1-t^2}\right| = \left|\frac{(1+t)^2}{(1+t)(1-t)}\right| = \left|\frac{1+t}{1-t}\right|.$$

*Third, using the tangent addition identity* $\tan(\phi+\psi) = (\tan\phi+\tan\psi)/(1-\tan\phi\tan\psi)$*,*

$$\left|\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right| = \left|\frac{\tan\frac{1}{2}\theta+\tan\frac{1}{4}\pi}{1-\tan\frac{1}{2}\theta\tan\frac{1}{4}\pi}\right| = \left|\frac{t+1}{1-t\cdot 1}\right| = \left|\frac{1+t}{1-t}\right|.$$

*Thus, all three expressions describe the same quantity.*

*There are also several approaches to resolving the integral, all shown in detail in the Wikipedia page, and of which we show one, the so-called Barrow's approach from the year 1670, using a partial fraction decomposition and the results we have in Example 2.37.*

*Proof (Barrow): Write*

$$\int \sec\theta d\theta = \int \frac{1}{\cos\theta}d\theta = \int \frac{\cos\theta}{\cos^2\theta}d\theta = \int \frac{\cos\theta}{1-\sin^2\theta}d\theta.$$

*Substituting* $u=\sin\theta, du=\cos\theta d\theta$*, reduces the integral to*

$$\begin{aligned}\int \frac{1}{1-u^2}du &= \int \frac{1}{(1+u)(1-u)}du = \int \frac{1}{2}\left(\frac{1}{1+u}+\frac{1}{1-u}\right)du \\ &= \frac{1}{2}(\ln|1+u|-\ln|1-u|)+C = \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|+C.\end{aligned}$$

*Therefore,*

$$\int \sec\theta d\theta = \frac{1}{2}\ln\frac{1+\sin\theta}{1-\sin\theta}+C.$$

*Taking the absolute value is not necessary because* $1+\sin\theta$ *and* $1-\sin\theta$ *are always non-negative for real values of* $\theta$*.* ■

### 2.5.3 Improper Integrals

Recall that the Riemann integral is designed for bounded functions on a closed, bounded interval domain. An extension, credited to Cauchy, is to let $f:(a,b]\to\mathbb{R}$ such that, $\forall c\in(a,b)$, $f$ is integrable on $[c,b]$, and define

$$\int_a^b f = \lim_{c\to a^+}\int_c^b f. \tag{2.202}$$

A similar definition holds when the limit is taken at the upper boundary. These are termed *improper integrals (of the second kind)*. If the limit in (2.202) exists, then $\int_a^b f$ is *convergent*, otherwise *divergent*.

**Example 2.52** *(Stoll, 2001, p. 241) Let* $f(x)=x^{-1/2}$*, for* $x\in(0,1]$*. Then*

$$\int_0^1 f = \lim_{c\to 0^+}\int_c^1 x^{-1/2}dx = \lim_{c\to 0^+}\left(2-2\sqrt{c}\right) = 2,$$

*and* $\int_0^1 f$ *is convergent. As* $f \in \mathcal{R}[c,1]$ *for* $c \in (0,1)$*, (2.165) implies that* $f^2 \in \mathcal{R}[c,1]$*. But that does* not *imply that the improper integral* $\int_0^1 f^2$ *is convergent. Indeed, recalling the integral definition of log from Example 2.47,*

$$\lim_{c\to 0^+}\int_c^1 x^{-1}dx = -\lim_{c\to 0^+}\ln c = \infty,$$

*i.e.,* $\int_0^1 f^2$ *is divergent.* ■

Integrals can also be taken over infinite intervals, i.e., $\int_a^\infty f(x)\,dx$, $\int_{-\infty}^b f(x)\,dx$ and $\int_{-\infty}^\infty f(x)\,dx$; these are (also) referred to as *improper integrals (of the first kind)*. If function $f$ is defined on $(-\infty,\infty)$, then $\int_{-\infty}^\infty f(x)\,dx$ is defined by

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^t f(x)\,dx + \lim_{b\to\infty}\int_t^b f(x)\,dx, \tag{2.203}$$

for any point $t\in\mathbb{R}$, when both limits on the rhs exist. An example that we will use often is

$$\int_0^\infty e^{-u}du = \lim_{b\to\infty}\int_0^b e^{-u}du = \lim_{b\to\infty}\left(1-e^{-b}\right) = 1, \tag{2.204}$$

having used (2.190) with $k=-1$.

We consider a few examples first, and then return to some basic theory.

**Example 2.53** *The unpleasant looking integral*

$$\int_0^1 \frac{e^{-((1/v)-1)}}{v^2}dv$$

*is easily handled by using the substitution* $u=(1/v)-1$*, so that* $v = 1/(1+u)$ *and* $dv = -(1+u)^{-2}du$*. Thus,*

$$\int_0^1 \frac{e^{-((1/v)-1)}}{v^2}dv = -\int_\infty^0 \frac{e^{-u}}{(1+u)^{-2}}(1+u)^{-2}du = \int_0^\infty e^{-u}du = 1,$$

*from (2.204).* ■

**Example 2.54** *Applying integration by parts to* $\int e^{at}\cos(bt)\,dt$ *with* $u=e^{at}$ *and* $dv = \cos(bt)\,dt$ *(so that* $du = ae^{at}dt$ *and* $v = (\sin bt)/b$*) gives*

$$\int e^{at}\cos(bt)\,dt = e^{at}\frac{\sin bt}{b} - \frac{a}{b}\int e^{at}\sin(bt)\,dt.$$

*Similarly,*

$$\int e^{at}\sin(bt)\,dt = -e^{at}\frac{\cos bt}{b} + \frac{a}{b}\int e^{at}(\cos bt)\,dt,$$

*so that*

$$\int e^{at}\cos(bt)\,dt = e^{at}\frac{\sin bt}{b} + e^{at}\frac{a\cos bt}{b^2} - \frac{a^2}{b^2}\int e^{at}(\cos bt)\,dt,$$

*or*

$$\int e^{at}\cos(bt)\,dt = \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt). \tag{2.205}$$

*This is given, for example, in Abramowitz and Stegun (1972, eq. 4.3.137). In the definite integral case over the positive real line, for $a = -1$ and $b = s$, it is easy to confirm that (2.205) reduces to*

$$\int_0^\infty e^{-t} \cos(st)\, dt = \frac{1}{1+s^2}. \tag{2.206}$$

*A similar derivation confirms that*

$$\int e^{at} \sin(bt)\, dt = \frac{e^{at}}{a^2+b^2} \left(a \sin bt - b \cos bt\right),$$

*with special case*

$$\int_0^\infty e^{-t} \sin(st)\, dt = \frac{s}{1+s^2}, \tag{2.207}$$

*for $a = -1$ and $b = s$. We will require (2.206) and (2.207) below.*

*Taking $a = -s$ and $b = 1$ yield*

$$\int_0^\infty e^{-st} \cos(t) dt = \frac{s}{s^2+1}, \quad \int_0^\infty e^{-st} \sin(t) dt = \frac{1}{s^2+1}, \tag{2.208}$$

*which also can be derived using Laplace transforms and basic complex analysis; see, e.g., Paolella, Intermediate Probability, Example 1.17.* ■

**Example 2.55** *The integral $\int_{-\infty}^{\infty} \exp(-x^2)\, dx$ is important, arising in conjunction with the Gaussian distribution (they are related by substituting $y = x/\sqrt{2}$). In this example, we only wish to verify that it is convergent. Its value will be computed in Examples 2.61 and 6.21 below.*

*Via symmetry, it suffices to study $\int_0^\infty \exp(-x^2)\, dx$. Let $f(x) = e^{-x^2}$, $x \in \mathbb{R}_{\geq 0}$. As $f$ is bounded and monotone on $[0, k]$ for all $k \in \mathbb{R}_{>0}$, it follows from (2.162) that $\int_0^k f$ exists. Alternatively, continuity of composite functions (2.48), and (2.163), also imply its existence. Thus, for examining the limit as $k \to \infty$, it suffices to consider $\int_1^k f$.*

*To proceed, we require the Taylor series expansion of $\exp(x)$. The general Taylor expansion is given in (2.323), and that for the exponential function is given in (2.272). From the latter, $e^{x^2} = 1 + x^2 + \frac{1}{2}(x^2)^2 + \cdots > x^2$ (for $x \in \mathbb{R}$), and it follows that, for $x > 0$,*

$$e^{x^2} > x^2 \Rightarrow x^2 > 2 \ln x \Rightarrow -x^2 < -2 \ln x \Rightarrow e^{-x^2} < x^{-2},$$

*recalling that $-2 \ln x = \ln(x^{-2})$ from (2.121). Thus, from integral monotonicity (2.166),*

$$\int_1^k e^{-x^2} dx < \int_1^k x^{-2} dx = \frac{k-1}{k} < 1, \quad \forall k > 1,$$

*and $\int_1^\infty e^{-x^2} dx$ is convergent. Alternatively, for $x > 1$, and that exp is strictly increasing,*

$$x^2 > x \Rightarrow -x^2 < -x \Rightarrow e^{-x^2} < e^{-x},$$

*so that, from integral monotonicity (2.166), result (2.204), and that $\forall x \in \mathbb{R}$, $\exp(x) > 0$,*

$$\int_1^k e^{-x^2}\, dx < \int_1^k e^{-x}\, dx = e^{-1} - e^{-k} < e^{-1} \approx 0.367879, \quad \forall k > 1. \tag{2.209}$$

*Thus, $\int_1^\infty e^{-x^2}dx$ and, hence, $\int_0^\infty e^{-x^2}dx$ are convergent.*

*To see how close (2.209) is to the true value, use of (2.171), the result in Example 2.61, and numeric integration gives*

$$\int_1^\infty e^{-x^2}dx = \int_0^\infty e^{-x^2}dx - \int_0^1 e^{-x^2}dx \approx \frac{\sqrt{\pi}}{2} - 0.746824 \approx 0.1394. \qquad \blacksquare$$

**Example 2.56** *To show that*

$$I := \int_0^\infty (1+t)\, e^{-t} \cos(st)\, dt = \frac{2}{(1+s^2)^2}, \tag{2.210}$$

*let $C = \int_0^\infty te^{-t}\cos(st)\,dt$ and $S = \int_0^\infty te^{-t}\sin(st)\,dt$. Set $u = te^{-t}$ and $dv = \cos(st)\,dt$ so that $du = (1-t)\,e^{-t}dt$ and $v = (\sin st)/s$. Then, from (2.207),*

$$\begin{aligned} C &= te^{-t}\frac{\sin st}{s}\bigg|_{t=0}^{\infty} - \int_0^\infty \frac{\sin st}{s}(1-t)\,e^{-t}dt \\ &= 0 \quad - \frac{1}{s}\int_0^\infty e^{-t}\sin(st)\,dt + \frac{1}{s}\int_0^\infty te^{-t}\sin(st)\,dt = -\frac{1}{1+s^2} + \frac{S}{s}. \end{aligned}$$

*Similarly, with $dv = \sin(st)\,dt$, $v = -(\cos st)/s$ and using (2.206),*

$$\begin{aligned} S &= -te^{-t}\frac{\cos st}{s}\bigg|_{t=0}^{\infty} + \int_0^\infty \frac{\cos st}{s}(1-t)\,e^{-t}dt \\ &= 0 \quad + \frac{1}{s}\int_0^\infty e^{-t}\cos(st)\,dt - \frac{1}{s}\int_0^\infty te^{-t}\cos(st)\,dt = \frac{1}{s(1+s^2)} - \frac{C}{s}. \end{aligned}$$

*Combining these yields*

$$C = \int_0^\infty te^{-t}\cos(st)\,dt = \frac{1-s^2}{(1+s^2)^2}, \tag{2.211}$$

*so that, from (2.206) and (2.211),*

$$I = \frac{1}{1+s^2} + \frac{1-s^2}{(1+s^2)^2} = \frac{2}{(1+s^2)^2},$$

*which is (2.210).* $\blacksquare$

**Example 2.57** *Let $f(x) = x/(1+x^2)$. Then, using the substitution $u = 1+x^2$, a straightforward calculation yields*

$$\int_0^c \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln\left(1+c^2\right) \quad \text{and} \quad \int_{-c}^0 \frac{x}{1+x^2}\,dx = -\frac{1}{2}\ln\left(1+c^2\right),$$

*which implies that*

$$\lim_{c\to\infty}\int_{-c}^c \frac{x}{1+x^2}\,dx = \lim_{c\to\infty} 0 = 0. \tag{2.212}$$

*This would seem to imply that*

$$0 = \lim_{c\to\infty}\int_{-c}^c \frac{x}{1+x^2}\,dx \overset{?}{=} \int_{-\infty}^\infty \frac{x}{1+x^2}\,dx,$$

*but the second equality is not true, because the limits in (2.203) do not exist. In (2.212), the order is conveniently chosen so that positive and negative terms precisely cancel, resulting in zero. An application of this is that the expectation of a Cauchy random variable does not exist.* $\blacksquare$

**Remarks**:
**(a)** A similar calculation shows that, for $c > 0$ and $k > 0$,

$$\lim_{c\to\infty} \int_{-c}^{kc} \frac{x}{1+x^2}\, dx = \ln k.$$

This expression could also be used for evaluating $\int_{-\infty}^{\infty} f(x)\, dx$, but results in a different value for each $k$. Thus, it also shows that $\int_{-\infty}^{\infty} f(x)\, dx$ does not exist.

**(b)** Notice that $f(x) = (1+x^2)^{-1}$ is an *even function*, i.e., it satisfies $f(-x) = f(x)$ for all $x$ (or is symmetric about zero). In this case, $f$ is continuous for all $x$, so that, for any finite $c > 0$, $\int_0^c f(x)dx = \int_{-c}^0 f(x)dx$. On the other hand, $g(x) = x$ is an *odd function*, i.e., satisfies $g(-x) = -g(x)$, and, as $g$ is continuous, for any finite $c > 0$, $\int_0^c g(x)dx = -\int_{-c}^0 g(x)dx$. Finally, as $h(x) = f(x)g(x)$ is also odd, $\int_{-c}^{c} h(x)dx = 0$. Thus, the result in (2.212) could have been immediately determined.

**(c)** The integral $\int_a^\infty \cos x\, dx$ also does not exist, because $\sin x$ does not have a limit as $x \to \infty$. Notice, however, that, for any value $t > 0$, the integral $\int_a^t \cos x\, dx$ is bounded. This shows that, if $\int_a^\infty f(x)dx$ does not exist, then it is not necessarily true that $\int_a^t f(x)dx$ increases as $t \to \infty$. ■

**Example 2.58** *(Example 2.57 cont.) We have seen that $\int_{-\infty}^{\infty} x\,(1+x^2)^{-1}\, dx$ does not exist, but, for any $z \in \mathbb{R}$,*

$$I = \int_{-\infty}^{\infty} f(x)\, dx := \int_{-\infty}^{\infty} \left( \frac{x}{1+x^2} - \frac{x}{1+(z-x)^2} \right) dx = -\pi z \tag{2.213}$$

*exists. We require (2.213) below in (2.216).*

*Recall (2.93), i.e.,* $(d/dx) \arctan(z-x) = -\left[1+(z-x)^2\right]^{-1}$*. Let*

$$F(x) = \frac{1}{2} \ln \left( \frac{1+x^2}{1+(z-x)^2} \right) + z \arctan(z-x),$$

*so that (having used Maple to verify the second to last equality),*

$$\begin{aligned} F'(x) &= \frac{1}{2} \frac{1+(z-x)^2}{1+x^2} \frac{\left(1+(z-x)^2\right) 2x + (1+x^2)\, 2\,(z-x)}{\left(1+(z-x)^2\right)^2} - \frac{z}{1+(z-x)^2} \\ &= \frac{x}{1+x^2} - \frac{x}{1+(z-x)^2} = f(x), \end{aligned}$$

*i.e., from (2.183), $F$ is an indefinite integral for $f$. We still need to address the bounds on the integral, with (2.213) being an improper integral. For fixed $z$, and using (2.48),*

$$\lim_{x\to\pm\infty} \ln \left( \frac{1+x^2}{1+(z-x)^2} \right) = \ln \left( \lim_{x\to\pm\infty} \frac{1+x^2}{1+(z-x)^2} \right) = \ln 1 = 0.$$

*From, e.g., a graph of* $\tan(x)$ *for $-\pi/2 < x < \pi/2$, and that its inverse,* arctan *is strictly increasing (see, e.g., the discussion following (2.219)), we see that, for fixed $z$,*

$$\lim_{x\to\infty} \arctan(z-x) = -\frac{\pi}{2} \qquad \text{and} \qquad \lim_{x\to-\infty} \arctan(z-x) = \frac{\pi}{2}, \tag{2.214}$$

*from which the value of $-\pi z$ for (2.213) follows.* ■

**Example 2.59** *(Example 2.58 cont.) Consider the integral*

$$I(s) = \int_{-\infty}^{\infty} \frac{1}{1+x^2} \frac{1}{1+(s-x)^2}\, dx.$$

*To resolve this, the first step is to use a* partial fraction decomposition *for the integrand,*

$$\frac{1}{1+x^2} \frac{1}{1+(s-x)^2} = \frac{Ax}{1+x^2} + \frac{B}{1+x^2} - \frac{Cx}{1+(s-x)^2} + \frac{D}{1+(s-x)^2},$$

*where*

$$A = \frac{2}{sR}, \quad B = \frac{1}{R}, \quad C = A, \quad D = \frac{3}{R},$$

*and* $R = s^2 + 4$, *which can be easily verified with symbolic math software. From linearity* (2.167), *we can integrate each of the four above pieces. With substitution* $u = s - x$, $du = -dx$, $x \to \infty \Rightarrow u \to -\infty$, *and* $x \to -\infty \Rightarrow u \to \infty$,

$$\int_{-\infty}^{\infty} \frac{dx}{1+(s-x)^2} = \int_{-\infty}^{\infty} \frac{du}{1+u^2}.$$

*From* (2.92), *namely, for* $g(y) = \arctan(y)$, $g'(y) = 1/(1+y^2)$, *we see that* $g(y)$ *is an indefinite integral of* $1/(1+y^2)$. *Thus, from* (2.183), $\int du/(1+u^2) = \arctan(u)$, *and, from* (2.214) *with* $z = 0$,

$$\int_{-\infty}^{\infty} \frac{dx}{1+(s-x)^2} = \int_{-\infty}^{\infty} \frac{du}{1+u^2} = \lim_{u\to\infty} \arctan u - \lim_{u\to-\infty} \arctan u = \pi.$$

*Thus, integration of the* $B$ *and* $D$ *terms gives*

$$\int_{-\infty}^{\infty} \left( \frac{B}{1+x^2} + \frac{D}{1+(s-x)^2} \right) dx = \pi (B + D) = \frac{4\pi}{R} = \frac{4\pi}{s^2+4}. \tag{2.215}$$

*For the remaining two terms, use of* (2.213) *leads to*

$$\begin{aligned}\int_{-\infty}^{\infty} \left( \frac{Ax}{1+x^2} - \frac{Cx}{1+(s-x)^2} \right) dx &= A \int_{-\infty}^{\infty} \left( \frac{x}{1+x^2} - \frac{x}{1+(s-x)^2} \right) dx \\ &= -A\pi s = \frac{2}{sR}\pi s = -\frac{2\pi}{s^2+4},\end{aligned} \tag{2.216}$$

*so that, adding* (2.215) *and* (2.216),

$$I(s) = \frac{2\pi}{s^2+4} = \frac{\pi}{2} \frac{1}{1+(s/2)^2}.$$

*This result is required for determining the density of the sum of two independent standard Cauchy random variables via the convolution formula.* ■

The indefinite integral $\int r^{-1} \sin r\, dr$ is termed the Si function. In preparation for the next example, we wish to consider $F(x) := \int_0^x r^{-1} \sin r\ dr$ for $x > 0$, and show that $\mathrm{Si}(0) = \lim_{x\to 0^+} F(x) = 0$. From (2.87), $\lim_{r\to 0} \sin(r)/r = 1$, so, if we define the integrand as $r^{-1} \sin r$, for $r > 0$, and 0, for $r = 0$, then the integrand is defined, bounded, and continuous on any closed interval $[0, x]$, $x > 0$. Thus, from the lhs of (2.172), $F(0) = 0$. We play around with some other ways to determine this.

For $r > 0$, $r^{-1} \sin r \leq |r^{-1} \sin r| \leq |r^{-1}| = r^{-1}$, so monotonicity of the integral implies, for $x > 0$,

$$F(x) = \int_0^x r^{-1} \sin r \; dr \leq \int_0^x r^{-1} dr.$$

For $0 < a, b < 1$, and recalling the integral representation of log in Example 2.47,

$$\int_a^b \frac{dx}{x} = \int_a^1 \frac{dx}{x} - \int_b^1 \frac{dx}{x} = \int_1^b \frac{dx}{x} - \int_1^a \frac{dx}{x} = \ln b - \ln a,$$

and, for $b = ka$ for $k > 1$,

$$\lim_{k \to 1^+} \lim_{a \to 0^+} (\ln b - \ln a) = \lim_{k \to 1^+} \lim_{a \to 0^+} (\ln ka - \ln a) = \lim_{k \to 1^+} \lim_{a \to 0^+} (\ln k + \ln a - \ln a) = \lim_{k \to 1^+} \ln k = 0.$$

Clearly, for $0 \leq x \leq \pi/2$, $\int_0^x r^{-1} \sin r \; dr \geq 0$, so the Squeeze Theorem implies that $\lim_{x \to 0^+} F(x) = 0$. See, e.g., https://ibmathsresources.com/2016/06/06/the-six-function/), for graphical illustration of the Si function.

Consider now a different approach, namely, using the Taylor series expansion of $\sin(x)/x$ about zero. From Maple, this is

$$\frac{\sin r}{r} = 1 - \frac{1}{6} r^2 + \frac{1}{120} r^4 - \frac{1}{5040} r^6 + O\left(r^8\right),$$

so ignoring higher-order terms,

$$F(x) \approx \int_0^x \left(1 - \frac{1}{6} r^2 + \frac{1}{120} r^4 - \frac{1}{5040} r^6\right) dr = x - \frac{1}{18} x^3 + \frac{1}{600} x^5 - \frac{1}{35\,280} x^7,$$

which is accurate for $x$ near zero, and from which it appears safe to conclude that

$$\lim_{x \to 0^+} F(x) = 0. \tag{2.217}$$

Clearly, this expansion cannot be used to approximate $\lim_{x \to \infty} F(x)$. How to do this is considered next.

**Example 2.60** *The integral*

$$S = \int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2} \tag{2.218}$$

*arises, for example, in the proof of the inversion theorem for continuous random variables (see, e.g., Paolella, Intermediate Probability, p. 31), which itself is of utmost importance in probability theory, and is heavily used in, e.g., quantitative risk management, advanced portfolio optimization, and option pricing.*

*We first wish to show that* (2.218) *converges. Recall from (2.87) that* $\lim_{x \to 0} x^{-1} \sin x = 1$, *so that* $\int_0^1 x^{-1} \sin x \; dx$ *is well defined, and it suffices to consider* $\lim_{M \to \infty} \int_1^M x^{-1} \sin x \; dx$. *Integration by parts with* $u = x^{-1}$ *and* $dv = \sin x \; dx$ *gives*

$$\int_1^M x^{-1} \sin x \, dx = -x^{-1} \cos x \Big|_1^M - \int_1^M x^{-2} \cos x \, dx.$$

*Clearly,* $\lim_{M\to\infty}(\cos M)\,/\,M = 0$*, so the first term is unproblematic. For the integral, note that*

$$\left|\frac{\cos x}{x^2}\right| \leq \frac{1}{x^2} \quad \textit{and} \quad \int_1^\infty \frac{dx}{x^2} < \infty,$$

*so that* $S$ *converges, having used the (improper integral) comparison test, (2.232).*

*Showing that the integral equals* $\pi/2$ *is a standard calculation via contour integration (see, e.g., Bak and Newman, 1997, p. 134), though derivation without complex analysis is also possible. As in Hijab (1997, p. 197) and Jones (2001, p. 192), we begin by defining* $F(x) = \int_0^x r^{-1}\sin r\; dr$ *for* $x > 0$*. Observe that, from FTC (ii, b) (2.179),* $F'(x) = x^{-1}\sin x$*.*

*Integration by parts (2.191) of*

$$I(b,s) := \int_0^b e^{-sx}\frac{\sin x}{x}dx$$

*with* $u = e^{-sx}$*,* $dv = x^{-1}\sin x\; dx$*,* $du = -se^{-sx}dx$ *and* $v = F(x)$ *gives, from (2.217),*

$$I(b,s) = e^{-sb}F(b) + s\int_0^b F(x)\,e^{-sx}dx = e^{-sb}F(b) + \int_0^{sb} F(y/s)\,e^{-y}\,dy,$$

*with* $y = sx$*. Letting* $b \to \infty$ *and using the boundedness of* $F(b)$ *as shown above gives*

$$I(b,s) \to I(\infty,s) = \int_0^\infty F(y/s)\,e^{-y}dy.$$

*Now taking the limit in* $s$*, and assuming the validity of the exchange of limit and integral,*

$$\lim_{s\to 0^+} I(\infty,s) = \int_0^\infty F(\infty)\,e^{-y}dy = F(\infty)\int_0^\infty e^{-y}dy = F(\infty)\,,$$

*from (2.204). But using (2.214) and the fact, proven in (2.310), that*

$$\int_0^\infty e^{-sx}\frac{\sin x}{x}dx = \arctan\left(s^{-1}\right),$$

*we have*

$$\lim_{s\to 0^+} I(\infty,s) = \lim_{s\to 0^+}\int_0^\infty e^{-sx}\frac{\sin x}{x}dx = \lim_{s\to 0^+}\left(\arctan\left(s^{-1}\right)\right) = \arctan(\infty) = \frac{\pi}{2},$$

*i.e.,*

$$\frac{\pi}{2} = \lim_{s\to 0^+} I(\infty,s) = F(\infty) = \int_0^\infty \frac{\sin r}{r}dr,$$

*as was to be shown. Other elementary proofs are outlined in Beardon (1997, p. 182) and Lang (1997, p. 343), while Goldberg (1964, p. 192) demonstrates that* $\int_0^\infty \frac{\sin x}{x}\,dx$ *does not converge absolutely, i.e.,* $\int_0^\infty \frac{|\sin x|}{x}\,dx$ *does not exist. As an aside, it can also be shown that* $\int_0^\infty (\sin ax)/x\,dx = \mathrm{sgn}(a)\pi/2$*.* ■

Recall the tan and arctan functions, and their derivatives, e.g., (2.91) and (2.92). Before commencing to the next example, we show some basic results for the arctangent function. In particular, we will explicitly require knowing that $\arctan 1 = \pi/4$. Some analysis books (e.g., Ghorpade and Limaye, 2018, p. 246) *define* the arctangent function by

$$\arctan x := \int_0^x \frac{dt}{1+t^2}. \tag{2.219}$$

The reason is that, from this, a rigorous, analytic (as opposed to geometric) definition of $\pi$ can be elicited; and then from which all the usual trigonometric results can be rigorously derived, without any appeal to the area of a unit circle. From (2.219) and (2.172), we immediately see that

$$\arctan 0 = 0. \tag{2.220}$$

The rest of this presentation is based on Ghorpade and Limaye (2018).

As $1/(1+t^2) \geq 0$ for all $t \in \mathbb{R}$, definition (2.219) (and (2.166) with $f = 0$ along with the definition of the Riemann integral, e.g., from (2.157)) implies that $\arctan x \geq 0$ if $x > 0$, while, from the latter equation in (2.172), $\arctan x \leq 0$ if $x < 0$. From FTC (ii,b) (2.179), the derivative of arctan is obviously positive on $\mathbb{R}$, so that arctan is strictly increasing on $\mathbb{R}$. Further,

$$(\arctan)'' x = -\frac{2x}{(1+x^2)^2}$$

is positive for all $x \in (-\infty, 0)$ and negative for all $x \in (0, \infty)$. Thus, arctan is strictly convex on $(-\infty, 0)$ and strictly concave on $(0, \infty)$; and 0 is a point of inflection.

For $x \in \mathbb{R}$, and with $s = -t$,

$$\arctan(-x) = \int_0^{-x} \frac{1}{1+t^2} dt = -\int_0^x \frac{1}{1+s^2} ds = -\arctan x.$$

Hence, arctan is an odd function. For $x \in (1, \infty)$,

$$\arctan x = \int_0^1 \frac{1}{1+t^2} dt + \int_1^x \frac{1}{1+t^2} dt$$

and since $1 \geq t^2$ for $t \in [0,1]$, while $t^2 \geq 1$ for $t \in [1, x]$, we see that

$$\int_0^1 \frac{1}{1+1} dt + \int_1^x \frac{1}{t^2+t^2} dt \leq \arctan x \leq \int_0^1 \frac{1}{1} dt + \int_1^x \frac{1}{t^2} dt.$$

The definite integrals above are easy to evaluate, and thus we obtain

$$1 - \frac{1}{2x} \leq \arctan x \leq 2 - \frac{1}{x} \quad \text{for all } x \in (1, \infty). \tag{2.221}$$

As arctan is strictly increasing and odd, the latter inequality in (2.221) implies that, $\forall x \in \mathbb{R}$, $-2 < \arctan x < 2$, i.e., arctan is a bounded function.

Function arctan is one-one because it is strictly increasing. We wish to show that it is also onto, and thus a bijection. Consider $y \in (-\pi/2, \pi/2)$. Since $\arctan x \to -\pi/2$ as $x \to -\infty$, there is some $x_0 \in \mathbb{R}$ such that $\arctan x_0 < y$, and since $\arctan x \to \pi/2$ as $x \to \infty$, there is some $x_1 \in \mathbb{R}$ such that $y < \arctan x_1$. From FTC (ii,a) (2.178), arctan is continuous on the interval $[x_0, x_1]$. Thus, the IVT (2.60) shows that $\exists x \in (x_0, x_1)$ such that $\arctan x = y$. Thus the function $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ is bijective.

For $x \in [1, \infty)$, the substitution $t = 1/s$ gives

$$\int_1^x \frac{dt}{1+t^2} = \int_{1/x}^1 \frac{ds}{1+s^2}, \quad \text{and hence} \quad \lim_{x\to\infty} \int_1^x \frac{dt}{1+t^2} = \int_0^1 \frac{ds}{1+s^2},$$

so that

$$\lim_{x\to\infty} (\arctan x - \arctan 1) = \lim_{x\to\infty} \int_1^x \frac{dt}{1+t^2} = \arctan 1.$$

But $\arctan x \to \pi/2$ as $x \to \infty$, so that

$$\arctan 1 = \pi/4. \tag{2.222}$$

**Example 2.61** *Example 2.55 showed that* $I = \int_0^\infty \exp(-x^2)\,dx$ *is convergent. Its value is commonly and quickly derived by use of polar coordinates (see §6.6, and Example 6.21), but it can be done without them. As in Weinstock (1990), let*

$$I = \int_0^\infty \exp\left(-x^2\right) dx$$

*and*

$$f(x) = \int_0^1 \frac{\exp\left(-x\left(1+t^2\right)\right)}{1+t^2} dt, \quad x > 0. \tag{2.223}$$

*From* (2.92) *and FTC (i)* (2.176)*; or from* (2.219)*; and recalling* (2.220) *and* (2.222)*, we have*

$$f(0) = \int_0^1 \left(1+t^2\right)^{-1} dt = \arctan(1) - \arctan(0) = \pi/4 \tag{2.224}$$

*and, as* $x > 0$*, and* $0 < t < 1 \Rightarrow e^{-x\cdot 1} < e^{-xt} < e^{-x\cdot 0} = 1$,

$$0 < f(x) = e^{-x} \int_0^1 \frac{e^{-xt^2}}{1+t^2} dt < e^{-x} \int_0^1 \frac{1}{1+t^2} dt = \frac{\pi}{4} e^{-x},$$

*so that, from the Squeeze Theorem,* $f(\infty) = 0$*. Differentiating (2.223) with respect to* $x$ *(and assuming we can interchange derivative and integral; see §6),* $f'(x)$ *is given by*

$$\int_0^1 \frac{d}{dx} \frac{\exp\left(-x\left(1+t^2\right)\right)}{1+t^2} dt = -\int_0^1 \exp\left(-x\left(1+t^2\right)\right) dt = -e^{-x} \int_0^1 \exp\left(-xt^2\right) dt.$$

*Now, with* $u = t\sqrt{x}$*,* $t = u/\sqrt{x}$ *and* $dt = x^{-1/2}\,du$*,*

$$f'(x) = -e^{-x} x^{-1/2} \int_0^{\sqrt{x}} \exp\left(-u^2\right) du = -e^{-x} x^{-1/2} g\left(\sqrt{x}\right), \tag{2.225}$$

*where* $g(z) := \int_0^z \exp\left(-u^2\right) du$*. From (2.224) and that* $f(\infty) = 0$*, integrating both sides of (2.225) from* $0$ *to* $\infty$ *and using FTC (i)* (2.176)*, gives, with* $z = \sqrt{x}$*,* $x = z^2$ *and* $dx = 2z\,dz$*,*

$$0 - \frac{\pi}{4} = f(\infty) - f(0) = -\int_0^\infty e^{-x} x^{-1/2} g\left(\sqrt{x}\right) dx = -2 \int_0^\infty e^{-z^2} g(z)\, dz$$

*or* $\int_0^\infty \exp(-z^2) g(z)\, dz = \pi/8$*. From FTC (ii,b)* (2.179)*,* $g'(z) = \exp(-z^2)$*, so*

$$\frac{\pi}{8} = \int_0^\infty g'(z)\, g(z)\, dz \overset{u=g(z)}{=} \int_0^I u\, du = \frac{I^2}{2},$$

*or* $I = \sqrt{\pi}/2$*.*

*It is now easy to show that* $J = \int_0^\infty \exp\left(-u^2/2\right) du = \sqrt{\pi}/\sqrt{2}$*: With* $u = x\sqrt{2}$*,* $x = u/\sqrt{2}$*,* $dx = du/\sqrt{2}$*,*

$$\frac{\sqrt{\pi}}{2} = I = \int_0^\infty \exp\left(-x^2\right) dx = \frac{1}{\sqrt{2}} \int_0^\infty \exp\left(-u^2/2\right) du = \frac{1}{\sqrt{2}} \frac{\sqrt{\pi}}{\sqrt{2}}. \tag{2.226}$$

*As* $\exp\left(-u^2/2\right)$ *is an even function,* $\int_{-\infty}^\infty \exp\left(-u^2/2\right) du = \sqrt{2\pi}$*. We will investigate two further ways of determining* $J$ *in Examples 6.21 and 6.22. Weinstock also gives similar derivations of the* Fresnel integrals $\int_0^\infty \cos y^2 dy$ *and* $\int_0^\infty \sin y^2 dy$*. Another way of calculating* $\int_{-\infty}^\infty \exp\left(-x^2\right) dx$ *without use of polar coordinates is detailed in Hijab (1997, §5.4).* ■

We return now to some theory. The first part of this material comes from Ghorpade and Limaye, pp. 91-2, and is included because it is anyway relevant, but also because it is needed for the proof of the Cauchy Criterion for improper integrals below, in (2.231).

Definition: Suppose $D \subseteq \mathbb{R}$ is such that $D$ is not bounded above. Then there is a sequence in $D$ that tends to $\infty$. Consider a function $f : D \to \mathbb{R}$. We say that a limit of $f$ as $x$ tends to infinity exists if there is a real number $\ell$ such that

$$(x_n) \text{ any sequence in } D \text{ and } x_n \to \infty \Longrightarrow f(x_n) \to \ell.$$

We then write

$$f(x) \to \ell \text{ as } x \to \infty \quad \text{ or } \quad \lim_{x\to\infty} f(x) = \ell.$$

Since there exists a sequence $(x_n)$ in $D$ such that $x_n \to \infty$, it follows from the uniqueness of limits (see the theorem around (2.2)) that, if $\lim_{x\to\infty} f(x)$ exists, then it is unique.

Proposition: Suppose $D \subseteq \mathbb{R}$ is not bounded above and $f : D \to \mathbb{R}$ is a function. Then $\lim_{x\to\infty} f(x)$ exists if and only if there is $\ell \in \mathbb{R}$ satisfying the following $\epsilon - \alpha$ condition: For every $\epsilon > 0$, there is $\alpha \in \mathbb{R}$ such that

$$x \in D \text{ and } x \geq \alpha \Longrightarrow |f(x) - \ell| < \epsilon. \tag{2.227}$$

> Proof: Assume that $\lim_{x\to\infty} f(x)$ exists and is equal to a real number $\ell$. Suppose for a moment that the $\epsilon - \alpha$ condition does not hold. This means that there is $\epsilon > 0$ such that for every $\alpha \in \mathbb{R}$, there is $x \in D$ satisfying $x \geq \alpha$, but $|f(x) - \ell| \geq \epsilon$. By choosing $\alpha = n$ for each $n \in \mathbb{N}$, we may obtain a sequence $(x_n)$ in $D$ such that $x_n \geq n$, but $|f(x_n) - \ell| \geq \epsilon$ for all $n \in \mathbb{N}$. Now $x_n \to \infty$ and $f(x_n) \nrightarrow \ell$. This contradicts the assumption that $\lim_{x\to\infty} f(x) = \ell$.
>
> Conversely, assume the $\epsilon - \alpha$ condition. Let $(x_n)$ be any sequence in $D$ such that $x_n \to \infty$. Let $\epsilon > 0$ be given. Then there is $\alpha \in \mathbb{R}$ such that
>
> $$x \in D \text{ and } x \geq \alpha \Longrightarrow |f(x) - \ell| < \epsilon.$$
>
> Since $x_n \to \infty$, there is $n_0 \in \mathbb{N}$ such that $x_n \geq \alpha$, and hence $|f(x_n) - \ell| < \epsilon$, for all $n \geq n_0$. Thus $f(x_n) \to \ell$. So $\lim_{x\to\infty} f(x)$ exists and equals $\ell$.

Proposition (Cauchy Criterion): Suppose $D \subseteq \mathbb{R}$ is not bounded above and $f : D \to \mathbb{R}$ is a function. Then $\lim_{x\to\infty} f(x)$ exists if and only if the following $\epsilon - \alpha$ condition holds: For every $\epsilon > 0$, there is $\alpha \in \mathbb{R}$ such that

$$x, y \in D, \;\; x \geq \alpha, \;\; y \geq \alpha \;\; \Longrightarrow \;\; |f(x) - f(y)| < \epsilon. \tag{2.228}$$

> Proof: Assume that $\lim_{x\to\infty} f(x)$ exists and is equal to a real number $\ell$. Let $\epsilon > 0$ be given. Then there is $\alpha \in \mathbb{R}$ such that
>
> $$x \in D \text{ and } x \geq \alpha \;\; \Longrightarrow \;\; |f(x) - \ell| < \frac{\epsilon}{2}.$$
>
> Hence for $x, y \in D$ satisfying $x \geq \alpha$ and $y \geq \alpha$, we obtain
>
> $$|f(x) - f(y)| \leq |f(x) - \ell| + |\ell - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Conversely, assume that the $\epsilon - \alpha$ condition holds. Let $\epsilon > 0$ be given. Then there is $\alpha \in \mathbb{R}$ such that

$$x, y \in D, \;\; x \geq \alpha \text{ and } y \geq \alpha \Longrightarrow |f(x) - f(y)| < \frac{\epsilon}{2}.$$

By our hypothesis, there is a sequence $\{x_n\}$ in $D$ such that $x_n \to \infty$. Hence there is an $n_0 \in \mathbb{N}$ such that $x_n \geq \alpha$ for all $n \geq n_0$. Consequently,

$$\forall n, m \geq n_0, \quad |f(x_n) - f(x_m)| < \frac{\epsilon}{2}.$$

Thus $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. By the Cauchy Criterion for sequences (2.17), there is $\ell \in \mathbb{R}$ such that $f(x_n) \to \ell$. Hence there is $n_1 \in \mathbb{N}$ such that $n_1 \geq n_0$ and $|f(x_{n_1}) - \ell| < \epsilon/2$. Since $x_{n_1} \geq \alpha$, it follows that

$$x \in D \text{ and } x \geq \alpha \Longrightarrow |f(x) - \ell| \leq |f(x) - f(x_{n_1})| + |f(x_{n_1}) - \ell| < \epsilon.$$

Consequently, by Proposition (2.227), $\lim_{x\to\infty} f(x)$ exists and is equal to $\ell$.

The remainder of this section is based on Ghorpade and Limaye, §9.4.

Let $a \in \mathbb{R}$, and let $f$ be defined on $[a, \infty)$ and such that $f$ is integrable on $[a, x]$ for every $x \geq a$. It is useful to define

$$F(x) = \int_a^x f(t)dt \quad \text{ for all } x \in [a, \infty),$$

with $F$ called the partial integral function, and $F(x)$ the partial integral of $f$.

Proposition: If $\int_{t\geq a} f(t)dt$ is convergent, then the set $\left\{\int_a^x f(t)dt : x \in [a, \infty)\right\}$ of partial integrals is bounded.

Proof: Let $F(x) := \int_a^x f(t)dt$ for $x \in [a, \infty)$ and note that since $\int_{t\geq a} f(t)dt$ is convergent, there exists $x_0 \geq a$ such that

$$\forall x \geq x_0, \quad |F(x)| \leq 1 + \left|\int_a^\infty f(t)dt\right|. \tag{2.229}$$

We add a bit to the previous proof. Convergence of $\int_a^\infty f(t)dt$ means, $\exists L \in \mathbb{R}$ such that $\lim_{x\to\infty} \int_a^x f(t)dt = L$, i.e., for any given $\epsilon > 0$, $\exists x_0 \geq a$ such that, for all $x \geq x_0$,

$$\left|L - \int_a^x f(t)dt\right| < \epsilon,$$

so that $F(x)$ is bounded for all $x \geq x_0$. By definition, $f$ is integrable on $[a, x]$ for every $x \geq a$, and from the definition of the Riemann integral in §2.5.1, $F(x)$ is finite for all $a \leq x < x_0$. Recall the reverse triangle inequality (1.21), namely, $\forall a, b \in \mathbb{R}$, $||a| - |b|| \leq |a + b|$ or $||a| - |b|| \leq |b - a|$. Then, for all $x \geq x_0$,

$$|\;|F(x)| - |L|\;| = \left|\;\left|\int_a^x f(t)dt\right| - |L|\;\right| \leq \left|L - \int_a^x f(t)dt\right| < \epsilon, \tag{2.230}$$

i.e., $\forall x \geq x_0$, $-\epsilon < |F(x)| - |L| < \epsilon$, or $|F(x)| < \epsilon + |L|$, which is (2.229).

Proposition (Cauchy Criterion): An improper integral $\int_{t\geq a} f(t)dt$ is convergent if and only if for every $\epsilon > 0$, there exists $x_0 \in [a, \infty)$ such that

$$\left|\int_x^y f(t)dt\right| < \epsilon \quad \text{ for all } y \geq x \geq x_0. \tag{2.231}$$

Proof: As in Ghorpade and Limaye, p. 394, let $F$ denote the partial integral function of $f$, and write $F(y) - F(x) = \int_x^y f(t)dt$ for all $y \geq x \geq x_0$. Now use the Cauchy Criterion for unbounded sets, (2.228).

Proposition (Comparison Test for Improper Integrals): Suppose $a \in \mathbb{R}$ and $f, g : [a, \infty) \to \mathbb{R}$ are such that both $f$ and $g$ are integrable on $[a, x]$ for every $x \geq a$ and $|f(t)| \leq g(t)$ for all large $t \in [a, \infty)$.

$$\textbf{If } \int_{t\geq a} g(t)dt \textbf{ is convergent, then } \int_{t\geq a} f(t)dt \textbf{ is absolutely convergent.} \tag{2.232}$$

Proof: Let $t_0 \in [a, \infty)$ be such that $|f(t)| \leq g(t)$ for all $t \in [t_0, \infty)$. Suppose $\int_{t\geq a} g(t)dt$ is convergent. Let $\epsilon > 0$ be given. Then by the Cauchy Criterion, there exists $x_0 \in [a, \infty)$ such that $\left|\int_x^y g(t)dt\right| < \epsilon$ for all $y \geq x \geq x_0$. Now let $x_1 := \max\{t_0, x_0\}$. Then

$$\forall y \geq x \geq x_1, \quad \int_x^y |f(t)|dt \leq \int_x^y g(t)dt = \left|\int_x^y g(t)dt\right| < \epsilon.$$

Hence by the Cauchy Criterion (2.231), $\int_{t\geq a} f(t)dt$ is absolutely convergent.

As a simple example, to prove the existence of the integral $\int_1^\infty \frac{dx}{1+x^2}$ we simply have to observe that, for any $x \geqslant 1$, $\frac{1}{1+x^2} \leqslant \frac{1}{x^2}$. Since $\int_1^\infty x^{-2}dx$ exists, the result then follows from (2.232).

Proposition (Limit Comparison Test for Improper Integrals): Let $a \in \mathbb{R}$ and let $f, g : [a, \infty) \to \mathbb{R}$ be such that both $f$ and $g$ are integrable on $[a, x]$ for every $x \geq a$. Suppose $f(t) > 0$ and $g(t) > 0$ for all large $t \in [a, \infty)$. Also suppose there exists $\ell \in \mathbb{R} \cup \{\infty\}$ such that $f(t)/g(t) \to \ell$ as $t \to \infty$.

(i) If $\ell \neq 0$ and $\ell \neq \infty$, then

$$\int_{t\geq a} f(t)dt \text{ is convergent } \iff \int_{t\geq a} g(t)dt \text{ is convergent.}$$

(ii) If $\ell = 0$ and $\int_{t\geq a} g(t)dt$ is convergent, then $\int_{t\geq a} f(t)dt$ is convergent.

(iii) If $\ell = \infty$ and $\int_{t\geq a} f(t)dt$ is convergent, then $\int_{t\geq a} g(t)dt$ is convergent.

A proof can be found in Ghorpade and Limaye, p. 402.

**Example 2.62** *Let $q \in \mathbb{R}$ and let $f : [1, \infty) \to \mathbb{R}$ be given by $f(t) := e^{-t}t^q$. Then $\int_{t\geq 1} f(t)dt$ is convergent. To see this, choose $k \in \mathbb{N}$ with $k > q + 1$, and define $g : [1, \infty) \to \mathbb{R}$ by $g(t) := t^{q-k}$. Then $f(t) > 0$ and $g(t) > 0$ for all $t \in [1, \infty)$, and moreover, by L'Hôpital's Rule,*

$$\frac{f(t)}{g(t)} = \frac{t^k}{e^t} \to 0 \quad \textit{as} \quad t \to \infty.$$

*Since $k - q > 1$, we see that $\int_{t\geq 1} g(t)dt$ is convergent. Hence by part (ii) of the previous proposition, we conclude that $\int_{t\geq 1} e^{-t}t^q dt$ is convergent.* ∎

## 2.6 Series

> Mathematics is an experimental science, and definitions do not come first, but later on. (Oliver Heaviside)

Of particular importance to us is differentiation and integration of infinite series of functions, along with the development of Taylor polynomials and Taylor series of functions. Their development, however, first requires establishing various notions and tools, of interest in themselves, such as sequences, Cauchy sequences, series of numbers, Cauchy products, and others, discussed herein.

### 2.6.1 Useful Results on Supremum

Recall from §1.1 the definitions of infimum and supremum. We repeat these now and give some useful results.

Let $S$ be a nonempty subset of $\mathbb{R}$. We say $S$ has an *upper bound* $M$ if $x \leq M\ \forall x \in S$, in which case $S$ is *bounded above* by $M$. Note that, if $S$ is bounded above, then it has infinitely many upper bounds. A fundamental property of $\mathbb{R}$ not shared by $\mathbb{Q}$ is that, if $S$ is a nonempty set that has an upper bound $M$, then $S$ possesses a unique *least upper bound*, or *supremum*, denoted $\sup S$. That is, $\exists U \in \mathbb{R}$ such that $U$ is an upper bound of $S$, and such that, if $V$ is also an upper bound of $S$, then $V \geq U$. If $S$ is not bounded above, then $\sup S = \infty$. Also, $\sup \emptyset = -\infty$. Similar terminology applies to the *greatest lower bound*, or *infimum* of $S$, denoted $\inf S$.

Theorem: If $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are two sequences of real numbers that are each bounded above, then

$$\sup_{n\in\mathbb{N}} \{a_n + b_n\} \leq \sup_{n\in\mathbb{N}} a_n + \sup_{n\in\mathbb{N}} b_n. \tag{2.233}$$

> Proof: For simplicity of notation, set $u = \sup a_n$ and $v = \sup b_n$. Since $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are each bounded above, we know that $u$ and $v$ are finite real numbers. Since $u$ is an upper bound for the $a_n$, we know that $a_n \leq u$ for every $n$. Similarly, $b_n \leq v$ for every $n$. Therefore $a_n + b_n \leq u + v$ for every $n$. Hence $u + v$ is an upper bound for $(a_n + b_n)_{n\in\mathbb{N}}$, so this sequence is bounded above and therefore has a finite supremum, which we will denote by $w = \sup(a_n + b_n)$. By definition, $w$ is the least upper bound for the sequence $(a_n + b_n)_{n\in\mathbb{N}}$. Since we saw above that $u + v$ is an upper bound for $(a_n + b_n)_{n\in\mathbb{N}}$, we must therefore have $w \leq u + v$, which is exactly what we wanted to prove.

Theorem: Let $\{a_n\}, \{b_n\} \in \mathbb{R}_{\geq 0}$, for $n \in \mathbb{N}$. Then

$$\sup_{n\in\mathbb{N}} a_n b_n \leq \sup_{n\in\mathbb{N}} a_n \sup_{n\in\mathbb{N}} b_n. \tag{2.234}$$

and, more generally, for each $n \in \mathbb{N}$,

$$\sup_{k\geq n} a_k b_k \leq \sup_{k\geq n} a_k \sup_{k\geq n} b_k. \tag{2.235}$$

> Proof: Clearly, (2.234) follows from (2.235), so we only need to prove the latter. For each $n \in \mathbb{N}$, $a_n \leq \sup_{k\geq n} a_k$ and $b_n \leq \sup_{k\geq n} b_k$, so that, as $\{a_n\}, \{b_n\} \in \mathbb{R}_{\geq 0}$, $a_n b_n \leq \left(\sup_{k\geq n} a_k\right)\left(\sup_{k\geq n} b_k\right)$. As the rhs is an upper bound, (2.235) follows.

Theorem: Let $A$ be a nonempty subset of $\mathbb{R}_{\geq 0}$, i.e., a set (countable or uncountable) of nonnegative real numbers. Then

$$\sup_{a \in A} a^2 = \left( \sup_{a \in A} a \right)^2, \tag{2.236}$$

i.e., "the sup of the square equals the square of the sup."

> Proof: If $\sup\{a\} = \infty$, then (2.236) is immediate. Assume $\sup\{a\} \in \mathbb{R}$. We need to demonstrate that (i) $\sup\{a^2\} \leq (\sup\{a\})^2$ and (ii) $(\sup\{a\})^2 \leq \sup\{a^2\}$. We suppress writing $a \in A$, i.e., we just write $\{a^2\}$ to mean $\{a^2\}_{a \in A}$. Similarly, we suppress always writing $\forall a \in A$, i.e., $\sup\{a\} \geq a$ is short for $\forall a \in A$, $\sup\{a\} \geq a$.
>
> For (i): Note that $\sup\{a\} \geq a$ (for all $a \in A$), hence $(\sup\{a\})^2 \geq a^2$ (for all $a \in A$), so $(\sup\{a\})^2$ is an upper bound for $\{a^2\} = \{a^2\}_{a \in A}$, hence it is greater than the least upper bound, i.e., $(\sup\{a\})^2 \geq \sup\{a^2\}$.
>
> For (ii): Let $N$ be large enough such that $\sup\{a\} - \frac{1}{N} > 0$ (if $\sup\{a\} = 0$ and all $a$ are non-negative, then it is clear what the set is and the result itself). Now for any $n > N$, the quantity $\sup\{a\} - \frac{1}{n}$ is not an upper bound of $\{a\}$, Hence, there is some $a_n$ such that $a_n > \sup\{a\} - \frac{1}{n}$. Square both sides(note that by non-negativity of both sides, this preserves sign) to see that $a_n^2 > \frac{1}{n^2} + (\sup\{a\})^2 - \frac{2\sup\{a\}}{n}$. Now, by definition of supremum, we have
>
> $$\sup\{a^2\} \geq a_n^2 > \frac{1}{n^2} + (\sup\{a\})^2 - \frac{2\sup\{a\}}{n}.$$
>
> This applies for all $n > N$. Since $(\sup\{a\})^2$ is a bounded quantity, letting $n \to \infty$, we see that $\sup\{a^2\} \geq (\sup\{a\})^2$. Hence, equality follows.

We will use (2.236) to prove an important result below, namely (2.283).

### 2.6.2 Series

We now give the definition of a series, and some basic properties of series.

Definition: (Series; and Convergent Series, Divergent Series). A *series* is an infinite sum of elements from a set, such as the real numbers, the complex numbers, vectors, matrices, functions, etc.. We say that a series

$$\sum_{n=1}^{\infty} c_n = c_1 + c_2 + \cdots$$

of real numbers *converges* if there is a real number $s$ such that the partial sums

$$s_N = \sum_{n=1}^{N} c_n = c_1 + c_2 + \cdots + c_N$$

converge to $s$ as $N \to \infty$. In this case we declare that the series $\sum_{n=1}^{\infty} c_n$ has the value $s$:

$$\sum_{n=1}^{\infty} c_n = \lim_{N \to \infty} s_N = \lim_{N \to \infty} \sum_{n=1}^{N} c_n = s.$$

If the series $\sum_{n=1}^{\infty} c_n$ does not converge, then we say that it *diverges*.

Theorem (The $n$th Term Test): If $\sum_{n=1}^{\infty} c_n$ is a convergent series of real numbers, then

$$\lim_{n\to\infty} c_n = 0. \tag{2.237}$$

Proof: Since the series converges, $x = \sum_{n=1}^{\infty} c_n$ is a real number. Let $s_N = \sum_{n=1}^{N} c_n$ be the $N$th partial sum of the series, and set $s_0 = 0$. Then, since $s_n$ converges to $x$ as $n \to \infty$, we have, using the linearity property of limits (2.23),

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = x - x = 0.$$

The converse of (2.237) is not true. The classic example is $\sum_{k=1}^{\infty} k^{-1}$, which is explicitly stated below in (2.241).

Theorem (Tails of Convergent Series): If $\sum_{n=1}^{\infty} c_n$ is a convergent series of real numbers, then

$$\lim_{N\to\infty} \left( \sum_{n=N}^{\infty} c_n \right) = 0. \tag{2.238}$$

Proof: Let $x = \sum_{n=1}^{\infty} c_n$, and for each $M$ let $s_M = \sum_{n=1}^{M} c_n$ be the $M$ th partial sum of this series. Since the series converges, we know that $\lim_{M\to\infty} s_M = x$. Now let $N$ be a fixed positive integer, and for each $M \geq N$ let

$$t_M = \sum_{n=N}^{M} c_n = \sum_{n=1}^{M} c_n - \sum_{n=1}^{N-1} c_n = s_M - s_{N-1}$$

be the $M$th partial sum of the infinite series $\sum_{n=N}^{\infty} c_n$. Keeping $N$ fixed, we have that

$$\sum_{n=N}^{\infty} c_n = \lim_{M\to\infty} t_M = \lim_{M\to\infty} (s_M - s_{N-1}) = x - s_{N-1}.$$

Therefore $\lim_{N\to\infty} \sum_{n=N}^{\infty} c_n = \lim_{N\to\infty} (x - s_{N-1}) = x - x = 0.$

In the following, we will often consider a sequence of (real-valued) functions, e.g., $f_k$, where $f_k : D \subset \mathbb{R} \to \mathbb{R}$, thus with common domain $D$. For example, let $f_k(x) = x^2 (1 + x^2)^{-k}$, for $x \in \mathbb{R}$ and $k = 0, 1, \ldots$, which we will investigate in Example 2.80. If function $f_k$ does not vary over its domain, then we can replace function sequence $\{f_k\}$ with the more comfortable sequence of real numbers notation $\{a_k\}$. The former, i.e., use of functions, is not just more general (thus killing two birds with one stone), but sequences and series of functions are of utmost importance in analysis.

Definition: Let $\{f_k\}$ be a sequence of (real) functions with common domain. Let

$$s_n = \sum_{k=1}^{n} f_k, \quad \text{and} \quad S = \sum_{k=1}^{\infty} f_k = \lim_{n\to\infty} \sum_{k=1}^{n} f_k = \lim_{n\to\infty} s_n. \tag{2.239}$$

$S$ is referred to as a *series* associated with the sequence $\{f_k\}$, and $s_n$ is its $n$th partial sum. Series $S$ *converges* if $\lim_{n\to\infty} s_n$ exists, i.e., if the limit is bounded, and *diverges* if the partial sums are not bounded. Further, we let 0 refer to the *zero function*, i.e., $z(D) = 0$.

Theorem: Let $\{f_k\}$ be a sequence of (real) functions with common domain $D$.

$$\textbf{If series } S = \sum_{k=1}^{\infty} f_k \textbf{ converges, then } \lim_{n\to\infty} f_n = 0. \tag{2.240}$$

This follows by applying the $n$th term test (2.237) to $f(x)$ for each $x \in D$.

The converse of (2.240), similar to series of real numbers, is not true.

Definition: If $\sum_{k=1}^{\infty} |f_k|$ converges, then $S$ is said to *converge absolutely*.

We prove below that, if $S$ is absolutely convergent, then $S$ is convergent. However, the converse is not true. For example, the *alternating (harmonic) series* $S = \sum_{k=1}^{\infty}(-1)^k k^{-1}$ is convergent, but not absolutely convergent. The harmonic series $S$ is *conditionally convergent*, i.e., $S = \sum_{k=1}^{\infty} k^{-1}$ diverges, as shown next.

Theorem: The harmonic series diverges, i.e.,

$$S = \sum_{k=1}^{\infty} k^{-1} \quad \text{diverges.} \tag{2.241}$$

Proof: Let $a_k = 1/k$ and define the partial sum $s_n = \sum_{k=1}^{n} a_k$. Observe that

$$\begin{aligned} s_{2^m} &= 1+\frac{1}{2} + \left(\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) + \dots \\ &\quad + \left(\frac{1}{2^{m-1}+1} + \dots + \frac{1}{2^m}\right) \\ &\geq 1 + \frac{1}{2} + 2\cdot\frac{1}{4} + 4\cdot\frac{1}{8} + \dots + 2^{m-1}\cdot\frac{1}{2^m} = 1 + \frac{m}{2} \to \infty \end{aligned}$$

as $m \to \infty$, which shows that the series $\sum_{k=1}^{\infty} 1/k$ diverges.

We also give a very short, clever proof using theory from infinite products in (2.261).

Theorem (Cauchy criterion, Cauchy, 1821): Let $\{f_k\}$ be a sequence of (real) functions with common domain $D$. For any given $\epsilon > 0$,

$$\sum_{k=1}^{\infty} f_k \textbf{ converges} \iff \exists N \in \mathbb{N} \textbf{ such that, } \forall n, m \geq N, \left|\sum_{k=n+1}^{m} f_k\right| < \epsilon. \tag{2.242}$$

Proof: Use (need to look ahead) (3.60) and write $\left|\sum_{k=n+1}^{m} f_k\right| = |s_m - s_n|$.

Theorem: Let $\{f_k\}$ be a sequence of (real) functions with common domain $D$. Any absolutely convergent series is convergent.

Proof: Suppose $\sum_{k=1}^{\infty} f_k$ is absolutely convergent. According to the Cauchy criterion (2.242) applied to the series $\sum_{k=1}^{\infty} |a_k|$, for $\varepsilon > 0$ there exists a number $N(\varepsilon)$ such that $n \geq m \geq N$ implies $\sum_{k=m}^{n} |a_k| < \varepsilon$. Now the triangle inequality yields for $n \geq m \geq N$

$$\left|\sum_{k=m}^{n} a_k\right| \leq \sum_{k=m}^{n} |a_k| < \varepsilon,$$

i.e. the Cauchy criterion holds for $\sum_{k=1}^{\infty} a_k$ which implies the convergence of $\sum_{k=1}^{\infty} a_k$ by (2.242).

The *geometric series* $S_0(c)$ and $S_1(c)$, and the *p-series* or *(Riemann's) zeta function*

$$S_0(c) = \sum_{k=0}^{\infty} c^k, \quad S_1(c) = \sum_{k=1}^{\infty} c^k, \;\; c \in [0,1)\,, \quad \text{and} \quad \zeta(p) = \sum_{k=1}^{\infty} \frac{1}{k^p}\;, \;\; p \in \mathbb{R}_{>1},$$

respectively, are important convergent series because they can often be used to help prove the convergence of other series via the tests outlined below. Indeed, for the geometric series $S_1 = S_1(c) = c + c^2 + c^3 + \cdots$, $cS_1 = c^2 + c^3 + \cdots$ and $S_1 - cS_1 = c - \lim_{k\to\infty} c^k = c$, for $c \in [0,1)$. Solving $S_1 - cS_1 = c$ implies

$$S_1 = \frac{c}{1-c}, \quad c \in [0,1)\,. \tag{2.243}$$

Trivially,

$$S_0 = 1 + \frac{c}{1-c} = \frac{1}{1-c}, \quad c \in [0,1)\,. \tag{2.244}$$

Further, for $-1 < c \leq 0$,

$$S_0(c) = 1 - c + c^2 - c^3 + \cdots = \frac{1}{1-(-c)} = \frac{1}{1+c}. \tag{2.245}$$

**Example 2.63** *For the zeta function, use (2.244) to express an upper bound of it as*

$$\begin{aligned}
\zeta(p) = \sum_{k=1}^{\infty} \frac{1}{k^p} &= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \\
&= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \cdots \\
&< 1 + \frac{2}{2^p} + \frac{4}{4^p} + \cdots = \sum_{i=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^i = \frac{1}{1 - \frac{1}{2^{p-1}}}.
\end{aligned}$$

*This is valid for $1/2^{p-1} < 1$ or $(p-1)\ln 2 > 0$ or $p > 1$. Thus, we can conclude that the zeta function converges for at least $p > 1$, but we know from (2.241) that it diverges for $p = 1$, and thus the zeta function converges for $p > 1$ and (by use of the comparison test for series (2.249)) diverges for $0 \leq p \leq 1$.* ■

**Example 2.64** *Let $\zeta(p) = \sum_{r=1}^{\infty} r^{-p}$. The well-known result*

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{2.246}$$

*is often proven via contour integration (Bak and Newman, 1997, p. 141) or in the context of Fourier analysis (Stoll, 2001, p. 413; Jones, 2001, p. 410). The first proof, by Euler in 1735, involves the use of infinite products; see Havil (2003, p. 39) for a simple account, or Hijab (1997, §5.6), Bak and Newman (1997, p. 223), or Little, Teo, van Brunt (An Introduction to Infinite Products, 2022, p. 94), for rigorous proofs. Before Euler solved it, the problem had been unsuccessfully attempted by Wallis, Leibniz, Jakob Bernoulli, and others (Havil, 2003, p. 38). Today, there are many known methods of proof.*[16] *It can also be shown that $\zeta(4) = \pi^4/90$ and $\zeta(6) = \pi^6/945$. In general, expressions exist for even $p$.* ■

[16] Matsuoka (1961) gives an elementary one, requiring two integration by parts of $\int_0^{\pi/2} \cos^{2n}(t)\,dt = (\pi/2)(2n-1)!!\,/\,(2n)!!$, where $(2n)!! = 2 \cdot 4 \cdots (2n)$, $0!! = 1$, $(2n+1)!! = 1 \cdot 3 \cdots (2n+1)$ and $(-1)!! = 1$. Kortam (1996) illustrates further simple proofs, Hofbauer (2002) and Harper (2003) each contribute yet another method, and Chapman (2003) provides 14 proofs (not including the previous two, but including that from Matsuoka).

**Example 2.65** *Recall* $e^\lambda = \lim_{n\to\infty}(1+\lambda/n)^n$ *from (2.137). From* (2.113), $e^\lambda = [\exp(1)]^\lambda$, *so that, to show convergence of the latter limit, it suffices to take* $\lambda = 1$ *and show that sequence* $s_n := (1+1/n)^n$ *converges. Applying the binomial theorem (1.34) to* $s_n$ *gives* $s_n = \sum_{i=0}^n \binom{n}{i}\left(\frac{1}{n}\right)^i$, *with each term expressible as*

$$\begin{aligned}\binom{n}{i} n^{-i} &= \frac{n_{[i]}}{i!} n^{-i} = \frac{n\,(n-1)\cdots(n-i+1)}{i!} n^{-i} \\ &= \frac{1}{i!}(1)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{i-1}{n}\right). \end{aligned} \tag{2.247}$$

*Similarly,* $s_{n+1} = \sum_{i=0}^{n+1}\binom{n+1}{i}(n+1)^{-i}$, *with*

$$\begin{aligned}\binom{n+1}{i}(n+1)^{-i} &= \frac{(n+1)\,n\,(n-1)\cdots(n-i+2)}{i!}(n+1)^{-i} \\ &= \frac{1}{i!}(1)\left(\frac{n}{n+1}\right)\left(\frac{n-1}{n+1}\right)\cdots\left(\frac{n-i+2}{n+1}\right) \\ &= \frac{1}{i!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{i-1}{n+1}\right). \end{aligned} \tag{2.248}$$

*As the quantity in (2.248) is larger than that in (2.247), it follows that* $s_n \le s_{n+1}$, *i.e.,* $s_n$ *is an increasing (or, nondecreasing) sequence. Also, for* $n \ge 2$,

$$s_n = \sum_{i=0}^n \binom{n}{i}\frac{1}{n^i} = \sum_{i=0}^n \frac{n\,(n-1)\cdots(n-i+1)}{n^i}\frac{1}{i!} \le \sum_{i=0}^n \frac{1}{i!}.$$

*Note that* $2! < 2^2$ *and* $3! < 2^3$, *but* $4! > 2^4$. *Assume* $k! > 2^k$ *holds for* $k \ge 4$. *It holds for* $k+1$ *because* $(k+1)! = (k+1)\,k! > (k+1)\,2^k > 2\cdot 2^k = 2^{k+1}$. *Thus,* $k! > 2^k$ *for* $k \ge 4$, *and*

$$\begin{aligned}\sum_{i=0}^n \frac{1}{i!} = \frac{8}{3} + \sum_{i=4}^n \frac{1}{i!} &< \frac{8}{3} + \sum_{i=4}^n \frac{1}{2^i} = \frac{8}{3} + \sum_{i=0}^n \frac{1}{2^i} - \sum_{j=0}^3 \frac{1}{2^j} = \frac{19}{24} + \sum_{i=0}^n \frac{1}{2^i} \\ &< \frac{19}{24} + \sum_{i=0}^\infty \frac{1}{2^i} = \frac{19}{24} + 2 < 2.8. \end{aligned}$$

*Thus,* $s_n$ *is a nondecreasing, bounded sequence, and is thus convergent.* ■

### 2.6.3 Tests for Convergence and Divergence

> With the exception of the geometric series, there does not exist in all of mathematics a single infinite series whose sum has been determined rigorously.
>
> (Niels Abel)

The following are some of the many conditions, or "tests", that can help determine if a series of nonnegative terms is convergent or divergent.

- **The *Comparison Test*** Let $S = \sum_{k=1}^\infty f_k$ and $T = \sum_{k=1}^\infty g_k$ with $0 \le f_k, g_k < \infty$ and $T$ convergent. Then:

  $$\textbf{If } \exists C > 0\textbf{: } f_k \le C g_k\textbf{, then } S \textbf{ converges.} \tag{2.249}$$

  This can be relaxed to requiring that that $f_k \le C g_k$ for all $k$ sufficiently large, i.e., $\exists\, K \in \mathbb{N}$ such that it holds for all $k \ge K$.

Proof: The proof when $f_k \leq C g_k$ for all $k \in \mathbb{N}$ is simply to note that

$$\sum_{k=1}^{n} f_k \leq \sum_{k=1}^{n} C g_k \leq C \sum_{k=1}^{\infty} g_k < \infty$$

is true for all $n$, so that the partial sum $\sum_{k=1}^{n} f_k$ is bounded. In a similar way, the comparison test can be used to show that a series diverges.

- **The *Ratio Test*** Let $S = \sum_{k=1}^{\infty} f_k$ with $0 \leq f_k < \infty$.

$$\textbf{If } \exists c \in (0,1)\textbf{: } f_{k+1}/f_k \leq c\textbf{, then } S \textbf{ converges}. \tag{2.250}$$

This can be relaxed to stating that, if $\exists c \in (0,1)$ such that $f_{k+1}/f_k \leq c$ for all $k$ sufficiently large, then $S$ converges.

Proof: Let $K$ be such that $f_{k+1} \leq c f_k$ for all $k \geq K$. Then $f_{K+1} \leq c f_K$, and $f_{K+2} \leq c f_{K+1} \leq c^2 f_K$, etc., and $f_{K+n} \leq c^n f_K$. Then, using geometric series result (2.243),

$$\begin{aligned}\sum_{n=1}^{\infty} f_{K+n} &= f_{K+1} + f_{K+2} + \cdots \\ &\leq c f_K + c^2 f_K + \cdots = \frac{c}{1-c} f_K,\end{aligned}$$

which is finite for $c \in (0,1)$.

More generally, allow $f_k$ to be negative or positive, and let $c = \lim_{k\to\infty} |f_{k+1}/f_k|$. If $c < 1$, then a similar argument shows that $S = \sum_{k=1}^{\infty} f_k$ converges absolutely. If $c > 1$ or $\infty$, then $\exists K \in \mathbb{N}$ such that $\forall k \geq K$, $|f_k| > |f_K|$. This implies, from the contrapositive of (2.240), that $\lim_{k\to\infty} |f_k| \neq 0$, and $S$ diverges.

- **The *Root Test*** Let $S = \sum_{k=1}^{\infty} f_k$ and $r = \lim_{k\to\infty} |f_k|^{1/k} \geq 0$.

$$\textbf{If } r < 1\textbf{, then } \sum_{k=1}^{\infty} |f_k| \textbf{ converges.} \tag{2.251}$$

Proof: If $r < 1$, then $\exists\, \epsilon > 0$ such that $r + \epsilon < 1$, and $\exists\, K \in \mathbb{N}$ such that $|f_k|^{1/k} < r + \epsilon$, or $|f_k| < (r+\epsilon)^k$, $\forall k \geq K$. It follows by the comparison test (2.249) with the geometric series $\sum_{k=1}^{\infty} (r+\epsilon)^k$ that $\sum_{k=1}^{\infty} |f_k|$ converges, i.e., $S$ is absolutely convergent.

$$\textbf{If } r > 1 \textbf{ or } \infty\textbf{, then } S \textbf{ diverges}. \tag{2.252}$$

Proof: If $r > 1$ or $\infty$, then $\exists\, \epsilon > 0$ such that $r - \epsilon > 1$, and $\exists\, K \in \mathbb{N}$ such that $|f_k|^{1/k} > r - \epsilon$, or $|f_k| > (r-\epsilon)^k$, $\forall k \geq K$. Thus, $\lim_{k\to\infty} |f_k| > 1$, and $S$ diverges.

$$\textbf{If } r = 1\textbf{, the test is inconclusive}. \tag{2.253}$$

Proof: Take the zeta function, with $f_k = k^{-p}$, and observe that, from the first limit result in Example 2.16,

$$\lim_{k\to\infty} f_k^{1/k} = \lim_{k\to\infty} \left(\frac{1}{k^p}\right)^{1/k} = \left(\frac{1}{\lim_{k\to\infty} k^{1/k}}\right)^p = 1$$

for any $p \in \mathbb{R}$. We know from Example 2.63 that $\zeta(p)$ converges for $p > 1$ and diverges otherwise, so that the root test is inconclusive.

Later, after we introduce the (lim inf and) lim sup of a sequence, we will extend the root test to allow it to have more "power". See (3.43).

- **The *Integral Test*** Let $f(x)$ be a nonnegative, decreasing function for all $x \geq 1$.

$$\textbf{If } \int_1^\infty f(x)\,dx < \infty\textbf{, then } S = \sum_{k=1}^{\infty} f(k) \textbf{ exists.} \tag{2.254}$$

Proof: This rests upon the fact that

$$f(k) \leq \int_{k-1}^{k} f(x)\,dx,$$

which is graphically obvious from Figure 9; the area of the rectangle from $k-1$ to $k$ with height $f(k)$ is $1 \times f(k) = f(k)$, which is less than or equal to the area under $f(x)$ between $x = k-1$ and $x = k$. Thus, from the domain additivity property of the Riemann integral (2.171),

$$f(2) + f(3) + \cdots + f(k) \leq \int_1^k f(x)\,dx \leq \int_1^\infty f(x)\,dx < \infty, \tag{2.255}$$

and the partial sums are bounded. To show divergence, note from Figure 9 that $f(k) \geq \int_k^{k+1} f(x)\,dx$, so that

$$f(1) + f(2) + \cdots + f(k) \geq \int_1^{k+1} f(x)\,dx.$$

If the latter integral diverges as $k \to \infty$, then so does the partial sum.

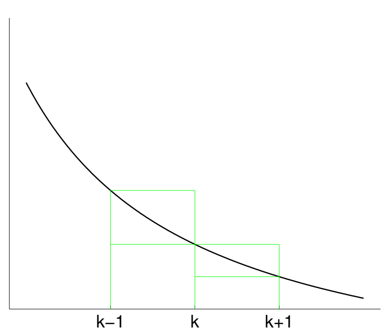



Figure 9: For continuous, positive, decreasing function $f$, $f(k) \leq \int_{k-1}^{k} f(x)dx$

- **The *Dirichlet Test*** Let $\{a_k\}$ and $\{b_k\}$ be sequences such that:
  - The partial sums of $\{a_k\}$ are bounded,
  - $\{b_k\}$ is positive and decreasing, i.e., $b_1 \geq b_2 \geq \cdots \geq 0$, and
  - $\lim_{k\to\infty} b_k = 0$.

  Then
$$\sum_{k=1}^{\infty} a_k b_k \textbf{ converges}. \tag{2.256}$$
  See, e.g., Stoll (2001, §7.2) for proof. As a special case, if $f_k$ is a positive, decreasing sequence with $\lim_{k\to\infty} f_k = 0$, then $\sum_{k=1}^{\infty}(-1)^k f_k$ converges, which is often referred to as the **alternating series test**. Observe how the partial sums of $(-1)^k$ are bounded, but this sequence, and the sequence of partial sums, are not convergent.

  If the sequence $\{b_k\}$ is positive and decreasing, and $\lim_{k\to\infty} b_k = 0$, then the Dirichlet test can also be used to prove that $\sum_{k=1}^{\infty} b_k \sin(kt)$ converges for all $t \in \mathbb{R}$, and that $\sum_{k=1}^{\infty} b_k \cos(kt)$ converges for all $t \in \mathbb{R}$, except perhaps for $t = 2z\pi$, $z \in \mathbb{Z}$. See Stoll (2001, p. 296-7) for proof.

**Example 2.66** *Let $f(x) = 1/x^p$ for $x \in \mathbb{R}_{\geq 1}$ and $p \in \mathbb{R}_{>0}$. As $f(x)$ is nonnegative and decreasing, the integral test (2.254) implies that $\zeta(p) = \sum_{x=1}^{\infty} x^{-p}$ exists if*
$$\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{x\to\infty}\left(\frac{x^{1-p}}{1-p}\right) - \frac{1}{1-p} < \infty,$$
*which is true for $1-p<0$, i.e., $p>1$, and does not exist otherwise. Thus, $\zeta(1)$ diverges, but $\zeta(p)$ converges for $p>1$, as also stated in Example 2.63.* ■

**Example 2.67** *Let $S(p) = \sum_{k=1}^{\infty} (\ln k)/k^p$. For $p > 1$, use the "standard trick" (Lang, 1997, p. 212) and write $p = 1+\epsilon+\delta$, $\delta > 0$. From (2.136) $\lim_{k\to\infty} (\ln k)/k^{\delta} = 0$, which implies that, for large enough $k$, $(\ln k)/k^{\delta} \leq 1$. Thus, for $k$ large enough and $C = 1$,*
$$\frac{\ln k}{k^p} = \frac{\ln k}{k^{\delta} k^{1+\epsilon}} \leq C \frac{1}{k^{1+\epsilon}},$$
*so that the comparison test (2.249) and the parameter range for convergence of the zeta function from Examples 2.63 and 2.66 imply that $S(p)$ converges for $p > 1$. A similar analysis shows that $S(p)$ diverges for $p < 1$. For $p = 1$, as $\ln k > 1$ for $k \geq 3$, the comparison test (2.249) with $\zeta(1)$ confirms that it also diverges. The integral test (2.254) also works in this case; see Stoll (2001, p. 286).* ■

**Example 2.68** *Continuing with the investigation of how inserting $\ln k$ affects convergence, consider now*
$$S(q) = \sum_{k=2}^{\infty} \frac{1}{(k \ln k)^q}.$$
*First, take $q = 1$. Let $f : D \to \mathbb{R}$ be given by $f(x) = (x\ln x)^{-1}$, with $D = \{x \in \mathbb{R} : x \geq 2\}$. It is straightforward to show that $f'(x) = -(1+\ln x)(x \ln x)^{-2} < 0$ on $D$, so that, from the*

*first derivative test, $f$ is decreasing on $D$, and the integral test (2.254) is applicable. But from (2.128) and FTC (i) (2.176), the improper integral*

$$\lim_{t\to\infty}\int_2^t \frac{dx}{x\ln x} = \lim_{t\to\infty}\left[\ln(\ln t) - \ln(\ln 2)\right]$$

*diverges, so that $S(1)$ diverges.*[17] *For $q > 1$, let $q = 1 + m$, so that (2.127) with $p = -m$ implies that*

$$\int_2^t \frac{dx}{x(\ln x)^{1+m}} = -\frac{(\ln x)^{-m}}{m}\bigg|_2^t = \frac{(\ln 2)^{-m}}{m} - \frac{1}{m(\ln t)^m}, \qquad m > 0,$$

*so that*

$$\int_2^\infty \frac{dx}{x(\ln x)^{1+m}} = \frac{(\ln 2)^{-m}}{m} < \infty,$$

*and $S(q)$ converges for $q > 1$.* ■

**Example 2.69** *Let $S = \sum_{k=2}^{\infty}(\ln k)^{-vk}$, for constant $v \in \mathbb{R}_{>0}$. As*

$$\lim_{k\to\infty}\left|(\ln k)^{-vk}\right|^{1/k} = \lim_{k\to\infty}\left(\frac{1}{\ln k}\right)^v = 0,$$

*the root test (2.251) shows that $S$ converges.* ■

**Example 2.70** *Let*

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n,$$

*which converges to* Euler's constant, *denoted $\gamma$, with $\gamma \approx 0.5772156649$. As in Beardon (1997, p. 176), let*

$$a_n = \ln\left(\frac{n}{n-1}\right) - \frac{1}{n}.$$

*That is, $a_2 = \ln 2 - \ln 1 - 1/2$, $a_3 = \ln 3 - \ln 2 - 1/3$, $a_4 = \ln 4 - \ln 3 - 1/4$, etc., so the log terms in $\sum_{i=2}^{n} a_i$ are "telescoping". Thus, $\sum_{i=2}^{n} a_i = \ln n - \ln 1 - 1/2 - 1/3 - \cdots - 1/n$, i.e., $\sum_{i=2}^{n} a_i = 1 - \gamma_n$. To see that $\gamma_n$ is convergent, it suffices to show that $\sum_{i=2}^{\infty} a_i$ converges. Observe that (use substitution $u = n - t$)*

$$\int_0^1 \frac{t}{n(n-t)}dt = \int_0^1\left(\frac{1}{n-t} - \frac{1}{n}\right)dt = a_n. \tag{2.257}$$

*Next, let $f(t) = t/(n-t)$ for $n \geq 1$ and $t \in (0,1)$. Clearly, $f(0) = 0$, $f(t)$ is increasing on $(0,1)$, and, as $f''(t) = 2n(n-t)^{-3} > 0$, it is convex. Thus, the area under its curve on $(0,1)$ is bounded by that of the right triangle with vertices $(0,0)$, $(1,0)$ and $(1, f(1))$, which has area $f(1)/2 = \frac{1}{2(n-1)}$. Thus, from these results and the first integral in (2.257), $a_n \geq 0$. Further,*

$$\int_0^1 \frac{t}{(n-t)}dt \leq \frac{1}{2(n-1)},$$

*or*

$$0 \leq a_n = \frac{1}{n}\int_0^1 \frac{t}{(n-t)}dt \leq \frac{1}{2n(n-1)} \leq \frac{1}{n^2}.$$

*By the comparison test (2.249) with the zeta function result in Examples 2.63 and 2.66, $\sum_{i=2}^{\infty} a_i$ and, thus, $\gamma_n$, converge. See also the next example, and Example 2.90.* ■

[17] The divergence is clearly very slow. The largest number in Matlab is obtained by `t=realmax`, which is about $1.7977 \times 10^{308}$, and $\ln(\ln t) = 6.565$.

**Example 2.71** *The following magnificent presentation is copied nearly verbatim from the (equally magnificent) Duren (Invitation to Classical Analysis, 2012, §2.3).*[18]

*Euler's constant is*

$$\gamma = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} \frac{1}{k} - \log n \right\}.$$

*It is named for Leonhard Euler, who first discussed it in 1734. The number $\gamma$ is an important constant that occurs frequently in mathematical formulas. The existence of the limit is not obvious. Our aim is to prove that the limit exists and to determine its approximate numerical value.*

*Consider the curve $y = 1/x$ for $1 \le x \le n$, where $n = 2, 3, \ldots$ The area under the curve is given by*

$$A_n = \int_1^n \frac{1}{x} dx = \log n.$$

*Now construct rectangular boxes of heights $1/k$ over the intervals $[k, k+1]$, as shown in the left panel of Figure 10.*

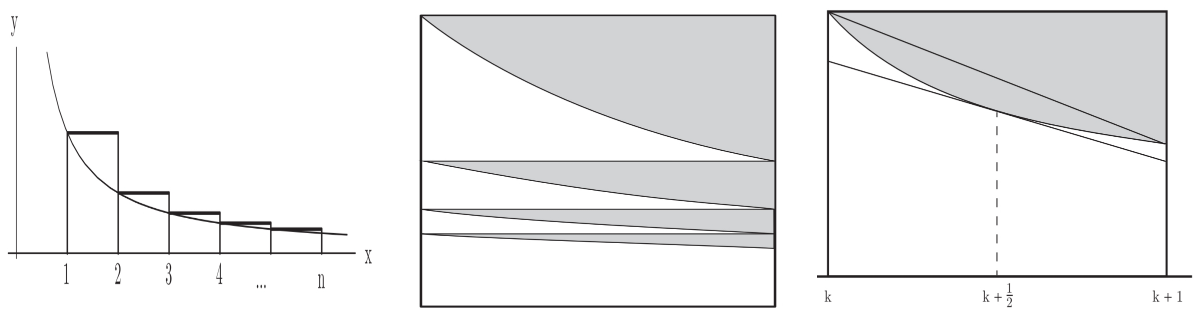



Figure 10: Left: The curve $y = 1/x$ and rectangular boxes. Middle: Geometric estimate of $S_{n-1} - A_n$. Right: Estimation of the area $\alpha_k$.

*Since*

$$\frac{1}{k+1} \le \frac{1}{x} \le \frac{1}{k} \quad \text{for} \quad k \le x \le k+1,$$

*it follows that*

$$\frac{1}{k+1} = \int_k^{k+1} \frac{1}{k+1} dx \le \int_k^{k+1} \frac{1}{x} dx \le \int_k^{k+1} \frac{1}{k} dx = \frac{1}{k}$$

*for $k = 1, 2, \ldots$. Adding these inequalities over $k = 1, 2, \ldots, n-1$, we have*

$$\sum_{k=1}^{n-1} \frac{1}{k+1} \le \int_1^n \frac{1}{x} dx \le \sum_{k=1}^{n-1} \frac{1}{k}.$$

*With the notation*

$$S_n = \sum_{k=1}^{n} \frac{1}{k}$$

*this says that $S_n - 1 \le A_n \le S_{n-1}$. The two inequalities can be rearranged to give*

$$0 \le S_{n-1} - A_n \le 1 - S_n + S_{n-1} = 1 - \frac{1}{n}.$$

[18]On the other hand, Duren's approach, and useful graphics, shown below, are basically the same as those in Mattuck (Introduction to Analysis, 1999, p. 96).

*This shows that the sequence $\{S_{n-1} - A_n\}$ is positive and is bounded above by 1.*

*Geometrically, the quantity $S_{n-1} - A_n$ is the sum of areas of those portions of the boxes that lie above the curve $y = 1/x$ from $x = 1$ to $n$. In order to estimate this total area, imagine that all of these boxes are slid to the left until they lie inside the first box, as shown in the middle panel of Figure 10, where the shaded regions have total area $S_{n-1} - A_n$. Since the regions are nonoverlapping and lie inside a square of area 1, this conceptual exercise gives a geometric interpretation of the inequality $S_{n-1} - A_n \leq 1$.*

*Next observe that*

$$A_{n+1} - A_n = \int_n^{n+1} \frac{1}{x} dx \leq \frac{1}{n} = S_n - S_{n-1},$$

*or $S_{n-1} - A_n \leq S_n - A_{n+1}$, which says that the sequence $\{S_{n-1} - A_n\}$ is nondecreasing. An appeal to the monotone boundedness theorem now shows that the sequence converges. Denoting its limit by $\gamma$, we have*

$$\gamma = \lim_{n\to\infty} (S_{n-1} - A_n) = \lim_{n\to\infty} \left( S_n - \frac{1}{n} - A_n \right) = \lim_{n\to\infty} (S_n - A_n) .$$

*This establishes the existence of Euler's constant, as well as that $0 \leq \gamma \leq 1$.*

*In fact, it is clear from the middle panel of Figure 10 that $\gamma$ is slightly larger than $1/2$. Our next task is to derive quantitative bounds on $\gamma$ by estimating the area*

$$\alpha_k = \frac{1}{k} - \int_k^{k+1} \frac{1}{x} dx$$

*of the region in the $k$ th box that lies above the curve $y = 1/x$. Since the curve is convex, that region contains a triangle of area*

$$\frac{1}{2} \left( \frac{1}{k} - \frac{1}{k+1} \right),$$

*and is contained in a trapezoid of area*

$$\frac{1}{k} - \frac{1}{k + \frac{1}{2}},$$

*constructed by drawing the tangent line to the curve at the point where $x = k + \frac{1}{2}$. (See the right panel of Figure 10, and note that the trapezoid above the tangent line is obtained by removing the lower trapezoid from the entire rectangle.)*

*Thus a comparison of areas shows that*

$$\frac{1}{2} \left( \frac{1}{k} - \frac{1}{k+1} \right) \leq \alpha_k \leq \frac{1}{k} - \frac{1}{k + \frac{1}{2}} = 2 \left( \frac{1}{2k} - \frac{1}{2k+1} \right).$$

*Summing these inequalities, we find*

$$\frac{1}{2} \sum_{k=1}^{n-1} \left( \frac{1}{k} - \frac{1}{k+1} \right) \leq \sum_{k=1}^{n-1} \alpha_k \leq 2 \sum_{k=1}^{n-1} \left( \frac{1}{2k} - \frac{1}{2k+1} \right),$$

*or*

$$\frac{1}{2} \left( 1 - \frac{1}{n} \right) \leq S_{n-1} - A_n \leq 2 \left( \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots - \frac{1}{2n-1} \right).$$

*Letting $n \to \infty$, we infer that*

$$0.5 \le \gamma \le 2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots\right) = 2(1 - \log 2) = 0.6137\dots,$$

*since, as shown below in (2.319),*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2.$$

*The actual value of Euler's constant is*

$$\gamma = 0.577215664901532\dots.$$

*It has been computed to thousands of decimal places and no periodicities have been detected, so it is strongly suspected to be an irrational number. In fact, it is generally conjectured to be a transcendental (or nonalgebraic) number, like the constants $\pi$ and $e$. However, no one has ever been able to prove that $\gamma$ is irrational!* ■

**Example 2.72** *To see that $\cos(z) = \sum_{k=0}^{\infty} (-1)^k z^{2k}/(2k)!$ converges, use the ratio test (2.250) to see that*

$$c = \lim_{k\to\infty} \left| \frac{(-1)^{k+1} \frac{z^{2(k+1)}}{(2(k+1))!}}{(-1)^k \frac{z^{2k}}{(2k)!}} \right| = \lim_{k\to\infty} \left| \frac{z^2}{(2k+1)(2k+2)} \right| = 0,$$

*for all $z \in \mathbb{R}$.* ■

**Example 2.73** *For $r \in \mathbb{N} \setminus \{1\} = \{2, 3, \dots\}$, let*

$$s(r) = \sum_{i=1}^{r-1} \frac{i}{(r-i)^2} \quad \text{and} \quad t(r) = \frac{s(r)}{r} = \sum_{i=1}^{r-1} \frac{i}{r(r-i)^2}.$$

*Next, define the two series*

$$S = \lim_{r\to\infty} s(r) = \lim_{r\to\infty} \sum_{i=1}^{r-1} \frac{i}{(r-i)^2} \quad \text{and} \quad T = \lim_{r\to\infty} t(r) = \lim_{r\to\infty} \sum_{i=1}^{r-1} \frac{i}{r(r-i)^2}.$$

*We will see that $S$ diverges to infinity, while $T$ is convergent, and we use the relationship between the two to arrive at an approximation for $S$ for large $r$.*

*With $j = r - i$ (so that, when $i = 1$, $j = r - 1$; and when $i = r - 1$, $j = 1$), we can write, assuming the two rhs limits exist,*

$$T = \lim_{r\to\infty} \sum_{j=1}^{r-1} \frac{r-j}{rj^2} = \lim_{r\to\infty} \left( \sum_{j=1}^{r-1} \frac{1}{j^2} - \sum_{j=1}^{r-1} \frac{1}{rj} \right) = \lim_{r\to\infty} \left( \sum_{j=1}^{r-1} \frac{1}{j^2} \right) - \lim_{r\to\infty} \sum_{j=1}^{r-1} \frac{1}{rj}.$$

*From (2.246), the first sum on the rhs converges to $\pi^2/6$. Using the comparison test (2.249), the second sum is bounded because, for $r \ge 2$,*

$$\sum_{j=1}^{r-1} \frac{1}{rj} < \sum_{j=1}^{r-1} \frac{1}{r} = \frac{r-1}{r} < 1.$$

*To see that it converges to zero, use the fact that* $(rj)^{-1}$ *is a positive, decreasing function in* $j$*, so that the conditions of the integral test* (2.254) *hold. Then (using* (2.255) *for the second inequality),*

$$0 \leq \sum_{j=2}^{r-1} \frac{1}{rj} = \frac{1}{r} \sum_{j=2}^{r-1} \frac{1}{j} \leq \frac{1}{r} \int_1^{r-1} \frac{1}{x} dx = \int_1^{r-1} \frac{1}{rx} dx,$$

*and, using* (2.197) *and l'Hôpital's rule,*

$$\lim_{r\to\infty} \int_1^{r-1} \frac{1}{rx} dx = \lim_{r\to\infty} \frac{\ln(r-1)}{r} = \lim_{r\to\infty} \frac{1}{r-1} = 0.$$

*The Squeeze Theorem* (2.9) *then shows that* $\lim_{r\to\infty} \sum_{j=1}^{r-1} (rj)^{-1} = 0$*, and, thus,* $T = \pi^2/6$*.*

*Now observe that, for all* $r \geq 2$*,*

$$1 = \frac{s(r)}{rt(r)} \quad \Rightarrow \quad 1 = \lim_{r\to\infty} \frac{s(r)}{rt(r)}.$$

*Clearly, the denominator* $rt(r)$ *diverges to infinity as* $r$ *increases, so that the rhs implies that* $s(r)$ *also diverges to infinity, i.e.,* $S = \infty$*. As both the numerator and denominator diverge to infinity in the limit, we cannot equate this with the ratio of limits. However, we can say that, as* $r$ *increases, the rhs approaches* $s(r)/[r\pi^2/6]$*, or, as* $r \to \infty$*,* $s(r)$ *approaches* $r\pi^2/6$*. That is, as* $r \to \infty$*,*

$$\sum_{i=1}^{r-1} \frac{i}{(r-i)^2} \quad \to \quad \frac{r\pi^2}{6}.$$

*This result is used in Paolella, Fundamental Probability, Example 6.8, for the calculation of the asymptotic variance of a particular random variable.* ■

### 2.6.4 Tannery's Theorem

Of great importance in analysis is knowing the conditions under which one can exchange limiting operations. For example, in §6.3, we consider the exchange of derivative and integral. Another case is the exchange of a limit and an infinite sum. We give here *Tannery's theorem (for series)*, after Jules Tannery (1848–1910), which is often of great use, as we will show with examples. It turns out to be a special case of the famous Lebesgue dominated convergence theorem (DCT), the latter requiring a study of measure theory and the Lebesgue integral to understand. In fact, Tannery's theorem for series is sometimes referred to as the discrete DCT. Fortunately, Tannery's theorem can be proven without invoking this machinery, and is in fact rather straightforward. The following presentation is taken from Loya (Amazing and Aesthetic Aspects of Analysis, 2018, §3.7.1).

Theorem (Tannery's theorem for series): For each natural number $n$, let $\sum_{k=1}^{m_n} a_k(n)$ be a finite sum such that $m_n \to \infty$ as $n \to \infty$. If for each $k$, $\lim_{n\to\infty} a_k(n)$ exists, and there is a convergent series $\sum_{k=1}^{\infty} M_k$ of nonnegative real numbers such that $|a_k(n)| \leq M_k$ for all $n \in \mathbb{N}$ and $1 \leq k \leq m_n$, then

$$\lim_{n\to\infty} \sum_{k=1}^{m_n} a_k(n) = \sum_{k=1}^{\infty} \lim_{n\to\infty} a_k(n). \tag{2.258}$$

In particular, both sides are well defined (the limits and sums converge) and are equal.

Proof: First of all, we remark that the series on the right converges. Indeed, if we put $a_k := \lim_{n\to\infty} a_k(n)$ (the limit exists by assumption), then taking $n \to \infty$ in the inequality $|a_k(n)| \le M_k$, we have $|a_k| \le M_k$ as well. Therefore, by the comparison test, $\sum_{k=1}^{\infty} |a_k|$ converges, and hence $\sum_{k=1}^{\infty} a_k$ converges as well.

Now to prove our theorem, let $\varepsilon > 0$ be given. It follows from Cauchy's criterion for series that there is an $\ell$ such that

$$M_{\ell+1} + M_{\ell+2} + \cdots < \frac{\varepsilon}{3}.$$

Since $m_n \to \infty$ as $n \to \infty$, we can choose $N_1$ such that for all $n > N_1$, we have $m_n > \ell$. Then using that $|a_k(n)| \le M_k$ and $|a_k| \le M_k$, observe that for every $n > N_1$,

$$\begin{aligned}\left|\sum_{k=1}^{m_n} a_k(n) - \sum_{k=1}^{\infty} a_k\right| &= \left|\sum_{k=1}^{\ell} (a_k(n) - a_k) + \sum_{k=\ell+1}^{m_n} a_k(n) - \sum_{k=\ell+1}^{\infty} a_k\right| \\ &\le \sum_{k=1}^{\ell} |a_k(n) - a_k| + \sum_{k=\ell+1}^{m_n} M_k + \sum_{k=\ell+1}^{\infty} M_k \\ &< \sum_{k=1}^{\ell} |a_k(n) - a_k| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \sum_{k=1}^{\ell} |a_k(n) - a_k| + \frac{2\varepsilon}{3}.\end{aligned}$$

Since for each $k, \lim_{n\to\infty} a_k(n) = a_k$, there is an $N_2$ such that for each $k = 1, 2, \ldots, \ell$ and for $n > N_2$, we have $|a_k(n) - a_k| < \varepsilon/(3\ell)$. Thus, if $n > \max\{N_1, N_2\}$, then

$$\left|\sum_{k=1}^{m_n} a_k(n) - \sum_{k=1}^{\infty} a_k\right| < \sum_{k=1}^{\ell} \frac{\varepsilon}{3\ell} + \frac{2\varepsilon}{3} = \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.$$

This completes the proof.

Our first example is a "non-example", showing that the necessity of having a convergent dominating series.

**Example 2.74** *(Loya, p. 218) For each $k, n \in \mathbb{N}$, let $a_k(n) := 1/n$ and let $m_n = n$. Then*

$$\lim_{n\to\infty} a_k(n) = \lim_{n\to\infty} \frac{1}{n} = 0 \Longrightarrow \sum_{k=1}^{\infty} \lim_{n\to\infty} a_k(n) = \sum_{k=1}^{\infty} 0 = 0.$$

*On the other hand,*

$$\sum_{k=1}^{m_n} a_k(n) = \sum_{k=1}^{n} \frac{1}{n} = \frac{1}{n} \cdot \sum_{k=1}^{n} 1 = 1 \Longrightarrow \lim_{n\to\infty} \sum_{k=1}^{m_n} a_k(n) = \lim_{n\to\infty} 1 = 1$$

*Thus, for this example,*

$$\lim_{n\to\infty} \sum_{k=1}^{m_n} a_k(n) \neq \sum_{k=1}^{\infty} \lim_{n\to\infty} a_k(n)$$

*It turns out there is no constant $M_k$ such that $|a_k(n)| \le M_k$ where the series $\sum_{k=1}^{\infty} M_k$ converges. Indeed, the inequality $|a_k(n)| = 1/n \le M_k$ for $1 \le k \le n$ implies (set $k = n$) that $1/k \le M_k$ for all $k$. Since $\sum_{k=1}^{\infty} 1/k$ diverges, the series $\sum_{k=1}^{\infty} M_k$ must also diverge.* ■

**Example 2.75** *(Wikipedia, Tannery's Theorem) We wish to prove that the limit of the binomial theorem (1.34), and the infinite series characterizations of the exponential $e^x$ (2.272), are equivalent. Note that*

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{n^k}.$$

*Define $a_k(n) = \binom{n}{k}\frac{x^k}{n^k}$. We have that $|a_k(n)| \le \frac{|x|^k}{k!}$ and that $\sum_{k=0}^{\infty}\frac{|x|^k}{k!} = e^{|x|} < \infty$, so Tannery's theorem can be applied and*

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\binom{n}{k}\frac{x^k}{n^k} = \sum_{k=0}^{\infty}\lim_{n\to\infty}\binom{n}{k}\frac{x^k}{n^k} = \sum_{k=0}^{\infty}\frac{x^k}{k!} = e^x. \qquad \blacksquare$$

Another example of the use of Tannery's theorem is given in Example 7.3 in the context of the digamma function.

### 2.6.5 Infinite Products

The tools for convergence of infinite series can be extended to convergence of *infinite products* by use of basic properties of the logarithm and continuity of the exponential function (see §2.3.4).

Definition: Let $\{z_n\}$ be a sequence of real or complex numbers, and let $\prod_{j=1}^{n} z_j = z_1 z_2 \cdots z_n$ for each $n \in \mathbb{N}$. The sequence of partial products $\{P_n\}$ is defined by $P_n = \prod_{j=1}^{n} z_j$ for each $n \in \mathbb{N}$. The limiting case, as $n \to \infty$, is called an *infinite product* and denoted $\prod_{j=1}^{\infty} z_j$.

Theorem: Let $a_k \ge 0$ such that $\lim_{k\to\infty} a_k = 0$. Then:

$$\textbf{If } S = \sum_{k=1}^{\infty} a_k \textbf{ converges, then so does } P = \prod_{k=1}^{\infty}(1+a_k). \tag{2.259}$$

Proof: Let $p_n = \prod_{k=1}^{n}(1+a_k)$. From (2.131), $\ln(1+a_k) \le a_k$, so

$$\ln p_n = \sum_{k=1}^{n}\ln(1+a_k) \le \sum_{k=1}^{n} a_k \le S,$$

so that, from the comparison test, $\ln P = \sum_{k=1}^{\infty}\ln(1+a_k)$ converges. Taking exponents gives the result.

As an example, the product $\prod_{j=1}^{\infty} 2^{-j}$ is such that, for all $j \ge 1$, $z_j = 2^{-j}$ cannot be expressed in the form $1+a_j$, for $a_j \ge 0$. Indeed,

$$P_n = \frac{1}{2}\cdot\frac{1}{2^2}\cdots\frac{1}{2^n} = 2^{-(1+2+\cdots+n)} = 2^{-\frac{n(n+1)}{2}},$$

and $P_n \to 0$ as $n \to \infty$. This product is said to *diverges to zero*; see, e.g., Little, Teo, van Brunt, An Introduction to Infinite Products, 2022, pp. 40-1. In this example, $a_j = 2^{-j} - 1 = -(2^j-1)/2^j < 0$ and $S = \sum_{j=1}^{\infty} a_j = -\frac{1}{2} - \frac{3}{4} - \frac{7}{8} - \cdots$ clearly diverges. As such, we might expect the next result, which generalizes (2.259).

Theorem: Let $\{a_n\}$ be a sequence of non-negative numbers. Let $S = \sum_{j=1}^{\infty} a_j$ and $P = \prod_{j=1}^{\infty}(1+a_j)$. Then

$$S \text{ and } P \text{ either both converge or both diverge.} \tag{2.260}$$

Proof: If $x \geq 0$, then, from (2.131) or (2.272), $1+x \leq e^x$. Thus, $\forall n \in \mathbb{N}$,

$$a_1 + a_2 + \cdots + a_n < (1+a_1)(1+a_2)\cdots(1+a_n) \leq \exp\{S_n\},$$

from (2.112), where $S_n = \sum_{j=1}^{n} a_j$. Thus, $S_n < P_n \leq e^{S_n}$, where $P_n = \prod_{j=1}^{n}(1+a_j)$. Note that both $\{S_n\}$ and $\{P_n\}$ are increasing sequences. Suppose $S_n \to S \in \mathbb{R}$. Then $S_n \leq S$ for all $n$. Therefore, from (2.110), $P_n \leq e^{S_n} \leq e^S$, and consequently $\{P_n\}$ is bounded above. Thus, from (2.4), $\{P_n\}$ converges to some limit $P$. Since $0 < P_1 \leq P_n \leq P$, we have $P \neq 0$ and therefore the product converges.

Suppose now that $\{P_n\}$ converges to a limit $P$. Then $P_n \leq P$ for all $n \in \mathbb{N}$. The sequence $\{S_n\}$ is increasing and bounded above by $P$; hence, $\{S_n\}$ converges and therefore the series converges.

Analogous to (2.237), we have:

Theorem (The $n$th Term Test for Products): Let $\{z_n\}$ be a sequence of numbers. If the product $\prod_{j=1}^{\infty}(1+z_j)$ converges, then $z_n \to 0$ as $n \to \infty$.

Theorem (2.260) gives rise to a very simple proof of the following result, which we stated in (2.241).

Corollary: The harmonic series $S = \sum_{k=1}^{\infty} k^{-1}$ diverges.

Proof: As in Little, Teo, van Brunt, p. 45, note that

$$\prod_{j=1}^{n}\left(1+\frac{1}{j}\right) = \prod_{j=1}^{n}\frac{j+1}{j} = n+1. \tag{2.261}$$

Thus the product $\prod_{j=1}^{\infty}(1+1/j)$ is divergent. It follows from (2.260) that the harmonic series is divergent.

Definition: A product $\prod_{j=1}^{\infty}(1+z_j)$ is called *absolutely convergent* if the product

$$\prod_{j=1}^{\infty}(1+|z_j|)$$

is convergent. The product is called *conditionally convergent* if $\prod_{j=1}^{\infty}(1+z_j)$ converges but the product (2.2.6) diverges.

Theorem: Assume $z_j \neq 0$ for all $j$. Let $P = \prod_{j=1}^{\infty} z_j$ and $S = \sum_{j=1}^{\infty} \log z_j$.

$$P \text{ converges if and only if } S \text{ converges.} \tag{2.262}$$

Moreover, if $S$ converges to $C$, then the product converges to $e^C$.

A proof can be found in, e.g., Little, Teo, van Brunt, p. 52.

Theorem (Cauchy's Test): Suppose $z_j \neq -1$ for all $j$ and $\sum_{j=1}^{\infty} |z_j|^2$ converges. Then $\sum_{j=1}^{\infty} z_j$ and $\prod_{j=1}^{\infty} (1+z_j)$ either both converge or both diverge.

A proof can be found in, e.g., Little, Teo, van Brunt, p. 61.

Theorem: Let $\{a_j\}$ be a real sequence such that $\sum_{j=1}^{\infty} a_j$ is convergent. If $\prod_{j=1}^{\infty} (1+a_j)$ converges, then so does $\sum_{j=1}^{\infty} a_j^2$.

> Proof: (Little, Teo, van Brunt, p. 62) Since $\sum_{j=1}^{\infty} a_j$ converges, we find that $\lim_{j\to\infty} a_j = 0$. Hence we may assume that $a_j > -1$ for all $j$, so that $1 + a_j > 0$. Moreover, L'Hôpital's rule shows that
>
> $$\begin{aligned}\lim_{j\to\infty} \frac{a_j - \log(1+a_j)}{a_j^2} &= \lim_{j\to\infty} \frac{1 - \frac{1}{1+a_j}}{2a_j} = \lim_{j\to\infty} \frac{a_j}{2a_j + 2a_j^2} \\ &= \lim_{j\to\infty} \frac{1}{2+4a_j} = \frac{1}{2}.\end{aligned}$$
>
> Therefore we can choose $N$ large enough so that
>
> $$a_j - \log(1+a_j) > \frac{a_j^2}{4}$$
>
> for all $j \geq N$. For each $n > N$ we then have
>
> $$\sum_{j=N}^{n} a_j - \frac{1}{4}\sum_{j=N}^{n} a_j^2 > \sum_{j=N}^{n} \log(1+a_j).$$
>
> As $\sum_{j=1}^{\infty} a_j$ converges, if $\sum_{j=1}^{\infty} a_j^2$ were divergent then $\sum_{j=1}^{\infty} \log(1+a_j)$ would also diverge. Theorem (2.262) then gives the contradiction that $\prod_{j=1}^{\infty} (1+a_j)$ would diverge. We conclude that $\sum_{j=1}^{\infty} a_j^2$ converges.

As an example, let $a_j = \frac{(-1)^j}{\sqrt{j}}$ for all $j$. Then $\sum_{j=1}^{\infty} a_j$ is a convergent alternating series but $\sum_{j=1}^{\infty} a_j^2$ is the harmonic series, which diverges. The previous theorem thus implies that $\prod_{j=1}^{\infty} (1+a_j)$ diverges.

In the next section, we detail the derivation, using "basic principles" of the famous Wallis' product, one expression of which is given below in (2.267) as

$$\lim_{n\to\infty} \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2n}{2n-1}\frac{2n}{2n+1} = \frac{\pi}{2}. \tag{2.263}$$

Here we show how this is simply obtained based on an infinite product expression of $\sin x$.

Theorem: For all $x \in \mathbb{R}$,

$$\sin x = x\prod_{j=1}^{\infty}\left(1 - \frac{x^2}{j^2\pi^2}\right), \tag{2.264}$$

and

$$\cos x = \prod_{j=1}^{\infty}\left(1 - \frac{4x^2}{(2j-1)^2\pi^2}\right). \tag{2.265}$$

The (long but basic) proof is detailed in Little, Teo, van Brunt, pp. 88-93, and taken from Venkatachaliengar (1962).[19]

Theorem (Wallis' product (2.263) derived from (2.264)):

> Proof: From (2.264),
>
> $$1 = \sin\frac{\pi}{2} = \frac{\pi}{2}\prod_{j=1}^{\infty}\left(1 - \frac{\left(\frac{\pi}{2}\right)^2}{j^2\pi^2}\right).$$
>
> Thus,
>
> $$\frac{2}{\pi} = \prod_{j=1}^{\infty}\left(1 - \frac{1}{(2j)^2}\right) = \prod_{j=1}^{\infty}\frac{(2j)^2 - 1}{(2j)^2} = \prod_{j=1}^{\infty}\frac{(2j-1)(2j+1)}{(2j)^2}.$$
>
> This equation implies
>
> $$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots,$$
>
> which is (2.263).

### 2.6.6 Wallis' Product and Stirling's Approximation

*Wallis' formula*, or *Wallis' product*, after John Wallis (1616–1703), is not only of interest in itself, but useful when deriving Stirling's approximation, both of which we detail in this subsection. As a bit of trivia, the sideways eight symbol, $\infty$, was introduced in 1655 by Wallis.

From (2.137), $\lim_{n\to\infty}(1 - t/n)^n = e^{-t}$. Following Keeping (1995, p. 392), for $x > 0$ and using $u = t/n$ and (1.64), $\Gamma(x)$ is given by

$$\lim_{n\to\infty}\int_0^n\left(1 - \frac{t}{n}\right)^n t^{x-1}dt = \lim_{n\to\infty} n^x\int_0^1 u^{x-1}(1-u)^n\ du = \lim_{n\to\infty} n^x\frac{\Gamma(x)\Gamma(n+1)}{\Gamma(x+n+1)}.$$

Dividing by $\Gamma(x)$ gives

$$1 = \lim_{n\to\infty}\frac{n^x\Gamma(n+1)}{\Gamma(x+n+1)}.$$

But with $x = 1/2$ and using (1.52) and (1.57),

$$\begin{aligned}
\frac{n^x\Gamma(n+1)}{\Gamma(x+n+1)} &= \frac{n^{1/2}n!}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{n^{1/2}n!}{\left(\frac{2n+1}{2}\right)\left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\cdots\frac{1}{2}\sqrt{\pi}} \\
&= \frac{n^{1/2}n!}{\frac{\left(\frac{2n+1}{2}\right)\left(\frac{2n}{2}\right)\left(\frac{2n-1}{2}\right)\left(\frac{2n-2}{2}\right)\left(\frac{2n-3}{2}\right)\cdots\frac{1}{2}\sqrt{\pi}}{\left(\frac{2n}{2}\right)\left(\frac{2n-2}{2}\right)\cdots 1}} = \frac{n^{1/2}n!}{\frac{(2n+1)!\sqrt{\pi}}{2^{2n+1}n!}} = \frac{2^{2n+1}n^{1/2}(n!)^2}{(2n+1)!\sqrt{\pi}},
\end{aligned}$$

or

$$\begin{aligned}
\sqrt{\pi} &= \lim_{n\to\infty}\frac{n^{1/2}2^{2n+1}(n!)^2}{(2n+1)!} = \lim_{n\to\infty}\frac{1}{n^{1/2}}\frac{2n}{(2n+1)}\frac{2^{2n}(n!)^2}{(2n)!} \\
&= \lim_{n\to\infty}\frac{1}{n^{1/2}}\frac{2^{2n}(n!)^2}{(2n)!} = \lim_{n\to\infty}\frac{1}{n^{1/2}}\frac{(2n)^2(2n-2)^2(2n-4)^2\cdots}{(2n)(2n-1)(2n-2)(2n-3)\cdots} \\
&= \lim_{n\to\infty}\frac{1}{n^{1/2}}\frac{(2n)(2n-2)(2n-4)\cdots 2}{(2n-1)(2n-3)\cdots 1},
\end{aligned}$$

[19] Venkatachaliengar, K., "Elementary proofs of the infinite product for sin z and allied formulae", Amer. Math. Monthly, vol. 69, pp. 541–5, 1962.

which is *Wallis' product*,

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n-1)}=\sqrt{\pi}. \tag{2.266}$$

Theorem: Wallis' product can also be expressed as

$$\lim_{n\to\infty}\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2n}{2n-1}\frac{2n}{2n+1}=\frac{\pi}{2}. \tag{2.267}$$

This is proven below. Before doing so, we first take care of two other things. To see that the product in (2.267) converges, use the infinite product result (2.259) with

$$\frac{2n}{2n-1}\frac{2n}{2n+1}=\frac{4n^2}{4n^2-1}=1+\frac{1}{4n^2-1}=:1+a_k.$$

To show that $\sum_{k=1}^{\infty}a_k$ converges, use the comparison test (2.249) with $\zeta(2)$.

Next consider how to obtain (2.266) from (2.267). As in Loya, express the latter as

$$\frac{\pi}{2}=\lim_{n\to\infty}\left\{\left(\frac{2}{1}\right)^2\cdot\left(\frac{4}{3}\right)^2\cdots\left(\frac{2n}{2n-1}\right)^2\cdot\frac{1}{2n+1}\right\}.$$

Then taking square roots,

$$\sqrt{\pi}=\lim_{n\to\infty}\sqrt{\frac{2}{2n+1}}\prod_{k=1}^{n}\frac{2k}{2k-1}=\lim_{n\to\infty}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{1+1/2n}}\prod_{k=1}^{n}\frac{2k}{2k-1}.$$

Using that $1/\sqrt{1+1/2n}\to 1$ as $n\to\infty$, we obtain (2.266).

There are several ways of proving (2.267); see, e.g., Keeping (1995, p. 392), Andrews, Askey and Roy (1999, p. 46), and the (charming and excellent) Loya (Amazing and Aesthetic Aspects of Analysis, 2018, §5.1.3). We present the approach as in Hijab (Introduction to Calculus and Classical Analysis, 4th ed, 2016, p 204-5).

> Proof: Begin with integrating by parts to obtain
>
> $$\int\sin^n x\,dx=-\frac{1}{n}\sin^{n-1}x\cos x+\frac{n-1}{n}\int\sin^{n-2}x\,dx,\quad n\geq 2.$$
>
> Evaluating at 0 and $\pi/2$ yields
>
> $$\int_0^{\pi/2}\sin^n x\,dx=\frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx,\quad n\geq 2.$$
>
> Since $\int_0^{\pi/2}\sin^0 x\,dx=\pi/2$ and $\int_0^{\pi/2}\sin^1 x\,dx=1$, by the last equation and induction,
>
> $$I_{2n}=\int_0^{\pi/2}\sin^{2n}x\,dx=\frac{(2n-1)\cdot(2n-3)\cdot\cdots\cdot 1}{2n\cdot(2n-2)\cdot\cdots\cdot 2}\cdot\frac{\pi}{2},$$
>
> and
>
> $$I_{2n+1}=\int_0^{\pi/2}\sin^{2n+1}x\,dx=\frac{2n\cdot(2n-2)\cdots\cdot 2}{(2n+1)\cdot(2n-1)\cdots\cdot 3}\cdot 1,$$

for $n \geq 1$. Since $0 < \sin x < 1$ on $(0, \pi/2)$, the integrals $I_n$ are decreasing in $n$. But, by the formula for $I_n$ with $n$ odd,

$$1 \leq \frac{I_{2n-1}}{I_{2n+1}} \leq 1 + \frac{1}{2n}, \quad n \geq 1.$$

Thus

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{I_{2n-1}}{I_{2n+1}} \leq 1 + \frac{1}{2n}, \quad n \geq 1,$$

or $I_{2n}/I_{2n+1} \to 1$, as $n \to \infty$. Since

$$\frac{I_{2n}}{I_{2n+1}} = \frac{(2n+1) \cdot (2n-1) \cdot (2n-1) \cdots 3 \cdot 3 \cdot 1}{2n \cdot 2n \cdot (2n-2) \cdots 4 \cdot 2 \cdot 2} \cdot \frac{\pi}{2},$$

we obtain (2.267).

As mentioned, the Wallis product is important for deriving *Stirling's approximation* to $n!$,

$$\Gamma(n) \approx \sqrt{2\pi} n^{n-1/2} \exp(-n). \tag{2.268}$$

The derivation of this famous result is often given in books on real analysis; e.g., Lang (1997, p. 120), Andrews, Askey and Roy (1999, §1.4), Kuttler (2021, Calculus of One and Many Variables, §10.1), and Duren (2012, §2.6). Our presentation is taken from Duren.

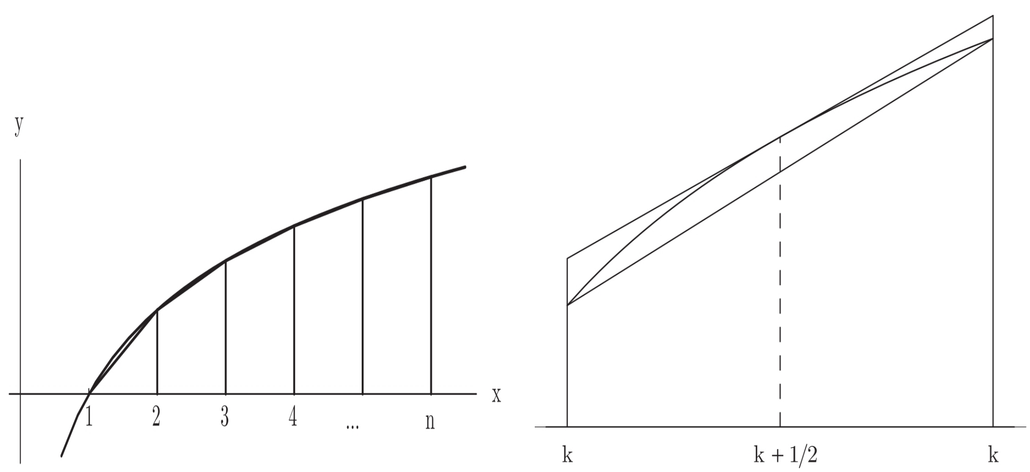



Figure 11: Left: The logarithmic curve and inscribed trapezoids. Right: Estimation of the area $\alpha_k$.

The asymptotic formula

$$n! \sim n^n e^{-n} \sqrt{2\pi n}, \quad n \to \infty,$$

is known as Stirling's formula.[20] It is of basic importance for instance in probability theory and combinatorics, because it gives precise information about the growth of the factorial function. The symbol "$\sim$" means that

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1.$$

We propose to prove Stirling's formula by showing that

$$1 < \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} < 1 + \frac{1}{4n}, \quad n = 1, 2, \ldots .$$

The error estimates are important in quantitative applications of the formula.

With the observation that

$$\log(n!) = \log 1 + \log 2 + \cdots + \log n$$

it is natural to base a proof on a careful study of the area under the logarithmic curve $y = \log x$ from $x = 1$ to $n$. (See the left panel of Figure 11.) An integration by parts calculates this area as

$$A_n = \int_1^n \log x dx = [x \log x - x]_1^n = n \log n - n + 1.$$

On the other hand, the area can be estimated geometrically. Since the logarithm is a concave function, the curve $y = \log x$ lies above each of its chords connecting successive points $(k, \log k)$, for $k = 1, 2, \ldots, n$. Thus $A_n$ is larger than the sum of areas of the trapezoids under those line segments. The total area of the trapezoids is

$$\begin{aligned} T_n &= \frac{1}{2}\log 2 + \frac{1}{2}(\log 2 + \log 3) + \cdots + \frac{1}{2}(\log(n-1) + \log n) \\ &= \log 2 + \log 3 + \cdots + \log(n-1) + \frac{1}{2}\log n \\ &= \log(n!) - \frac{1}{2}\log n. \end{aligned}$$

Now let $\alpha_k$ denote the area of the small region bounded by the curve $y = \log x$ and the line segment joining the two points $(k, \log k)$ and $(k+1, \log(k+1))$, for $k = 1, 2, \ldots, n-1$. Then the total area under the curve is

$$A_n = T_n + E_n, \qquad \text{where } E_n = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}.$$

Inserting the expressions for $A_n$ and $T_n$, we can write this relation in the form

$$\log(n!) = \left(n + \frac{1}{2}\right)\log n - n + 1 - E_n,$$

or

$$n! = C_n n^{n+\frac{1}{2}} e^{-n}, \qquad \text{where } C_n = e^{1-E_n}.$$

The sequence $\{E_n\}$ is increasing, since each term $\alpha_k$ is positive. We now show that the sequence $\{E_n\}$ has an upper bound and is therefore convergent. In order to estimate $\alpha_k$, we construct the tangent line to the curve $y = \log x$ at the point where $x = k + \frac{1}{2}$ (see the right panel of Figure 11) and compare areas:

$$\begin{aligned} \alpha_k &< \log\left(k + \frac{1}{2}\right) - \frac{1}{2}(\log k + \log(k+1)) \\ &= \frac{1}{2}\log\left(\frac{k+\frac{1}{2}}{k}\right) - \frac{1}{2}\log\left(\frac{k+1}{k+\frac{1}{2}}\right) \\ &= \frac{1}{2}\log\left(1 + \frac{1}{2k}\right) - \frac{1}{2}\log\left(1 + \frac{1}{2k+1}\right) \\ &< \frac{1}{2}\log\left(1 + \frac{1}{2k}\right) - \frac{1}{2}\log\left(1 + \frac{1}{2(k+1)}\right). \end{aligned}$$

Adding these inequalities, we find that

$$E_n = \sum_{k=1}^{n-1} \alpha_k < \sum_{k=1}^{n-1} \left\{ \frac{1}{2} \log\left(1 + \frac{1}{2k}\right) - \frac{1}{2} \log\left(1 + \frac{1}{2(k+1)}\right) \right\}$$
$$= \frac{1}{2} \log \frac{3}{2} - \frac{1}{2} \log\left(1 + \frac{1}{2n}\right) < \frac{1}{2} \log \frac{3}{2},$$

since the dominant series telescopes. Thus $E_n$ increases to a finite limit $E = \sum_{k=1}^{\infty} \alpha_k$, and so $C_n = e^{1-E_n}$ decreases to a limit $C = e^{1-E} > 0$. In particular, $C_n > C$, and so $1 < C_n/C = e^{E-E_n}$. But

$$E - E_n = \sum_{k=n}^{\infty} \alpha_k < \sum_{k=n}^{\infty} \left\{ \frac{1}{2} \log\left(1 + \frac{1}{2k}\right) - \frac{1}{2} \log\left(1 + \frac{1}{2(k+1)}\right) \right\}$$
$$= \frac{1}{2} \log\left(1 + \frac{1}{2n}\right),$$

again because the dominant series telescopes. Therefore,

$$1 < C_n/C = e^{E-E_n} < \sqrt{1 + \frac{1}{2n}} < 1 + \frac{1}{4n}.$$

In summary, we have shown that

$$0 < C < C_n = \frac{n!}{n^{n+\frac{1}{2}} e^{-n}} < C\left(1 + \frac{1}{4n}\right).$$

In order to finish the proof of Stirling's formula with error estimates, it now remains only to show that $C = \sqrt{2\pi}$. This is where we invoke the Wallis product formula. It gives

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{2^{2n} (n!)^2}{(2n)! \sqrt{n}} = \lim_{n\to\infty} \frac{2^{2n} \left(C_n n^{n+\frac{1}{2}} e^{-n}\right)^2}{\left(C_{2n} (2n)^{2n+\frac{1}{2}} e^{-2n}\right) \sqrt{n}}$$
$$= \frac{1}{\sqrt{2}} \lim_{n\to\infty} \frac{C_n^2}{C_{2n}} = \frac{C^2}{\sqrt{2} C} = \frac{C}{\sqrt{2}}.$$

Thus $C = \sqrt{2\pi}$ and the proof is complete.

**Example 2.76** *We now provide a vastly faster and easier, albeit heuristic, derivation of Stirling's approximation using probability theory. Let* $S_n \sim \mathrm{Gam}(n, 1)$ *for* $n \in \mathbb{N}$*, so that, for large* $n, S_n \overset{app}{\sim} \mathrm{N}(n, n)$*. The definition of convergence in distribution, and the continuity of the c.d.f. of* $S_n$ *and that of its limiting distribution, informally suggest the limiting behavior of the p.d.f. of* $S_n$*, i.e.,*

$$f_{S_n}(s) = \frac{1}{\Gamma(n)} s^{n-1} \exp(-s) \approx \frac{1}{\sqrt{2\pi n}} \exp\left(-\frac{(s-n)^2}{2n^2}\right).$$

*Choosing* $s = n$ *leads to* $\Gamma(n+1) = n! \approx \sqrt{2\pi}(n+1)^{n+1/2} \exp(-n-1)$*. From* (2.137),

*$\lim_{n\to\infty}(1+\lambda/n)^n = e^\lambda$, so*

$$(n+1)^{n+1/2} = n^{n+1/2}\left(1+\frac{1}{n}\right)^{n+1/2} \approx n^{n+1/2}e,$$

*and substituting this into the previous expression for $n!$ yields Stirling's approximation $n! \approx \sqrt{2\pi}n^{n+1/2}e^{-n}$.*

*As an aside, Stirling's approximation also drops out of an application of a saddlepoint approximation; see Paolella, Intermediate Probability.* ■

### 2.6.7 Cauchy Product

> Cauchy is mad and there is nothing that can be done about him, although, right now, he is the only one who knows how mathematics should be done.
>
> (Niels Abel)

Consider the product of the two series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$. Multiplying their values out in tabular form

$$\begin{array}{c|cccc} & b_0 & b_1 & b_2 & \cdots \\ \hline a_0 & a_0b_0 & a_0b_1 & a_0b_2 & \cdots \\ a_1 & a_1b_0 & a_1b_1 & a_1b_2 & \\ a_2 & a_2b_0 & a_2b_1 & a_2b_2 & \\ \vdots & \vdots & & & \ddots \end{array}$$

and summing the off–diagonals suggests that the product is given by

$$a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots .$$

Definition: The *Cauchy product* of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is $\sum_{n=0}^{\infty} c_n$, where

$$c_n = a_0b_n + a_1b_{n-1} + \cdots + a_{n-1}b_1 + a_nb_0.$$

Thus, $c_n$ is the sum of all products $a_ib_j$, where $i \geq 0, j \geq 0$, and $i + j = n$; thus,

$$c_n = \sum_{r=0}^{n} a_r b_{n-r} = \sum_{r=0}^{n} b_r a_{n-r}. \tag{2.269}$$

Theorem: If $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ are absolutely convergent series with sums $A$ and $B$, respectively, then their Cauchy product

$$\sum_{k=0}^{\infty} c_k, \qquad c_k = a_0b_k + a_1b_{k-1} + \cdots + a_kb_0, \tag{2.270}$$

is absolutely convergent with sum $AB$.

For proof, see, e.g., Trench, Introduction to Real Analysis, 2013, p. 225.

**Example 2.77** *Let $S = \sum_{k=0}^{\infty} a^k = (1-a)^{-1}$ for $a \in [0,1)$. As this is absolutely convergent, (2.270) with $a_k = b_k = a^k$ implies that $c_k = a^0a^k + a^1a^{k-1} + \cdots + a^ka^0 = (k+1)\,a^k$ and $(1-a)^{-2} = S^2 = \sum_{k=0}^{\infty} c_k = 1 + 2a + 3a^2 + \cdots$.* ■

The Cauchy product result can be generalized. Let $x_{nm} := x(n,m)$ be a function of $n,m \in \mathbb{N}$ with $x_{nm} \in \mathbb{R}_{\geq 0}$. Then

$$\lim_{N\to\infty}\sum_{n=0}^{N}\sum_{m=0}^{N} x_{nm} = \lim_{N\to\infty}\sum_{m=0}^{N}\sum_{n=0}^{N} x_{nm} = \sum_{s=0}^{\infty}\sum_{\substack{m\geq 0, n\geq 0\\ m+n=s}} x_{nm}, \tag{2.271}$$

if the unordered sum converges or, equivalently, if the terms are absolutely summable (see e.g., Beardon, 1997, §5.5; Browder, 1996, §2.5). As before, the values to be summed can be shown in a table as

$$\begin{array}{cc|cccc} & & & n & & \\ & & 0 & 1 & 2 & \cdots \\ \hline & 0 & x_{00} & x_{01} & x_{02} & \cdots \\ m & 1 & x_{10} & x_{11} & x_{12} & \\ & 2 & x_{20} & x_{21} & x_{22} & \\ & \vdots & \vdots & & & \ddots \end{array}$$

and, if the double sum is absolutely convergent, then the elements can be summed in any order, i.e., by columns, by rows, or by summing the off-diagonals.

**Example 2.78** *It is instructive to show some of the properties of the exponential starting from its power series expression*

$$f(x) = \exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \tag{2.272}$$

*Most students are familiar with this from their basic calculus class, and we will justify this rigorously below in §2.6.11. In particular, it is easily determined from (2.323) with $c = 0$. Clearly, $f(0) = 1$, and observe that, for all $x \in \mathbb{R}$,*

$$\lim_{k\to\infty}\left|\frac{x^{k+1}/(k+1)!}{x^k/k!}\right| = \lim_{k\to\infty}\left|\frac{x}{k+1}\right| = 0,$$

*so that the ratio test (2.250) shows absolute convergence of the series in (2.272). Differentiating (2.272) termwise*[21] *shows that $f(x) = f'(x)$. Thus, from (2.109) and (2.111), with $s_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, $s_n(x) \to \exp(x)$ for all $x \in \mathbb{R}$.*

*Next, to show $f(x+y) = f(x)f(y)$ from (2.272), use the binomial theorem (1.34) to get*

$$\begin{aligned} \exp(x+y) &= \sum_{s=0}^{\infty}\frac{(x+y)^s}{s!} = \sum_{s=0}^{\infty}\frac{1}{s!}\sum_{n=0}^{s}\binom{s}{n}x^n y^{s-n} \\ &= \sum_{s=0}^{\infty}\frac{1}{s!}\sum_{n=0}^{s}\frac{s!}{n!(s-n)!}x^n y^{s-n} = \sum_{s=0}^{\infty}\sum_{n=0}^{s}\frac{x^n}{n!}\frac{y^{s-n}}{(s-n)!} \\ &= \sum_{s=0}^{\infty}\sum_{\substack{m\geq 0, n\geq 0\\ m+n=s}}\frac{x^n}{n!}\frac{y^m}{m!}. \end{aligned}$$

*It follows from (2.271) that*

$$\exp(x+y) = \lim_{N\to\infty}\sum_{n=0}^{N}\sum_{m=0}^{N}\frac{x^n}{n!}\frac{y^m}{m!} = \lim_{N\to\infty}\sum_{n=0}^{N}\frac{x^n}{n!}\sum_{m=0}^{N}\frac{y^m}{m!} = \exp(x)\exp(y). \tag{2.273}$$

[21]See Example 2.81 and results (2.311) and (2.320) for the justification of termwise differentiation.

*With $x = y$, (2.273) obviously implies $\exp(2x) = [\exp(x)]^2$. That $[\exp(x)]^n = \exp(nx)$ follows from having confirmed the $n = 1$ and $n = 2$ cases, and use of induction: Assuming the result for $n$, $[\exp(x)]^{n+1} = [\exp(x)]^n \exp(x) = \exp(nx)\exp(x)$, and (2.273) implies $[\exp(x)]^{n+1} = \exp((n+1)x)$. The start of a direct proof of this, using the multinomial theorem, was given in Example 1.16.* ■

Theorem: If $\sum_\infty |a_n| < \infty$ and $\sum b_n$ converges (perhaps conditionally), with $\sum_{n=0}^\infty a_n = A$ and $\sum_{n=0}^\infty b_n = B$, then the Cauchy product

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)$$

converges to $AB$. For proof, see, e.g., Trench, 2013, p. 233.

**Example 2.79** *If*

$$a_n = b_n = \frac{(-1)^{n+1}}{\sqrt{n+1}},$$

*then $\sum_{n=0}^\infty a_n$ and $\sum_{n=0}^\infty b_n$ converge conditionally. From (2.269), the general term of their Cauchy product is*

$$c_n = \sum_{r=0}^{n} \frac{(-1)^{r+1}(-1)^{n-r+1}}{\sqrt{r+1}\sqrt{n-r+1}} = (-1)^n \sum_{r=0}^{n} \frac{1}{\sqrt{r+1}} \frac{1}{\sqrt{n-r+1}},$$

*so*

$$|c_n| \geq \sum_{r=0}^{n} \frac{1}{\sqrt{n+1}} \frac{1}{\sqrt{n+1}} = \frac{n+1}{n+1} = 1$$

*Therefore, the Cauchy product diverges, from (the contrapositive of) (2.237).* ■

### 2.6.8 Sequences of Functions

> It is true that a mathematician who is not also something of a poet will never be a perfect mathematician. (Karl Weierstrass)

We first encountered a sequence of (real) functions $\{f_k\}$ with common domain, and its associated series and partial sums, in definition (2.239). Also there, we stated and proved some basic results, such as (2.240): If series $S = \sum_{k=1}^\infty f_k$ converges, then $\lim_{n\to\infty} f_n = 0$. Recall also the Cauchy criterion for sequences of functions (2.242): $\sum_{k=1}^\infty f_k$ converges $\Leftrightarrow$ $\exists N \in \mathbb{N}$ such that, $\forall n, m \geq N$, $\left|\sum_{k=n+1}^m f_k\right| < \epsilon$.

With the exception of defining the (pointwise) convergence of series, this section is on convergence of sequences. We begin here with the concept of pointwise convergence.

Definition: Let $\{f_n\}$ be a sequence of functions with the same domain $D$. The function $f$ is the *pointwise limit* of sequence $\{f_n\}$, or $\{f_n\}$ *converges pointwise* to $f$, if,

$$\forall x \in D, \quad \lim_{n\to\infty} f_n(x) = f(x). \tag{2.274}$$

This is denoted $f_n \to f$. That is, $\forall x \in D$ and for every given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that, $\forall n > N$, $|f_n(x) - f(x)| < \epsilon$. It is helpful to read $\forall x$ in (2.274) as “for each”, and not “for all”, to emphasize that $N$ depends on both $x$ and $\epsilon$.

Definition: Let $\{f_n\}$ be a sequence of functions with the same domain $D$, with associated series $\sum_{n=1}^{\infty} f_n$. If, for each $x \in D$, the sequence of partial sums $s_k(x) = \sum_{n=1}^{k} f_n(x)$ converges pointwise to a value $S(x)$, then the series is said to *converge pointwise (on $D$) to (the function)* $S = \sum_{n=1}^{\infty} f_n$. We write $\sum_{n=1}^{\infty} f_n \to S$ as shorthand for: $s_k \to S$, where $S = \sum_{n=1}^{\infty} f_n$.

**Example 2.80** *(Stoll, 2001, p. 320) Let $f_k(x) = x^2 (1+x^2)^{-k}$, for $x \in \mathbb{R}$ and $k = 0, 1, \ldots$, and observe that $f_k(x)$ is continuous. Then, from (2.244),*

$$S(x) := \sum_{k=0}^{\infty} f_k(x) = x^2 \sum_{k=0}^{\infty} \frac{1}{(1+x^2)^k} = 1 + x^2, \quad x \neq 0,$$

*and $S(0) = 0$. Thus, $S(x)$ converges pointwise on $\mathbb{R}$ to the function $f(x) = (1+x^2)\mathbb{I}(x \neq 0)$, and is not continuous at zero.* ■

The above example shows that the pointwise limit may not be continuous even if each element in the sequence is continuous. Similarly, differentiability or integrability of $f_n$ does not ensure that the pointwise limit shares that property. A standard example for the latter is

$$f_n(x) = nx(1-x^2)^n, \quad x \in [0,1], \tag{2.275}$$

with $\lim_{n\to\infty} f_n(x) = 0$ for $x \in (0,1)$ and $f_n(0) = f_n(1) = 0$, so that the pointwise limit is $f(x) = 0$, but $\int_0^1 f_n \neq \int_0^1 f$. We address this topic in §2.6.10.

Definition: Let $\{f_n(x)\}$ be a sequence of functions with the same domain $D$. The function $f$ is the *uniform limit* on $D$ of the sequence $\{f_n\}$, (or $\{f_n\}$ is *uniformly convergent* to $f$), if, for every given $\epsilon > 0$,

$$\exists N \in \mathbb{N} \text{ such that, } \forall n > N, \forall x \in D, \quad |f_n(x) - f(x)| < \epsilon. \tag{2.276}$$

We will write either $f_n \to f$ uniformly, or $f_n \rightrightarrows f$. The difference between pointwise and uniform convergence parallels that between continuity and uniform continuity; in the uniform limit, $N$ is a function of $\epsilon$, but not of $x$. It should be clear from the definitions that, for $\{f_n\}$ a sequence of functions with common domain, $f_n \rightrightarrows f \implies f_n \to f$.

Definition: Let $\{f_n\}$ be a sequence of functions with common domain $D$. Sequence $\{f_n\}$ is said to be *uniformly Cauchy* if, for every given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\forall n, m > N, \ \forall x \in D, \ |f_n(x) - f_m(x)| < \epsilon. \tag{2.277}$$

Theorem: Let $\{f_n\}$ be a sequence of functions with common domain $D$.

$$\{f_n\} \textbf{ is uniformly convergent} \iff \{f_n\} \textbf{ is uniformly Cauchy.} \tag{2.278}$$

Proof: Suppose $f_n \to f$ uniformly on $D$. Let $\epsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \text{ and } x \in D \implies |f_n(x) - f(x)| < \frac{\epsilon}{2}.$$

Hence, for all $m, n \geq n_0$ and all $x \in D$, we obtain

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Conversely, suppose $\{f_n\}$ is uniformly Cauchy on $D$. Then $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ for each $x \in D$. Thus, from (2.17), for $x \in D$, $\{f_n(x)\}$ converges to a real number, which we denote by $f(x)$. Let $\epsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that

$$m, n \geq n_0 \text{ and } x \in D \implies |f_m(x) - f_n(x)| < \epsilon.$$

Fix $n \geq n_0$ and let $m \to \infty$. Thus we obtain $|f(x) - f_n(x)| \leq \epsilon$ for all $x \in D$. This implies that $f_n \to f$ uniformly on $D$.

Theorem: Let $\{f_n\}$ be a sequence of functions with common domain $D$. If $f, f_n \in \mathcal{C}^0\,[a,b]$, $\forall n \in \mathbb{N}$, where $[a,b] \subset D$. Let

$$m_n = \max_{x \in [a,b]} |f_n(x) - f(x)|,$$

which is well-defined from (2.59). Then

$$f_n(x) \rightrightarrows f(x) \textbf{ on } [a,b] \iff \lim_{n \to \infty} m_n = 0. \tag{2.279}$$

Proof: First assume $f_n(x) \rightrightarrows f(x)$ so that, by definition, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that, $\forall x \in [a,b]$ and $\forall n > N$, $D_n(x) := |f_n(x) - f(x)| < \epsilon$. By assumption, $f_n$ and $f$ are continuous on $[a,b]$, so that $D_n(x)$ is as well, from (2.38), (2.39), and (2.45). From (2.59), $\exists x_n \in [a,b]$ such that $x_n = \arg\max_x D_n(x)$. Thus, $\forall n > N$, $m_n = D_n(x_n) = \max_x D_n(x) < \epsilon$, i.e., $\lim_{n\to\infty} m_n = 0$.

Now assume $m_n = \max_x D_n(x) \to 0$ as $n \to \infty$. Then, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that, $\forall n > N$, $m_n < \epsilon$, so that, $\forall x \in [a,b]$ and $n > N$, $D_n(x) = |f_n(x) - f(x)| \leq m_n < \epsilon$.

In the above theorem, $\max_x |f_n(x) - f(x)|$ can be generalized to use of the supremum, notably if we do not assume continuity and a closed, bounded interval. This gives rise to an important norm for functions, and, when using the difference of two functions, a distance measure. We define this now, and present a result that does not involve continuity.

Definition: The *uniform norm* for a function $f : D \to \mathbb{R}$ is given by $\|f\|_{\mathrm{u}} = \sup_{t \in D} |f(t)|$. It is also known as the sup norm, supremum norm, Chebyshev norm, infinity norm, or, when the supremum is in fact the maximum, the max norm.

Theorem: Let $\{f_n\}$ be a sequence of functions on domain $D$. Then $f_n \to f$ uniformly as in definition (2.276) if and only if

$$\lim_{n\to\infty} \|f - f_n\|_{\mathrm{u}} = \lim_{n\to\infty} \sup_{t \in D} |f(t) - f_n(t)| = 0. \tag{2.280}$$

This means that, for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that, $\forall n \geq N$, $\|f - f_n\|_{\mathrm{u}} < \epsilon$.

Proof:
($\Rightarrow$) Assume (2.276). Fix any $\epsilon > 0$. Then $\epsilon/2 > 0$, so $\exists N \in \mathbb{N}$ such that

$$n \geq N \implies |f(x) - f_n(x)| < \frac{\epsilon}{2} \text{ for every } x \in D.$$

Consequently, if $n \geq N$, then

$$\|f - f_n\|_{\mathrm{u}} = \sup_{t \in D} |f(t) - f_n(t)| \leq \frac{\epsilon}{2} < \epsilon.$$

($\Leftarrow$) Assume (2.280). Fix $\epsilon > 0$. Then $\exists N \in \mathbb{N}$ such that, for all $n \geq N$, $\|f - f_n\|_{\mathrm{u}} < \epsilon$. Consequently, if $n \geq N$, then, for every $x \in D$, we have

$$|f(x) - f_n(x)| \leq \sup_{t \in D} |f(t) - f_n(t)| = \|f - f_n\|_{\mathrm{u}} < \epsilon.$$

Theorem: Let $\{f_n\}$ be a sequence of functions on domain $D$.

$$\textbf{If } f_n \in \mathcal{C}^0(D)\textbf{, } \forall n \in \mathbb{N}\textbf{, and } f_n \rightrightarrows f\textbf{, then } f \in \mathcal{C}^0(D). \tag{2.281}$$

We state and prove this in more general metric space terms. Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces, and assume that $\{f_n\}$ is a sequence of continuous functions $f_n : X \to Y$ converging uniformly to a function $f$. Then $f$ is continuous.

Proof: As in Fitzpatrick, Thm 9.31; and Lindstrøm, Prop 4.2.4. Let $a \in X$. Given an $\epsilon > 0$, we must find a $\delta > 0$ such that $d_Y(f(x), f(a)) < \epsilon$ whenever $d_X(x, a) < \delta$. Since $\{f_n\}$ converges uniformly to $f$, there is an $N \in \mathbb{N}$ such that, when $n \geq N$, $d_Y(f(x), f_n(x)) < \frac{\epsilon}{3}$ for all $x \in X$. Since $f_N$ is continuous at $a$, there is a $\delta > 0$ such that $d_Y(f_N(x), f_N(a)) < \frac{\epsilon}{3}$ whenever $d_X(x, a) < \delta$. If $d_X(x, a) < \delta$, the triangle inequality implies

$$\begin{aligned} d_Y(f(x), f(a)) &\leq d_Y(f(x), f_N(x)) + d_Y(f_N(x), f_N(a)) + d_Y(f_N(a), f(a)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

and hence $f$ is continuous at $a$.

Stoll, Thm 8.3.1 and Coro 8.3.2(a) proves this result with the sequential definition of continuity.

Definition: Let $\{f_n\}$ be a sequence of functions on domain $D$. The sequence $\{f_n\}$ is said to be *monotone decreasing (increasing)* if, $\forall n \in \mathbb{N}$ and $\forall x \in D$, $f_{n+1}(x) \leq f_n(x)$ ($f_{n+1}(x) \geq f_n(x)$), and *monotone* if it is either monotone decreasing or monotone increasing.

The monotonicity of $\{f_n\}$ is key for pointwise convergence to imply uniform convergence:

Theorem (Dini's Theorem): If (i) the $f_n : D \to \mathbb{R}$ are continuous, (ii) the $f_n$ are monotone, (iii) $D$ is a closed, bounded interval, and (iv) $f_n \to f$ to $f \in \mathcal{C}^0$, then

$$f_n \rightrightarrows f \textbf{ on } D. \tag{2.282}$$

For proof, see, e.g., Browder (1996, p. 64), Stoll (2021, p. 355), or Ghorpade and Limaye (2018, p. 432).

**Example 2.81** *Let $f_n(x) = x^n/n!$, $n = 0, 1, \ldots$ and $s_n(x) = \sum_{k=0}^{n} f_k(x)$. Then (i) $f_n(x), s_n(x) \in \mathcal{C}^0$ for $x \in \mathbb{R}$ and all $n \in \mathbb{N}$, (ii) for each $x \geq 0$, $s_n(x)$ is monotone increasing in $n$, (iii) $\forall r \in \mathbb{R}_{>0}$, $D = [0, r]$ is a closed, bounded interval, and (iv) from Example 2.78, $s_n \to \exp(x)$ for all $x \in \mathbb{R}$. Thus, from Dini's theorem (2.282), $\forall r \in \mathbb{R}_{>0}$ and $\forall x \in [0, r]$, $\lim_{n\to\infty} s_n(x) = \sum_{k=0}^{\infty} x^k/k! \rightrightarrows \exp(x)$.* ■

**Example 2.82** *Let $E := [-1, 1]$ and $f_n(x) := \sqrt{x^2 + (1/n^2)}$ for $n \in \mathbb{N}$ and $x \in E$. Then $f_n \to f$ on $E$, where $f(x) := \sqrt{x^2} = |x|$ for $x \in E$. Note that $[-1, 1]$ is a closed and bounded subset of $\mathbb{R}$, $f_n \geq f_{n+1}$ for all $n \in \mathbb{N}$ on $E$, and each $f_n$ is continuous on $E$, and so is $f$. Hence by Dini's theorem (2.282), $f_n \to f$ uniformly on $E$.* ■

**Example 2.83** *(Ghorpade and Limaye, p. 433) We give examples to show that none of the hypotheses in the Dini Theorem for sequences of continuous functions can be omitted.*

*(a) Let* $E := [0, 2]$*, and for each* $n \in \mathbb{N}$*, let* $f_n(x) := 1 - |2nx - 3|$ *if* $(1/n) \le x \le (2/n)$ *and* $f_n(x) := 0$ *otherwise. In this example, the set* $E$ *is closed and bounded, but the sequence* $(f_n)$ *is not monotonic.*

*(b) Let* $E := (0, 1]$*, and for each* $n \in \mathbb{N}$*, let* $f_n(x) := 1/(nx + 1)$ *for* $x \in E$*. In this example, the sequence* $(f_n)$ *is monotonic and the set* $E$ *is bounded but* $E$ *is not closed.*

*(c) Let* $E := [1, \infty)$*, and for each* $n \in E$*, let* $f_n(x) := x/(x + n)$ *for* $x \in E$*. In this example, the sequence* $(f_n)$ *is monotonic and the set* $E$ *is closed, but* $E$ *is not bounded.*

*(d) Let* $E := [0, 1]$*, and for* $n \in E$*, let* $f_n(x) := 1 - nx$ *if* $0 \le x \le (1/n)$ *and* $f_n(x) := 0$ *otherwise. Here the set* $E$ *is closed and bounded, the sequence* $(f_n)$ *is monotonic, and it converges to a discontinuous function* $f : E \to \mathbb{R}$ *given by* $f(0) := 1$ *and* $f(x) := 0$ *if* $x \in (0, 1]$*.*

*In (a), (b), and (c) above,* $f_n \to f$ *on* $E$*, where* $f := 0$ *on* $E$*, but the sequence* $(f_n)$ *does not converge to* $f$ *uniformly, since* $\sup\{|f_n(x) - f(x)| : x \in E\} = 1$ *for each* $n \in \mathbb{N}$*. In (d) above,* $(f_n)$ *does not converge uniformly to* $f$*, since* $|f_n(1/2n) - f(1/2n)| = 1/2$ *for each* $n \in \mathbb{N}$*.* ■

We end this section with a short detour on another form of convergence of functions, namely mean square. We will not make subsequent use of this, but it is of paramount importance in, e.g., Fourier (more generally, harmonic) analysis, where clear, useful, convenient results can be derived using mean square convergence, but far less so for pointwise and uniform convergence. We restrict attention to continuous functions, which, thus, are Riemann integrable.

Definition: Denote by $C(I)$ the space of continuous functions on $I = [a, b]$.

(a) The *mean square norm*, or $L^2$ norm, denoted $\|f\|_2$, of $f \in C(I)$ is given by

$$\|f\|_2 = \left( \int_I |f(x)|^2 dx \right)^{1/2}.$$

(b) By the *mean square distance*, or simply the distance, between functions $f, g \in C(I)$, we mean the quantity

$$\|f - g\|_2 = \left( \int_I |f(x) - g(x)|^2 dx \right)^{1/2}.$$

(c) If $f, f_1, f_2, \ldots \in C(I)$ satisfy

$$\lim_{N\to\infty} \|f_N - f\|_2 = 0,$$

then we say the sequence $f_1, f_2, \ldots$ converges to $f$, or the $f_N$'s converge to $f$ in *mean square norm*.

More generally,

Definition: Given $1 \le p \le \infty$, for each $f \in C[a, b]$,

$$\|f\|_p = \begin{cases} \left( \int_a^b |f(t)|^p dt \right)^{1/p}, & \text{if } 1 \le p < \infty, \\ \sup_{t\in[a,b]} |f(t)|, & \text{if } p = \infty. \end{cases}$$

It is shown in books on measure theory, metric space analysis, and functional analysis, that $\|\cdot\|_p$ is a norm on $C[a,b]$. We call $\|\cdot\|_p$ the $L^p$-norm on $C[a,b]$.

Theorem: Let $I=[a,b]$, let $f_1, f_2, \ldots$ be a sequence of functions in $C(I)$; and let $f \in C(I)$. If this sequence converges uniformly to $f$, then it converges in norm to $f$.

> Proof: By basic properties of the Riemann integral,
>
> $$\begin{aligned} \|f_N - f\|_2^2 &= \int_I |f_N(x)-f(x)|^2\,dx \le \sup_{x\in I} |f_N(x)-f(x)|^2 \int_I dx \\ &= (b-a)\left(\sup_{x\in I}|f_N(x)-f(x)|\right)^2, \end{aligned} \tag{2.283}$$
>
> the last step because "the sup of the square equals the square of the sup", from (2.236).
>
> If the $f_N$'s converge uniformly to $f$, then the rhs of (2.283) goes to zero as $N\to\infty$. By the Squeeze Theorem (2.9), then, so does $\|f_N - f\|^2$, and from Example 2.3, so does $\|f_N - f\|$. So the $f_N$'s converge in norm to $f$.

We return to the function used in Example 2.19 to show that pointwise convergence does not imply convergence in $L^2$ norm.

**Example 2.84** *(Stade, Fourier Analysis, p. 158) For domain $D=[0,2\pi]$, $N\in\mathbb{N}$, let $f_N : D\to\mathbb{R}$ be the function defined by $f_N(x) = N\left(\frac{x}{2\pi}\right)^N \sqrt{2\pi - x}$. Also let $f(x)=0$ for all $x\in[0,2\pi]$. Then, with substitution $x = 2\pi u$,*

$$\begin{aligned} \lim_{N\to\infty}\|f_N - f\|^2 &= \lim_{N\to\infty}\int_0^{2\pi}|f_N(x)-f(x)|^2\,dx = \lim_{N\to\infty} N^2\int_0^{2\pi}\left(\frac{x}{2\pi}\right)^{2N}(2\pi - x)dx \\ &= 4\pi^2\lim_{N\to\infty}N^2\int_0^1 u^{2N}(1-u)dx = 4\pi^2\lim_{N\to\infty}N^2\left[\frac{u^{2N+1}}{2N+1}-\frac{u^{2N+2}}{2N+2}\right]_0^1 \\ &= 4\pi^2\lim_{N\to\infty}N^2\frac{1}{(2N+1)(2N+2)} = \pi^2 \neq 0. \end{aligned}$$

*Thus, $\|f_N - f\| \nrightarrow 0$, so the $f_N$'s do not converge to $f$ in norm. Notice that $\|f_N - f\|$ does converge to something.* ■

### 2.6.9 Series of Functions and the Weierstrass $M$-Test

The previous section, §2.6.8 concentrated on convergence of sequences. It is best to review the definitions of pointwise and uniform convergence, as we now turn to series. We begin by repeating a definition from the previous section on pointwise convergence of a series, and then turn to the definition of uniform convergence of series.

Definition: Let $\{f_n\}$ be a sequence of functions with the same domain $D$, with associated series $\sum_{n=1}^{\infty} f_n$. If, for each $x\in D$, the sequence of partial sums $s_k(x) = \sum_{n=1}^{k} f_n(x)$ converges pointwise to a value $S(x)$, then the series is said to *converge pointwise (on $D$) to (the function) $S=\sum_{n=1}^{\infty} f_n$*. We write $\sum_{n=1}^{\infty} f_n \to S$ as shorthand for: $s_k \to S$, where $S=\sum_{n=1}^{\infty} f_n$.

Definition: Let $\{f_n\}$ be a sequence of functions with the same domain $D$. The series $\sum_{n=1}^{\infty} f_n$ is said to *converge uniformly (on $D$) to the function $S$* if the associated sequence of partial sums converges uniformly on $D$. In this case, we write $\sum_{n=1}^{\infty} f_n \rightrightarrows S$.

Theorem: Let $\{f_n\}$ be a sequence of functions with the same domain $D$.

$$\textbf{If, } \forall n \in \mathbb{N},\ f_n \in \mathcal{C}^0 \textbf{ and } \sum_{n=1}^{\infty} f_n(x) \rightrightarrows S(x)\textbf{, then } S \in \mathcal{C}^0\textbf{.} \tag{2.284}$$

Proof: Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$. For each $n \in \mathbb{N}$, $S_n$ is continuous on $D$, from (2.38). Since $\{S_n\}$ converges uniformly to $S$ on $D$, (2.281) implies $S$ is also continuous on $D$.

Theorem (Weierstrass $M$-test): Let $\{f_n\}$ be a sequence of functions with the same domain $D$. If there exists a sequence of (nonnegative) constants $M_n$ such that, $\forall x \in D$ and $\forall n \in \mathbb{N}$, $|f_n(x)| \le M_n$, and $\sum_{n=1}^{\infty} M_n < \infty$, then

$$\sum_{n=1}^{\infty} f_n \text{ is uniformly convergent on } D. \tag{2.285}$$

Proof: Let partial sum $S_n(x) = \sum_{k=1}^{n} f_k(x)$. Then, for $n > m$ and $\forall x \in D$,

$$0 \le |S_n(x) - S_m(x)| = \left| \sum_{k=m+1}^{n} f_k(x) \right| \le \sum_{k=m+1}^{n} |f_k(x)| \le \sum_{k=m+1}^{n} M_k. \tag{2.286}$$

As the series $M_n$ is convergent, the Cauchy criterion (2.242) implies that the rhs of (2.286) can be made arbitrarily close to zero, i.e., for any $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that, for $n > m > N$, $\sum_{k=m+1}^{n} M_k < \epsilon$. Thus, from definition (2.277), $\{S_n\}$ is a uniformly Cauchy sequence, and the result now follows from theorem (2.278).

**Example 2.85** *Let $f_n(x) = (-1)^n x^{2n}/(2n)!$, $x \in \mathbb{R}$. Then, $\forall L \in \mathbb{R}_{>0}$,*

$$|f_n(x)| = \left| \frac{x^{2n}}{(2n)!} \right| \le \left| \frac{L^{2n}}{(2n)!} \right| =: M_n, \quad x \in D = [-L, L].$$

*By the ratio test (2.250), $M_{n+1}/M_n = L^2/(2n+1)(2n+2) \to 0$ as $n \to \infty$, so that $\sum_{n=0}^{\infty} M_n < \infty$, and, from the Weierstrass $M$-test, $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly on $D = [-L, L]$. This justifies the definitions*

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \qquad \textit{and} \qquad \sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!},$$

*as in (2.89).*[22] *Next, for $x \neq 0$, we wish to know if*

$$\frac{\sin x}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}.$$

[22] Observe that, while $\lim_{n\to\infty} x^{2n}/(2n)! = 0$ for all $x \in \mathbb{R}$, for any fixed $n$, $\lim_{x\to\infty} x^{2n}/(2n)! = \infty$. This means that, although the series converges, evaluation of the truncated sum will be numerically problematic because of the limited precision with which numbers are digitally stored. Of course, the relations $\cos(x+\pi) = -\cos x$, $\cos(x+2\pi) = \cos x$ and $\cos(-x) = \cos x$; and $\sin(x+\pi) = -\sin x$, $\sin(x+2\pi) = \sin x$ and $\sin(-x) = -\sin x$; and $\sin x = -\cos(x+\pi/2)$, imply that only the series for cosine is required, with $x$ restricted to $[0, \pi/2]$. For $x \in [0, \pi/2]$, it is enough to sum the $\cos x$ series up to $n = 10$ to ensure 15 digit accuracy.

*As the series for* $\sin x$ *is convergent, it follows from the definition of convergence that, for any* $\epsilon > 0$ *and* $x \neq 0$, $\exists N = N(x, \epsilon) \in \mathbb{N}$ *such that,* $\forall k > N$,

$$\left| \sin x - \sum_{n=0}^{k} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right| < \epsilon |x|$$

*or, recalling that* $|ab| = |a| |b|$,

$$\left| \frac{\sin x}{x} - \sum_{n=0}^{k} \frac{1}{x} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right| < \epsilon,$$

*so that, for* $x \neq 0$,

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}. \tag{2.287}$$

*With* $0^0 = 1$, *the rhs equals 1 for* $x = 0$, *which coincides with the limit as* $x \to 0$ *of the lhs. As above, the Weierstrass* $M$*-test shows that this series is uniformly convergent on* $[-L, L]$ *for any* $L \in \mathbb{R}_{>0}$. ■

**Example 2.86** *(Example 2.81 cont.) Again let* $f_n(x) = x^n/n!$, $n = 0, 1, \ldots,$ *and* $s_n(x) = \sum_{k=0}^{n} f_k(x)$. *For all* $L \in \mathbb{R}_{>0}$,

$$|f_n(x)| = \frac{|x^n|}{n!} \leq \frac{L^n}{n!} = M_n, \quad x \in [-L, L],$$

*and* $\sum_{n=0}^{\infty} M_n < \infty$ *from Example 2.78, which showed that* $s_n(L) \to \exp(L)$ *absolutely. Thus, the Weierstrass* $M$*-test implies that* $\sum_{n=0}^{\infty} f_n(x)$ *converges uniformly on* $[-L, L]$, *where* $L$ *is an arbitrary positive real number. It is, however,* not *true that* $\sum_{n=0}^{\infty} x^n/n!$ *converges uniformly on* $(-\infty, \infty)$. *Also, as required below,*

$$\sum_{r=0}^{\infty} \frac{(-z \ln z)^r}{r!} \rightrightarrows e^{-z \ln z}, \quad z > 0, \tag{2.288}$$

*by taking* $x = -z \ln z \in \mathbb{R}$. ■

**Example 2.87** *Let* $f_k(x) = (x/R)^k =: a^k$ *for a fixed* $R \in \mathbb{R}_{>0}$. *Then* $G(x) = \sum_{k=1}^{\infty} f_k(x)$ *is a geometric series that converges to* $S(x) = (x/R)/(1 - x/R)$ *for* $|x/R| < 1$, *or* $|x| < R$. *Let* $S_n$ *be the partial sum of sequence* $\{f_k\}$. *For* $G(x)$ *to converge uniformly to* $S(x)$, *it must be the case that, for any* $\epsilon > 0$, *there is a value* $N \in \mathbb{N}$ *such that,* $\forall n > N$,

$$|S_n - G| = \left| \sum_{k=1}^{n} \left( \frac{x}{R} \right)^k - \frac{x/R}{1 - x/R} \right| = \left| \frac{a - a^{n+1}}{1 - a} - \frac{a}{1 - a} \right| = \left| \frac{a^{n+1}}{1 - a} \right| < \epsilon. \tag{2.289}$$

*But, for any* $n$,

$$\lim_{x \to R^-} \frac{a^{n+1}}{1 - a} = \lim_{a \to 1^-} \frac{a^{n+1}}{1 - a} = \infty,$$

*so that the inequality in (2.289) cannot hold. Now choose a value* $b$ *such that* $0 < b < R$ *and let* $M_k = (b/R)^k$, *so that* $\sum_{k=1}^{\infty} M_k = (b/R)/(1 - b/R) < \infty$. *Then, for* $|x| \leq b$, $\left|(x/R)^k\right| \leq (b/R)^k = M_k$, *and use of the Weierstrass* $M$*-test shows that the series* $G(x)$ *converges uniformly on* $[-b, b]$ *to* $S(x)$. *See also Example 2.98.* ■

**Remarks**:

**(a)** When using the Maple engine that accompanies Scientific Workplace 4.0, evaluating $\lim_{n\to\infty}\sum_{k=1}^{n}(-1)^k$ yields the interval $-1..0$, yet evaluating $\sum_{k=1}^{\infty}(-1)^k$ produces $-1/2$. Presumably, the latter result is obtained because Maple computes $\lim_{x\to-1^+}\sum_{k=1}^{\infty}x^k = \lim_{x\to-1^+}x/(1-x) = -1/2$, which is itself correct, but,

$$\sum_{k=1}^{\infty}(-1)^k \neq \lim_{x\to-1^+}\sum_{k=1}^{\infty}x^k.$$

From (2.284), this would be true if $\sum_{k=1}^{\infty}x^k$ were uniformly convergent for $x=-1$, which it is not. While this is probably a mistake in Maple, it need not be one, as the next remark shows.

**(b)** The series $\sum_{k=1}^{\infty}a_k$ is said to be *Abel summable* to $L$ if $\lim_{x\to1^-}f(x)=L$, where $f(x)=\sum_{k=1}^{\infty}a_kx^k$ for $0\leq x<1$ (after Neils Henrik Abel, 1802–1829; see Goldberg, 1964, p. 251). For example, with $a_k=(-1)^k$, the series $f(x)=\sum_{k=1}^{\infty}a_kx^k=-x+x^2-x^3+\cdots$ converges for $|x|<1$ to $-x/(x+1)$. Then the series $-1+1-1+1-\cdots$ is clearly divergent, but is Abel summable to $\lim_{x\to1^-}f(x)=-1/2$. ■

**Example 2.88** *The contrapositive of (2.284) implies*

$$f_n\in\mathcal{C}^0,\ f\notin\mathcal{C}^0 \implies \sum_{n=1}^{\infty}f_n(x)\not\rightrightarrows f.$$

*In words, if the $f_n$ are continuous but $f$ is not, then $\sum_{n=1}^{\infty}f_n(x)$ is not uniformly convergent to $f$. In Example 2.80, $f(x)$ is not continuous at $x=0$, so that $\sum_{k=0}^{\infty}f_k(x)$ is not uniformly convergent on any interval containing zero.* ■

### 2.6.10 Sequences and Series of Functions: Integration and Differentiation

Recall from (2.281) and (2.284) that uniform convergence of sequences and series of continuous functions implies continuity of the limiting function. Similar result hold for integrability of sequences and series. In the following proof, and that of (2.281), the reader should explicitly observe how the uniform convergence assumption is used, and why it is necessary.

Theorem: If, $\forall n \in \mathbb{N}$, $f_n \in \mathcal{R}[a,b]$ and $f_n(x) \rightrightarrows f(x)$ on $[a,b]$, then $f \in \mathcal{R}[a,b]$ and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx. \tag{2.290}$$

Proof: Let $\epsilon_n = \max_{x\in[a,b]} |f_n(x) - f(x)|$, so that

$$\forall x \in [a,b], \quad f_n(x) - \epsilon_n \le f(x) \le f_n(x) + \epsilon_n. \tag{2.291}$$

From (2.279), uniform convergence of $\{f_n\}$ implies that $\lim_{n\to\infty} \epsilon_n = 0$. Recall the definitions of $\underline{S}(f,\pi)$ and $\overline{S}(f,\pi)$ from §2.5.1, and their properties, such as $\sup_\pi \underline{S}(f,\pi) \le \inf_\pi \overline{S}(f,\pi)$. As $f_n \in \mathcal{R}[a,b]$,

$$\sup_\pi \underline{S}(f_n,\pi) = \inf_\pi \overline{S}(f_n,\pi) = \int_a^b f_n(x)\,dx.$$

Further, from (2.167), $f_n(x) - \epsilon_n \in \mathcal{R}[a,b]$, so

$$\sup_\pi \underline{S}(f_n - \epsilon_n,\pi) = \inf_\pi \overline{S}(f_n - \epsilon_n,\pi) = \int_a^b (f_n(x) - \epsilon_n)\,dx,$$

and, from (2.291),

$$\underline{S}(f_n - \epsilon_n,\pi) < \underline{S}(f,\pi), \quad \text{and} \quad \overline{S}(f,\pi) < \overline{S}(f_n + \epsilon_n,\pi).$$

The previous two equations and monotonicity of the supremum (1.7) imply, $\forall n \in \mathbb{N}$,

$$\int_a^b (f_n(x) - \epsilon_n)\,dx \le \sup_\pi \underline{S}(f,\pi) \le \inf_\pi \overline{S}(f,\pi) \le \int_a^b (f_n(x) + \epsilon_n)\,dx. \tag{2.292}$$

The difference of the two inner values is less than or equal to the difference of the two outer values, so that

$$0 \le \inf_\pi \overline{S}(f,\pi) - \sup_\pi \underline{S}(f,\pi) \le \int_a^b (f_n(x) + \epsilon_n)\,dx - \int_a^b (f_n(x) - \epsilon_n)\,dx = \int_a^b 2\epsilon_n dx,$$

but $\lim_{n\to\infty} \epsilon_n = 0$, so that, from the Squeeze Theorem (2.9),

$$\lim_{n\to\infty} \left[\inf_\pi \overline{S}(f,\pi) - \sup_\pi \underline{S}(f,\pi)\right] = 0,$$

so that, from (2.161), $f \in \mathcal{R}[a,b]$. We can now write

$$\begin{aligned}\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| &= \left|\int_a^b [f_n(x) - f(x)]\,dx\right| \\ &\le \int_a^b |f_n(x) - f(x)|\,dx \le \int_a^b \epsilon_n\,dx = \epsilon_n(b-a).\end{aligned}$$

Taking the limit (and recalling the definition of sequence convergence (2.1)), we obtain (2.290).

Theorem: If, $\forall n \in \mathbb{N}$, $f_n \in \mathcal{R}[a,b]$ and $\sum_{n=1}^{\infty} f_n(x) \rightrightarrows S(x)$ for $x \in [a,b]$, then $S \in \mathcal{R}[a,b]$ and

$$\int_a^b \left( \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x) \right) dx = \int_a^b S(x)\, dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} \int_a^b f_k(x)\, dx,$$

i.e.,

$$\int_a^b S(x)\, dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\, dx. \tag{2.293}$$

Proof: Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$, so that, $\forall n \in \mathbb{N}$, $S_n \in \mathcal{R}[a,b]$. From the previous result, $S \in \mathcal{R}[a,b]$ and, from (2.290) applied to $S(x)$ and $S_n(x)$,

$$\int_a^b \sum_{k=1}^{\infty} f_k(x)\ dx = \int_a^b S(x)\ dx = \lim_{n\to\infty} \int_a^b S_n(x)\ dx = \lim_{n\to\infty} \sum_{k=1}^{n} \int_a^b f_k(x)\ dx, \tag{2.294}$$

which is (2.293), having used finite additivity (2.167) of the Riemann integral.

**Example 2.89** *Recall from Example 2.15 that* $\lim_{x\to 0} x^x = 1$. *The integral* $I = \int_0^1 x^{-x} dx$ *was shown to be equal to* $\sum_{r=1}^{\infty} r^{-r}$ *by Johann Bernoulli in 1697. To see this, as in Havil (2003, p. 44), use* (2.122) *and* (2.272) *to write*

$$I = \int_0^1 e^{-x\ln x} dx = \int_0^1 \sum_{r=0}^{\infty} \frac{(-x\ln x)^r}{r!} dx = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \int_0^1 (x\ln x)^r\, dx,$$

*where the exchange of sum and integral is justified by (2.288) and (2.293). The result now follows from (2.194), i.e.,* $\int_0^1 (x\ln x)^r\, dx = (-1)^r r! \ /\ (r+1)^{r+1}$, *or*

$$I = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{(-1)^r r!}{(r+1)^{r+1}} = \sum_{r=0}^{\infty} \frac{1}{(r+1)^{r+1}}. \qquad \blacksquare$$

**Example 2.90** *(Browder, 1996, p. 113) Let*

$$f_n(x) = \frac{x}{n(x+n)} = \frac{1}{n} - \frac{1}{x+n},$$

*for* $x \in I = [0,1]$ *and* $n \in \mathbb{N}$. *As* $f_n'(x) = 1/(x+n)^2 > 0$ *for* $x \in I$, $f_n$ *is strictly increasing on its domain, from* (2.100)*; and* $\max f_n$ *occurs at* $x = 1$. *Thus,* $0 \le f_n(x) \le 1/n(n+1)$, *and, from the comparison test with* $g_n = n^{-2}$, $\sum_{n=1}^{\infty} [n(n+1)]^{-1}$ *converges. This series has the "telescoping property", i.e.,*

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}, \qquad \text{so that} \qquad \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1} \to 1. \tag{2.295}$$

*Thus, from the Weierstrass M-test,* $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly on* $[0,1]$ *to a function, say* $S(x)$, *which, by (2.284), is continuous. Note that* $S(0) = 0$ *and, from continuity* (2.36) *applied to* $S$, *continuity of* $\sum_{n=1}^{k} f_n$, *and that* $f_n$ *is strictly increasing on its domain,*

$$S(1) = \lim_{x\nearrow 1} S(x) = \lim_{x\nearrow 1} \lim_{k\to\infty} \sum_{n=1}^{k} f_n(x) \le \lim_{k\to\infty} \sum_{n=1}^{k} f_n(1) = \lim_{k\to\infty} \sum_{n=1}^{k} \left[ \frac{1}{n} - \frac{1}{1+n} \right] = 1.$$

*Thus, $0 \leq S(x) \leq 1$. From (2.293),*

$$\sum_{n=1}^{\infty} \int_0^1 f_n(x)\, dx = \int_0^1 S(x)\, dx =: \gamma. \tag{2.296}$$

*From $0 \leq S(x) \leq 1$ and continuity of $S$, the IVT (2.60) implies that $\int_0^1 S(x)\, dx < \int_0^1 dx = 1$, so that $0 < \gamma < 1$.*

*From (2.188) (or perform the substitution $u = x + n$), $\int_0^1 1/(x+n)dx = \ln(n+1) - \ln n$, so*

$$\int_0^1 f_n = \int_0^1 \left(\frac{1}{n} - \frac{1}{x+n}\right) dx = \frac{1}{n} - \ln\frac{n+1}{n},$$

*and (2.296) implies*

$$\gamma = \lim_{N\to\infty} \sum_{n=1}^{N} \int_0^1 f_n(x)\, dx = \lim_{N\to\infty} \left[\sum\nolimits_{n=1}^{N} n^{-1} - \ln(N+1)\right],$$

*or, as $\lim_{N\to\infty} [\ln(N+1) - \ln N] = 0$,*

$$\gamma = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \ln N\right),$$

*which is Euler's constant from Example 2.70.* ■

**Example 2.91** *Let $S(x) = 1 - x + x^2 - x^3 + \cdots$. For $-1 < x \leq 0$, $S(x) = 1 + y + y^2 + \cdots$, where $y = -x$, and so converges to $1/(1-y) = 1/(1+x)$ from (2.244). For $0 \leq x < 1$, the alternating series test (page 125) shows that $S(x)$ converges; and from (2.245), converges to $1/(1+x)$. Thus, $S(x)$ converges to $1/(1+x)$ for $|x| < 1$.*

*Similar to the derivation in Example 2.87, for every $b \in [0,1)$, $S(x)$ is uniformly convergent for $x \in [-b,b]$. So, from (2.293),*

$$\int_0^b \frac{1}{1+x}\, dx = \int_0^b 1\, dx - \int_0^b x\, dx + \int_0^b x^2\, dx - \cdots.$$

*For the first integral, let $u = 1 + x$ so that $\int_0^b (1+x)^{-1}\, dx = \int_1^{b+1} u^{-1}\, du = \ln(1+b)$. Thus,*

$$\ln(1+b) = b - \frac{b^2}{2} + \frac{b^3}{3} - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{b^n}{n}, \qquad 0 \leq b < 1. \tag{2.297}$$

*Example 2.101 will show that (2.297) also holds for $b = 1$.*

*Now let $c = -b \in (-1, 0]$. Again using substitution $u = 1 + x$,*

$$\int_c^0 \frac{1}{1+x} dx = \int_{1+c}^1 \frac{1}{u} du = -\int_1^{1+c} \frac{1}{u} du = -\ln(1+c),$$

*and $\int_c^0 x^k dx = x^{k+1}/(k+1)\big|_c^0 = -c^{k+1}/(k+1)$. Thus,*

$$\begin{aligned} -\ln(1-b) &= -\ln(1+c) = -c - \left(-\frac{c^2}{2}\right) + \left(-\frac{c^3}{3}\right) - \cdots \\ &= -c + \frac{c^2}{2} - \frac{c^3}{3} + \cdots = b + \frac{b^2}{2} + \frac{b^3}{3} + \cdots \end{aligned}$$

*or*

$$\ln(1-b) = -b - \frac{b^2}{2} - \frac{b^3}{3} + \cdots = -\sum_{n=1}^{\infty} \frac{b^n}{n}, \quad 0 \le b < 1. \tag{2.298}$$

*We can combine* (2.297) *and* (2.298) *to get*

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad -1 < x \le 1, \tag{2.299}$$

*and*

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \quad -1 \le x < 1, \tag{2.300}$$

*which are known as the* Newton-Mercator series. ∎

**Example 2.92** *Similar to Example 2.91,*

$$\frac{1}{1+y^2} = 1 - y^2 + y^4 - y^6 + \cdots, \quad |y| < 1, \tag{2.301}$$

*and, for every $b \in [0,1)$, the rhs is uniformly convergent for $y \in [-b, b]$. Thus, from* (2.293)*, termwise integration of the rhs is permitted. From (2.92) and the FTC (2.176),*

$$\int_0^t \frac{1}{1+y^2}\, dy = \arctan(t) - \arctan(0) = \arctan(t), \tag{2.302}$$

*so that*

$$\arctan(t) = t - \frac{t^3}{3} + \frac{t^5}{5} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} t^{2n+1}, \quad t \in [0,1), \tag{2.303}$$

*from which* $\arctan(t)$*, $t \in [0,1)$, can be computed to any degree of accuracy up to machine precision.*[23] *Example 2.102 below consider the case when $t = 1$.* ∎

Another useful result is the *bounded convergence theorem*, which involves the interchange of limit and integral using only pointwise convergence, but requires that $f$ also be integrable on $I = [a, b]$, and that the $f_n$ are bounded for all $n$ and all $x \in I$.

Theorem (Bounded Convergence Theorem for Riemann Integrals): If, $\forall n \in \mathbb{N}$, $f_n \in \mathcal{R}[a,b]$ with $f_n(x) \to f(x)$, $f \in \mathcal{R}[a,b]$, and $\exists M \in \mathbb{R}_{>0}$ such that, $\forall x \in [a,b]$ and $\forall n \in \mathbb{N}$, $|f_n(x)| \le M$, then

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx. \tag{2.304}$$

This is explicitly proven in Stoll (2001, §10.6) and is a special case of *Lebesgue's dominated convergence theorem*, detailed in, e.g., Browder (1996, §10.2), Stoll (2001, §10.7), Pugh (2002, §6.4), and any book on measure theory and the Lebesgue integral. Interestingly, measure theory is not required to prove (2.304): The Arzela Bounded Convergence Theorem proves the result using only basic analysis, and can be found in, e.g., Ghorpade and Limaye (2018, Prop. 10.40).

Paralleling results (2.290) and (2.293), let $\{f_n(x)\}$ be a sequence of functions with $n$th partial sum $S_n(x)$, such that $S_n(x) \to S(x)$. If the conditions leading to (2.304) apply to $\{S_n(x)\}$ and $S(x)$, then (2.293) also holds, the proof of which is the same as (2.294).

[23] One could use so-called virtual precision arithmetic, or VPA, to obtain far higher accuracy. It works by using software, and traditional computing machines that allocate 64 bits per number, to allow computations with any (up to some limit) precision.

**Example 2.93** *With* $f_n(x) = nx(1-x^2)^n$ *as in (2.275), use of a symbolic software package easily shows that* $f_n'(x) = -n(1-x^2)^{n-1}(x^2(1+2n)-1)$ *and solving* $f_n'(x_m) = 0$ *yields* $x_m = (1+2n)^{-1/2}$, *so that*

$$f_n(x_m) = \frac{n}{\sqrt{1+2n}}\left(\frac{2n}{1+2n}\right)^n \quad \text{and} \quad \lim_{n\to\infty} f_n(x_m) = \infty.$$

*Thus,* $\nexists M$ *such that,* $\forall x \in [0,1]$ *and* $\forall n \in N$, $|f_n(x)| \le M$, *and the contrapositive of the bounded convergence theorem* (2.304) *implies that* $\lim_{n\to\infty}\int_0^1 f_n \neq \int_0^1 f$, *as determined with a direct calculation of the integrals, as mentioned just after (2.275).* ■

**Example 2.94** *For a fixed* $x \in \mathbb{R}_{>0}$ *and all* $t \in \mathbb{R}$, *define*[24]

$$h(t) := \frac{\exp\left(-x\left(1+t^2\right)\right)}{1+t^2} = e^{-x}\frac{e^{-xt^2}}{1+t^2}, \tag{2.305}$$

*which is the integrand in (2.223). Interest centers on developing computational formulae for* $\int_0^1 h$ *and comparing their efficacy.*

***Method 1.*** *From (2.272) with* $x$ *replaced by* $-xt^2$,

$$h(t) = \frac{e^{-x}}{1+t^2}\left(1 - xt^2 + \frac{x^2t^4}{2!} - \cdots\right) = e^{-x}\sum_{k=0}^{\infty}(-1)^k\frac{x^k}{k!}\frac{t^{2k}}{1+t^2},$$

*and termwise integration, valid from Example 2.86 and* (2.293), *gives*

$$\int_0^1 h(t)\ dt = e^{-x}\sum_{k=0}^{\infty}(-1)^k\frac{x^k}{k!}J_k,$$

*where*

$$J_k := \int_0^1 \frac{t^{2k}}{1+t^2}dt, \quad k \in \mathbb{N},$$

*which seems resilient to use of a transformation or integration by parts. It is, however, quite simple, with the following "trick":*

$$\int \frac{t^{2k}}{1+t^2}\ dt = \int \frac{t^{2k-2}(t^2+1-1)}{1+t^2}\ dt = \int \frac{t^{2k-2}(1+t^2)}{1+t^2}\ dt - \int \frac{t^{2k-2}}{1+t^2}\ dt,$$

*giving the recursion*

$$\int_a^b \frac{t^{2k}}{1+t^2}\ dt = \left.\frac{t^{2k-1}}{2k-1}\right|_a^b - \int_a^b \frac{t^{2k-2}}{1+t^2}\ dt.$$

*With* $a = 0$ *and* $b = 1$,

$$J_k = \frac{1}{2k-1} - J_{k-1}. \tag{2.306}$$

*From (2.224),* $J_0 = \pi/4$, *and iterating (2.306) gives*

$$J_1 = 1 - \pi/4, \quad J_2 = 1/3 - 1 + \pi/4, \quad J_3 = 1/5 - 1/3 + 1 - \pi/4$$

[24] This example was contributed by my friend and colleague Professor Walther Paravicini.

*and the general formula*

$$J_k = (-1)^k \left( \sum_{m=1}^{k} \frac{(-1)^m}{2m-1} + \pi/4 \right),$$

*so that*

$$\int_0^1 h(t)\ dt = e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} \ \left( \sum_{m=1}^{k} \frac{(-1)^m}{2m-1} + \pi/4 \right). \tag{2.307}$$

*This sum converges very fast because of the* $k!$ *in the denominator. To illustrate, take* $x = 0.3$. *Accurate numeric integration of* $h$, *and also evaluation of (2.307) truncating the infinite sum at* $U = 200$, *gives* 0.5378448777, *which we will deem correct to 10 digits. With only* $U = 4$, *(2.307) yields* 0.5378456, *accurate to 5 digits.*

***Method 2.*** *We make the* ansatz *that* $h(t)$ *can be expressed as the series* $h(t) = \sum_{k=0}^{\infty} a_k t^{2k}$, *and calculate the* $a_k$. *With* $j = k - 1$,

$$\begin{aligned} e^{-xt^2} &= e^x(1+t^2)\sum_{k=0}^{\infty} a_k t^{2k} = e^x \left( a_0 + \sum_{k=1}^{\infty} a_k t^{2k} + \sum_{k=0}^{\infty} a_k t^{2k+2} \right) \\ &= e^x \left( a_0 + \sum_{j=0}^{\infty} a_{j+1} t^{2j+2} + \sum_{k=0}^{\infty} a_k t^{2k+2} \right) = e^x \left( a_0 + \sum_{j=0}^{\infty} (a_{j+1} + a_j)\, t^{2j+2} \right) \\ &= e^x a_0 + e^x \sum_{k=1}^{\infty} (a_k + a_{k-1})\, t^{2k}. \end{aligned}$$

*With* $t = 0$, *it follows immediately that* $a_0 = e^{-x}$. *By comparison with* $\exp(-xt^2) = \sum_{k=0}^{\infty} (-xt^2)^k/k!$, *we see that*

$$e^x(a_k + a_{k-1}) = \frac{(-1)^k}{k!} x^k, \qquad k = 1, 2, \ldots .$$

*Iterating on*

$$e^x a_k = -e^x a_{k-1} + \frac{(-1)^k}{k!} x^k$$

*with* $a_0 = e^{-x}$ *gives*

$$\begin{aligned} e^x a_1 &= -1 + \frac{(-1)^1}{1!} x^1 = -\left(1 + \frac{x^1}{1!}\right), \\ e^x a_2 &= +\left(1 + \frac{x^1}{1!}\right) + \frac{(-1)^2}{2!} x^2 = +\left(1 + \frac{x^1}{1!} + \frac{x^2}{2!}\right), \end{aligned}$$

*and, in general,*

$$e^x a_k = (-1)^k \left(1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!}\right) = (-1)^k \sum_{j=0}^{k} \frac{x^j}{j!}.$$

*Thus,*

$$h(t) = \sum_{k=0}^{\infty} a_k t^{2k} = e^{-x} \sum_{k=0}^{\infty} (-1)^k \left( \sum_{j=0}^{k} \frac{x^j}{j!} \right) t^{2k}$$

*and, as* $\int_0^1 t^{2k} dt = 1/(2k+1)$,

$$\int_0^1 h(t)\ dt = e^{-x} \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} \left( \sum_{j=0}^{k} \frac{x^j}{j!} \right). \tag{2.308}$$

*Whereas (2.307) has a* $k!$ *in the denominator, (2.308) has only* $2k+1$*, so we expect it to converge much slower. Indeed, with* $x = 0.3$*, use of* 1000 *terms in the sum results in* 0.53809*, which is correct only to three digits. The formula is useless for numeric purposes.*

***Method 3****. Expanding the numerator of the middle term in (2.305) as a power series in* $-x(1+t^2)$ *gives*

$$h(t) = \frac{1}{1+t^2} \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!} \left(1+t^2\right)^k = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!} \left(1+t^2\right)^{k-1},$$

*so that*

$$\int_0^1 h(t)\ dt = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!} I_k,$$

*where* $I_k := \int_0^1 (1+t^2)^{k-1}\ dt$*. From (2.224),* $I_0 = \pi/4$*, and for* $k > 0$*, use of the binomial formula gives*

$$I_k = \int_0^1 \sum_{m=0}^{k-1} \binom{k-1}{m} t^{2m}\ dt = \sum_{m=0}^{k-1} \binom{k-1}{m} \frac{1}{2m+1},$$

*yielding*

$$\int_0^1 h(t)\ dt = \frac{\pi}{4} + \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k!} \sum_{m=0}^{k-1} \binom{k-1}{m} \frac{1}{2m+1}. \tag{2.309}$$

*Like (2.307), (2.309) converges fast: with* $x = 0.3$*, truncating the infinite sum at* $U = 6$ *gives* 0.5378453*, which is accurate to 5 digits. Based on this value of* $x$*, it appears that (2.307) converges the fastest.* ■

**Example 2.95** *Consider evaluating the improper integral* $\int_0^\infty e^{-sx} x^{-1} \sin x\, dx$ *for* $s \in \mathbb{R}_{>1}$*. From (2.287),* (2.11)*, and that* $x > 0$*,*

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} < \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!} < \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} < \sum_{n=0}^{\infty} \frac{x^n}{(n)!} = e^x,$$

*so that, as* $x > 0$ *and* $s > 1$*,*

$$e^{-sx} \frac{\sin x}{x} < e^{-sx} e^x = e^{-x(s-1)} < e^0 = 1.$$

*The conditions in the bounded convergence theorem* (2.304) *are fulfilled, and termwise integration can be performed. Recalling the gamma function* (1.51)*, using* $\Gamma(n) = (n-1)!$ *for* $n \in \mathbb{N}$*, and the easy-to-verify (use* $u = mx$*)*

$$I = \int_0^\infty x^n \mathrm{e}^{-mx} dx = m^{-1} \int_0^\infty (u/m)^n \mathrm{e}^{-u} du = m^{-(n+1)} \Gamma(n+1), \quad m > 0,$$

*and (2.303), this gives*

$$\int_0^\infty e^{-sx}\frac{\sin x}{x}dx = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}\int_0^\infty e^{-sx}x^{2n}dx = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}\frac{\Gamma(2n+1)}{s^{2n+1}}$$
$$= \sum_{n=0}^\infty \frac{(-1)^n}{2n+1}\left(\frac{1}{s}\right)^{2n+1} = \arctan\left(s^{-1}\right), \tag{2.310}$$

*which was used in Example 2.60.* ■

We now turn to the conditions that allow for interchange of limits and differentiation, beginning with some illustrations of what conditions are *not* sufficient.

**Example 2.96** *Let $f_n(x) = (\sin nx)/n$ so that, $\forall x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = 0 =: f(x)$. Then $f'(x) = 0$ but $f_n'(x) = \cos nx$ and, $\forall n \in \mathbb{N}$, $f_n'(0) = 1$, so that $\exists x \in \mathbb{R}$ such that $\lim\limits_{n\to\infty} \frac{d}{dx} f_n(x) \neq \frac{d}{dx}\lim\limits_{n\to\infty} f_n(x)$. Given the previous results on interchange of limit and integral, one might expect that uniform convergence is sufficient. But, $\forall x \in \mathbb{R}$,*

$$|f_n(x) - f_m(x)| = \left|\frac{\sin nx}{n} - \frac{\sin mx}{m}\right| \leq \left|\frac{1}{n} - \frac{-1}{m}\right| = \left|\frac{1}{n} + \frac{1}{m}\right|,$$

*so $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that, $\forall n, m > N$, $|f_n(x) - f_m(x)| < \epsilon$, i.e., $f_n$ is uniformly Cauchy and, by (2.278), $f_n$ is uniformly convergent. Thus, uniform convergence is not enough to ensure the interchange of limit and derivative.*

*Observe that $f_n'(x) = \cos nx$ is not convergent (pointwise or uniformly). It turns out that uniform convergence of $\{f_n'\}$ is necessary for interchange.* ■

**Example 2.97** *Let $I = [-1, 1]$ and $f_n(x) = (x^2 + n^{-1})^{1/2}$ for $x \in I$, so that $f_n \in \mathcal{C}^1$ with $f_n'(x) = x(x^2 + n^{-1})^{-1/2}$. Figure 12 shows $f_n$ and $f_n'$ for several $n$. In the limit, $f_n(x) \to f(x) := |x|$, which is not differentiable at $x = 0$. In fact, $f_n(x) \rightrightarrows f(x)$, because, as shown next, $m_n = \max_{x\in I}|f_n(x) - f(x)| = n^{-1/2}$ and result (2.279).*

*To derive $m_n$, first note that $f_n(x) - f(x)$ is symmetric in $x$, so we can restrict attention to $x \in [0, 1]$, in which case $d(x) = |f_n(x) - f(x)| = (x^2 + n^{-1})^{1/2} - x > 0$, for $x \in [0, 1]$. Its first derivative is $d'(x) = x(x^2 + n^{-1})^{-1/2} - 1$, which is strictly negative for all $x \in [0, 1]$ and $n \in \mathbb{N}$. (At $x = 1$, $d'(x) = \sqrt{n/(n+1)} - 1$.) Thus, $d(x)$ reaches its maximum on $[0, 1]$ at $x = 0$, so that*

$$\max_{x\in[0,1]} |d(x)| = \max_{x\in I} |f_n(x) - f(x)| = d(0) = n^{-1/2}.$$

*Also, $f_n'(x) \to x/|x|$ for $x \neq 0$, but the convergence cannot be uniform at $x = 0$, because, for any $n \in \mathbb{N}$,*

$$\begin{aligned}\lim_{x\to0^+}\left|f_n'(x) - \frac{x}{|x|}\right| &= \lim_{x\to0^+}\left|\frac{x}{\sqrt{x^2+n^{-1}}} - \frac{x}{|x|}\right| = \lim_{x\to0^+}\frac{x}{|x|} - \lim_{x\to0^+}\frac{x}{\sqrt{x^2+n^{-1}}}\\ &= 1 - \lim_{x\to0^+}\frac{x/x}{\sqrt{x^2/x^2 + 1/nx^2}} = 1 - 0 = 1.\end{aligned}$$ ■

The following theorem gives the desired result. Recall from §2.3.1 that $\mathcal{C}^1$ is the class of continuously differentiable functions, i.e., $f$ is differentiable and $f'(x)$ is continuous on $D$.

Theorem: Let $f : D \to \mathbb{R}$, where $I = [a, b] \subset D$. Let $f_n \in \mathcal{C}^1(I)$ such that $f_n'(x) \rightrightarrows g(x)$ and $f_n(x) \to f(x)$ on $I$. Then

$$g \in \mathcal{C}^0(I) \text{ and } \forall x \in I,\ f'(x) = g(x). \tag{2.311}$$

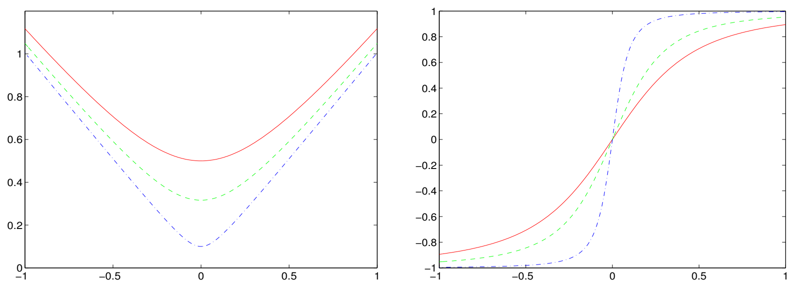


Figure 12: Function $f_n(x) = (x^2 + n^{-1})^{1/2}$ (left) and $f_n'$ (right), for $n = 4$ (solid), $n = 10$ (dashed) and $n = 100$ (dash-dot).

> Proof: That $g \in \mathcal{C}^0$ follows directly from (2.281). For $a, x \in I$, (2.163) and FTC (2.176) imply that, $\forall n \in \mathbb{N}$, $\int_a^x f_n' = f_n(x) - f_n(a)$. As $f_n'(x) \rightrightarrows g(x)$,taking the limit as $n \to \infty$ and using (2.290) and that $f_n(x) \to f(x)$ on $I$ gives $\int_a^x g = \lim_{n\to\infty}[f_n(x) - f_n(a)] = f(x) - f(a)$. As $g \in \mathcal{C}^0$, differentiating this via FTC (2.179) yields $g(x) = f'(x)$.

**Example 2.98** *Again consider the geometric series $S(x) = \sum_{k=1}^{\infty} x^k$, which, from (2.243), converges pointwise for $x \in (-1, 1)$ to $S(x) = x/(1-x)$. That is, with $S_n(x) = \sum_{k=1}^{n} x^k$ the nth partial sum, $S_n(x) \to S(x)$, and, being a polynomial, $S_n \in \mathcal{C}^1(I)$ for $I = [-1, 1]$. To apply (2.311), we need to show that $\exists g$ such that $S_n'(x) \rightrightarrows g(x)$. With $S_n'(x) = \sum_{k=1}^{n} kx^{k-1}$, for $r \in (0, 1)$ and $\epsilon > 0$ such that $r + \epsilon < 1$, the binomial theorem (1.34) implies*

$$(r+\epsilon)^k = \sum_{i=0}^{k} \binom{k}{i} \epsilon^i r^{k-i} = r^k + k\epsilon r^{k-1} + \cdots + \epsilon^k. \tag{2.312}$$

*The positivity of each term implies that $k\epsilon r^{k-1} < (r+\epsilon)^k$, so that the Weierstrass M-test (2.285) implies that $\sum_{k=1}^{\infty} kx^{k-1}$ is also uniformly convergent for $x \in [-r, r]$.*

*Thus, from (2.311) with $g(x) = S'(x) = \frac{d}{dx}\left(\lim_{n\to\infty} \sum_{k=1}^{n} x^k\right)$,*

$$S_n'(x) = \sum_{k=1}^{n} kx^{k-1} \quad \rightrightarrows \quad g(x) = \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{1}{(1-x)^2}. \tag{2.313}$$

*As this holds $\forall x \in [-r, r]$, where $r$ is an arbitrary number from the open interval $(0, 1)$, (2.313) holds for all $|x| < 1$. (If this were not true, then there would exist an $x \in (0, 1)$, say $x_0$, for which it were not true, but the previous analysis applies to all $x \in [0, x_0 + \epsilon]$, where $\epsilon$ is such that $x_0 + \epsilon < 1$, which always exists.) Thus, $S'(x) = (1-x)^{-2}$ on $(-1, 1)$.*

*To add some intuition and informality, let*

$$D = \lim_{n\to\infty} \sum_{k=1}^{n} kx^{k-1} = 1 + 2x + 3x^2 + \cdots \quad \textit{and} \quad xD = x + 2x^2 + 3x^3 + \cdots,$$

*so that $D - xD = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k = (1-x)^{-1}$, which, for $|x| < 1$, converges, and $D = (1-x)^{-2}$. Also see the next section.* ■

The assumptions in result (2.311) can be somewhat relaxed, though we won't require it. In particular, as proven, e.g., in Stoll (2001, pp. 340-1) and other analysis books:

Theorem: Let $\{f_n\}$ be a sequence of differentiable functions on $I = [a, b]$. If $f_n'(x) \rightrightarrows g(x)$ on $I$ and $\exists\, x_0 \in I$ such that $\{f_n(x_0)\}$ converges, then $f_n(x) \rightrightarrows f(x)$ on $I$, and, $\forall x \in I$, $f'(x) = g(x)$.

### 2.6.11 Power and Taylor Series

> I regard as quite useless the reading of large treatises of pure analysis: too large a number of methods pass at once before the eyes. It is in the works of applications that one must study them. (Joseph-Louis Lagrange)

Definition: A series of the form $\sum_{k=0}^{\infty} a_k x^k$ for sequence $\{a_k\}$ is a *power series* in $x$ with coefficients $a_k$. More generally, $S(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$ is a power series in $(x-c)$, where $c \in \mathbb{R}$.

Subsequently, we will occasionally refer to and use the "lim sup" of a sequence. We postpone its development until §3.3, but the interested reader can quickly read ahead, requiring only about two pages of material there, in order to proceed here. An alternative temporary approach is, in the below, replace lim sup with lim. The benefit of using lim sup (and lim inf) is that they always exist for all sequences, albeit possibly in the extended real line $\mathbb{X}$, whereas the usual limit may not exist.

Definition: With $f_k = a_k (x-c)^k$, the *exponential growth rate* of power series $S$ is

$$g(x) := \limsup |f_k|^{1/k} = |x-c| \limsup |a_k|^{1/k}, \tag{2.314}$$

having used (3.39) in the second equality. From the root test (2.251), $S$ converges absolutely if $g(x) < 1$ and diverges for $g(x) > 1$.

See (3.43) and the subsequent text for further discussion on this formulation.

Definition: The *radius of convergence* of $S$ is

$$R = 1/\limsup |a_k|^{1/k}. \tag{2.315}$$

If $\limsup |a_k|^{1/k} = \infty$, we take $R = 0$. If $\limsup |a_k|^{1/k} = 0$, we take $R = \infty$. When $R = 0$, the power series $\sum a_k (x-c)^k$ converges only for $x = c$. On the other hand, if $R = \infty$, then the power series converges for all $x \in \mathbb{R}$.

If $a_k \neq 0$ for all $k$ and $\lim_{k\to\infty} |a_{k+1}| / |a_k|$ exists, then from (3.42), the radius of convergence of $\sum a_k x^k$ is also given by

$$\frac{1}{R} = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}. \tag{2.316}$$

This formulation is particularly useful if the coefficients involve factorials.

Definition: Power series $S(x)$ converges if $g(x) = |x-c| R^{-1} < 1$, or

$$S(x) \textbf{ converges if } |x-c| < R\textbf{, and diverges if } |x-c| > R. \tag{2.317}$$

When working with series involving factorials, the following result becomes very useful: If $\lim_{k\to\infty} |a_{k+1}/a_k|$ exists, then, from (3.31), $\liminf |a_{k+1}/a_k| = \limsup |a_{k+1}/a_k|$ and, from (3.42), these equal $\limsup |a_k|^{1/k}$, so that the radius of convergence is $R = 1/\lim_{k\to\infty} |a_{k+1}/a_k|$.

**Example 2.99** *Consider the power series of the form*

$$S(x) = \sum_{k=0}^{\infty} a_k x^k, \quad \textit{where} \quad a_k = \frac{(-1)^k}{m^k (k!)^p} \quad \textit{and} \quad m, p \in \mathbb{R}_{>0}.$$

*As*

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{m^k (k!)^p}{m^{k+1} ((k+1)!)^p} = \lim_{k\to\infty} \frac{1}{m(1+k)^p} = 0,$$

*we have $R = \infty$.* ■

The next result relates power series and uniform convergence.

Theorem: If power series $S$ has radius of convergence $R > 0$, then, $\forall b \in (0, R)$,

$$S \textbf{ converges uniformly for all } x \textbf{ with } |x - c| \leq b. \tag{2.318}$$

> Proof: Choose $\epsilon > 0$ such that $b + \epsilon \in (b, R)$, which implies $\limsup |a_k|^{1/k} = R^{-1} < (b+\epsilon)^{-1}$. From (3.28), $\exists N \in \mathbb{N}$ such that, $\forall n \geq N$, $|a_k|^{1/k} < (b+\epsilon)^{-1}$, so that, $\forall n \geq N$ and $|x - c| \leq b$, $\left|a_k (x-c)^k\right| \leq |a_k| b^k < (b/(b+\epsilon))^k$. As $\sum_{k=1}^{\infty} (b/(b+\epsilon))^k < \infty$, the result follows from the Weierstrass $M$-test.

**Example 2.100** *In Example 2.98, the uniform convergence of $\sum_{k=1}^{\infty} kx^{k-1}$ was shown via the binomial theorem and the Weierstrass $M$-test. The following way is easier: As $\sum_{k=1}^{\infty} kx^{k-1} = \sum_{j=0}^{\infty} (j+1) x^j$, let $a_j = j+1$, so that, from (3.31) and a small extension of the first limit result in Example 2.16, $\limsup |a_j|^{1/j} = \lim_{j\to\infty} (j+1)^{1/j} = 1$, and $R = 1$. Thus, $\sum_{k=1}^{\infty} kx^{k-1}$ converges, from (2.317) with $c = 0$, for $x \in (-1, 1)$, and (2.318) implies that $\sum_{k=1}^{\infty} kx^{k-1}$ is uniformly convergent on $[-r, r]$ for each $r \in (0, 1)$.* ■

Theorem (Abel): **Suppose $S(x) = \sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R = 1$. If $\sum_{k=0}^{\infty} a_k < \infty$, then**

$$\lim_{x \to 1^-} S(x) = S(1) = \sum_{k=0}^{\infty} a_k.$$

See, e.g., Goldberg (1964, §9.6) or Stoll (2021, p. 380) for proof. Naturally, Abel's theorem can also be stated for general $c$ and $R > 0$.

**Example 2.101** *Let $S(x) = \sum_{k=1}^{\infty} (-1)^{k+1} x^k / k$. From Example 2.15, $\lim |1/k|^{1/k} = 1$. Results (3.31) and (2.315) then imply that the radius of convergence of $S$ is $R = 1$. From the alternating series test (*Dirichlet Test, *page 125), $S(1) = \sum_{k=1}^{\infty} (-1)^{k+1}/k$ is also convergent. Abel's theorem and (2.297) thus imply that*

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots. \tag{2.319}$$

*Another interesting method of proof of (2.319) is given in Loya (2017, §4.7.5), along with other expressions for it.* ■

**Example 2.102** *From Example 2.92,*

$$S(y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} y^{2n+1} = \arctan(t), \quad t \in [0, 1).$$

*Recall from (2.222) that $\arctan 1 = \pi/4$. From the alternating series test, $S(1)$ converges, so that, from Abel's theorem,*

$$\frac{\pi}{4} = \arctan(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

*giving a (rather inefficient) method of calculating pi.* ■

Of great use is the following result for termwise differentiation of power series. Recall the exponential growth rate of power series $S$, from (2.314); and the radius of convergence of $S$, from (2.315).

Theorem: Let $f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$ for $|x-c| < R$, where $R > 0$ is the radius of convergence of $f$. Then $d(x) = \sum_{k=1}^{\infty} k a_k (x-c)^{k-1}$ has radius of convergence $R$ and

$$f'(x) = d(x) \textbf{ for } x \textbf{ such that } |x-c| < R. \tag{2.320}$$

> Proof: Using the two limit results in Example 2.16, and, crucially, (3.39), the exponential growth rate of $d(x)$ is, for $x \neq c$,
>
> $$\begin{aligned} \limsup \left| k a_k (x-c)^{k-1} \right|^{1/k} &= \limsup |k|^{1/k} \limsup \left| (x-c)^{k-1} \right|^{1/k} \limsup |a_k|^{1/k} \\ &= 1 \cdot \limsup \left| \frac{(x-c)^k}{x-c} \right|^{1/k} \limsup |a_k|^{1/k} \\ &= \frac{|x-c|}{\lim_{k\to\infty} |x-c|^{1/k}} \limsup |a_k|^{1/k} = |x-c| \limsup |a_k|^{1/k}, \end{aligned}$$
>
> so that $d(x)$ has the same radius of convergence as does $f(r)$, namely $R$. That $f'(x) = d(x)$ for $|x-c| < R$ follows directly from the results in (2.318) and (2.311).

Thus, the result in Example 2.98 could have been obtained immediately via (2.320). It also implies the following.

Corollary: If $f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$ and $g(x) = \sum_{k=0}^{\infty} b_k (x-c)^k$ are power series with radius of convergence $R$ that are equal for $|x-c| < R$, then $a_k = b_k$ for $k = 0, 1, \ldots$.

> Proof: Note that $f, g \in \mathcal{C}^\infty$ so $f^{(n)}(x) = g^{(n)}(x)$ for $n \in \mathbb{N}$ and $|x-c| < R$. In particular,
>
> $$f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1) a_k (x-c)^{k-n},$$
>
> and, as $0^0 = 1$, $f^{(n)}(c) = n! a_n$. Thus, $n! a_n = f^{(n)}(c) = g^{(n)}(c) = n! b_n$ for $n = 0, 1, \ldots$, which implies that $a_n = b_n$ for $n = 0, 1, \ldots$.

Repeated use of (2.320) implies that $f \in \mathcal{C}^\infty(x-R, x+R)$, i.e., that $f$ is infinitely differentiable on $(x-R, x+R)$. The converse, however, does not hold, i.e., there exist functions in $\mathcal{C}^\infty(I)$ that cannot be expressed as a power series for particular $c \in I$. The ubiquitous example is to use $c = 0$ and the function given by $f(x) = \exp(-1/x^2)$ for $x \neq 0$ and $f(0) = 0$. See, e.g., Stoll, 2021, p. 385; and Ghorpade and Limaye, p. 387.

Definition: Let $I$ be an open interval. A function $f : I \to \mathbb{R}$ is said to be *analytic* in $I$ if, $\forall c \in I$, there exists a sequence $\{a_k\}$ in $\mathbb{R}$ and a $\delta > 0$ such that, $\forall x$ with $|x-c| < \delta$, $f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$. Thus, the class of analytic functions is a proper subset of $\mathcal{C}^\infty$.

Recall from (2.64) that, for a differentiable function $f$, $f(x+h) \approx f(x) + h f'(x)$, accurate for $h$ near zero, i.e., knowledge of a function and its derivative at a specified point, $x$, can be used to approximate the function at other points near $x$. By replacing $x$ with $c$ and then setting $h = x - c$, this can be written as

$$f(x) \approx T_1(x) := f(c) + (x-c) f'(c), \tag{2.321}$$

where $T_1$ is referred to as the first order Taylor polynomial, as given below in (2.324). The above use of "$\approx$" will be made more precise below, in (2.330) and (2.331).

For example, with $f(x) = e^x$ and $c = 0$, (2.321) reads $e^x \approx e^0 + xe^0 = 1 + x$, which is accurate for $x \approx 0$. When evaluated at $x = c$, (2.321) is exact, and taking first derivatives of both sides w.r.t. $x$ gives $f'(x) \approx f'(c)$, which is again exact at $x = c$. One might imagine that accuracy is improved if terms involving higher derivatives are taken into account. This is the nature of a *Taylor polynomial*, which was developed by Brooks Taylor, 1685–1731 (though variants were independently discovered by others, such as Gregory, Newton, Leibniz, Johann Bernoulli and de Moivre). It was only in 1772 that Joseph-Louis Lagrange (1736–1813) recognized the importance of the contribution, proclaiming it the basic principle of the differential calculus. Lagrange is also responsible for characterizing the error term. The first usage of the term Taylor series appears to be by Simon Lhuilier (1750–1840) in 1786.

Definition: Let $f : I \to \mathbb{R}$, where $I$ is an open interval, and let $c \in I$. If $f^{(n)}(x)$ exists for all $x \in I$, then the $n$th order *Taylor polynomial* of $f$ at $c$ is (using several common notations)

$$T_n(f,c)(x) = T_n(x; f, c) = T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x-c)^k, \tag{2.322}$$

and if $f \in \mathcal{C}^{\infty}(I)$, then the *Taylor series* of $f$ at $c$ is

$$T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k. \tag{2.323}$$

When $c = 0$, (2.323) is also referred to as the *Maclaurin series*, after Colin Maclaurin (1698–1746). The first three Taylor polynomials $T_0, T_1, T_2$, are given specifically by

$$\begin{aligned} T_0(x) &= T_0(f,c)(x) = f(c), \\ T_1(x) &= T_1(f,c)(x) = f(c) + f'(c)(x-c), \\ T_2(x) &= T_2(f,c)(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2. \end{aligned} \tag{2.324}$$

As in (2.321), and using the previous $T_i(x)$, $i = 0, 1, 2$, as cases in point, observe that $T_n(c) = f(c)$, $T_n'(c) = T_n'(x)|_{x=c} = f'(c)$, up to $T_n^{(n)}(c) = f^{(n)}(c)$, so that locally (i.e., for $x$ near $c$), $T_n(c)$ behaves similarly to $f(x)$. In applications, these are used for effective approximation of $f$.

Definition: The *remainder* between $f$ and $T_n(x)$ is defined as $R_n(x) := f(x) - T_n(x)$, and *Taylor's formula with remainder* is given by

$$f(x) = T_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x-c)^k + R_n(x). \tag{2.325}$$

Clearly, $f(x) = T(x)$ iff $\lim_{n\to\infty} R_n(x) = 0$.

Theorem: Let $f : I \to \mathbb{R}$, where $I$ is an open interval, and let $c \in I$. If $f^{(n+1)}(x)$ exists for all $x \in I$, then the *Lagrange form of the remainder* is

$$R_n(x) = \frac{f^{(n+1)}(\zeta)}{(n+1)!} (x-c)^{n+1}, \quad \zeta \text{ between } x \text{ and } c. \tag{2.326}$$

Recall (2.10): For $a \in \mathbb{R}$, $\lim_{n\to\infty} \frac{a^n}{n!} = 0$. This implies $\lim_{n\to\infty} R_n(x) = 0$.

Proof: (Bartle and Sherbert, 1st edition, 1983, p. 222) Assume $x \neq c$ and let $J = [x,c] \cup [c,x]$, i.e., $J = [x,c]$ if $x < c$, and $[c,x]$ if $c < x$. Then, for $t \in J$, let

$$P_n(t) := f(x) - f(t) - (x-t) f'(t) - \frac{(x-t)^2 f''(t)}{2!} \cdots - \frac{(x-t)^n f^{(n)}(t)}{n!}, \qquad (2.327)$$

with $P_n(x) = 0$. Then $P_1'(t) = -f'(t) - [(x-t) f''(t) + f'(t)(-1)] = -(x-t) f''(t)$, which can be written as $-(x-t)^n f^{(n+1)}(t)/n!$ for $n = 1$. Now use induction: assume this holds for $n-1$; then

$$\begin{aligned} P_n'(t) &= \frac{d}{dt}\left(P_{n-1}(t) - \frac{(x-t)^n f^{(n)}(t)}{n!}\right) \\ &= -\frac{(x-t)^{n-1} f^{(n)}(t)}{(n-1)!} - \frac{(x-t)^n f^{(n+1)}(t) + f^{(n)}(t)\, n\,(x-t)^{n-1}(-1)}{n!} \\ &= -\frac{(x-t)^n f^{(n+1)}(t)}{n!}. \end{aligned}$$

Now let

$$G(t) := P_n(t) - \left(\frac{x-t}{x-c}\right)^{n+1} P_n(c), \quad t \in J,$$

so that $G(c) = 0$ and $G(x) = P_n(x) = 0$. The mean value theorem then implies that there exists a $\zeta \in J$ (actually, the interior of $J$) such that

$$\frac{G(c) - G(x)}{c - x} = G'(\zeta),$$

so that $0 = G'(\zeta) = P_n'(\zeta) + (n+1)\frac{(x-\zeta)^n}{(x-c)^{n+1}} P_n(c)$. Thus,

$$\begin{aligned} P_n(c) &= -\frac{1}{n+1}\frac{(x-c)^{n+1}}{(x-\zeta)^n} P_n'(\zeta) = \frac{1}{n+1}\frac{(x-c)^{n+1}}{(x-\zeta)^n}\frac{(x-\zeta)^n f^{(n+1)}(\zeta)}{n!} \\ &= \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-c)^{n+1}, \end{aligned}$$

and (2.327) reads

$$\frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-c)^{n+1} = f(x) - f(c) - (x-c) f'(c) - \frac{(x-c)^2 f''(c)}{2!} \cdots - \frac{(x-c)^n f^{(n)}(c)}{n!},$$

as was to be shown.

Remarks:

1. This proof, like other variants of it, are somewhat "rabbit-out-of-the-hat", in the sense that it is not at all clear how one stumbles upon choosing $P_n(t)$ and $G(t)$. Such elegant proofs are just the result of concerted effort and much trial and error, and abound in mathematics, old and new. Indeed, referring to Gauss' style of mathematical proof, Niels Abel said that "He is like the fox, who effaces his tracks in the sand with his tail". In defense of his style, Gauss exclaimed that "no self-respecting architect leaves the scaffolding in place after completing

the building". As encouragement, Gauss also said "If others would but reflect on mathematical truths as deeply and continuously as I have, then they would also make my discoveries".

2. For fun, the reader can look at the "evolution" of the above proof, taken from Bartle and Sherbert (4th edition, 2011, p. 189). It is the same proof, but they shorten it and put more work on the reader. (I still have and cherish the first edition, having had it since my undergraduate studies and it having been my first exposure to the subject. The 4th edition contains further topics, and less typos.)

**Example 2.103** *Let $f(x) = \sin x$, so that, from the conditions in (2.88), $f'(x) = \cos x$ and $f''(x) = -\sin x$. Thus, $f^{(2n)}(x) = (-1)^n \sin x$ and $f^{(2n+1)}(x) = (-1)^n \cos x$, for $x \in \mathbb{R}$ and $n \in \mathbb{N} \cup 0$. As $\sin 0 = 0$ and $\cos 0 = 1$, the $n$th order Taylor polynomial for $c = 0$ is thus*

$$T_n(x) = 0 + x - 0 - \frac{1}{6}x^3 + 0 + \frac{1}{120}x^5 + \cdots = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

*As $|\sin x| \le 1$ and $|\cos x| \le 1$, the remainder satisfies $|R_n(x)| \le |x|^{n+1} / (n+1)!$, which goes to zero as $n \to \infty$. Thus,*

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1},$$

*which is its definition; see also Example 2.85.* ■

Another useful way of expressing (2.325) and (2.326) is (obtained by replacing $c$ with $x$, and $x$ with $x+h$, where $x \in I$ and $h$ is small enough such that the "perturbation" $x + h \in I$; and we also switch from $n$ to $k$, because we will refer to this formula in the multivariate section, and $n$ is reserved for something else)

$$f(x+h) - \left[ f(x) + f'(x)h + \cdots + \left(\frac{1}{k!}\right) f^k(x)h^k \right] = \frac{f^{k+1}(x+\theta h)}{(k+1)!} \cdot h^{k+1} \tag{2.328}$$

for some $0 < \theta < 1$, this being the analog of $\zeta$ between $x$ and $c$. Dividing by $h^k$ shows that

$$\lim_{h \to 0} \frac{f(x+h) - \left[ f(x) + f'(x)h + \cdots + (1/k!) f^k(x)h^k \right]}{h^k} = 0. \tag{2.329}$$

Observe that, for $n = 1$ in (2.325), we can express (2.321) as

$$f(x) = f(c) + f'(c)(x-c) + r(x), \quad r(x) = \frac{1}{2} f''(\zeta)(x-c)^2, \tag{2.330}$$

with (as $f''(\zeta)$ exists, by assumption)

$$\lim_{x \to c} \frac{r(x)}{|x-c|} = \frac{1}{2} f''(\zeta) \lim_{x \to c} \frac{(x-c)^2}{|x-c|} = \frac{1}{2} f''(\zeta) \lim_{x \to c} |x-c| = 0. \tag{2.331}$$

Thus, $f(c) + f'(c)(x-c)$ is an (affine, for $f(c) \neq 0$) linear approximation to $f(x)$ with the property that, not only is the error term $r(x)$ in (2.330) such that $\lim_{x \to c} r(x) = 0$, but also (2.331), i.e., the limit of $r(x)$ after dividing by the linear quantity that itself goes to zero, is zero. In this sense, it is the best linear approximation to the function $f$ at the point $c$. Representation (2.330) and (2.331) will be of use in the multivariate function case, when, in two dimensions, $T_1(x)$ in (2.322) will have two terms, one for each of the two input variables, and will represent the "best" affine two-dimensional plane approximation to a differentiable (defined in (5.34)) function, at a point $(x_0, y_0)$ in the interior of its domain.

Theorem: The remainder $R_n(x)$ can be expressed in integral form, provided $f^{(n+1)}(x)$ exists for each $x \in I$ and, $\forall a, b \in I$, $f^{(n+1)} \in \mathcal{R}[a,b]$. In particular,

$$R_n(x) = \frac{1}{n!} \int_c^x f^{(n+1)}(t)(x-t)^n \, dt, \quad x \in I.$$

This is proven in most all books on real analysis.

# 3 Selected Topics from Real Analysis

The previous §2 covered many of the most important topics in univariate calculus. Still, in a course in (univariate) real analysis, several other topics are covered that become very relevant when pursuing "next level" topics, including measure theory and the Lebesgue integral (as required in probability theory, among other areas), topology, and functional analysis. Given our stated goals at the beginning of this document regarding preparing students for various quantitative subjects, we cover in this section a selection of topics that are vital and prerequisite for learning (among other things) measure theory.

## 3.1 Denseness, Open and Closed Sets, Nested Interval Theorem

Definition: A set $S$ of real numbers is said to be *dense* in $\mathbb{R}$ provided that every interval $I = (a, b)$, where $a < b$, contains a member of $S$.

Theorem (Density of the Rationals): The set of rational numbers is dense in $\mathbb{R}$.

> Proof: See, e.g., Fitzpatrick, p. 15.

Lemma: The product of an irrational number and a rational number is irrational.

> Let $z$ be irrational and $x > 0$ rational, say $x = m/n$, $m, n \in \mathbb{N}$. If $zx$ were rational, then $zx = k/h$, for some $k, h \in \mathbb{N}$, which implies $z = k/(xh) = (k/h) \times (n/m)$, the latter expression being rational, but as $z$ is irrational, we must have $zx$ is also irrational.

Corollary (Density of the Irrationals): The set of irrational numbers is dense in $\mathbb{R}$.

> Proof: The density of the irrationals follows from the density of the rationals and the existence of positive irrational numbers. Indeed, given an interval $(a, b)$, choose any positive irrational number $z$; for instance, choose $z = \sqrt{2}$. By the density of the rationals there is a rational number $x$ in the interval $(a/z, b/z)$ so that $zx$ lies in the interval $(a, b)$ and $zx$ is irrational since it is the product of an irrational number and a rational number.

Proposition: A set $S$ is dense in $\mathbb{R}$ iff every number $x$ is the limit of a sequence in $S$.

> Proof: First, assume that the set $S$ is dense in $\mathbb{R}$. Fix a number $x$. Let $n$ be an index. By the denseness of $S$ in $\mathbb{R}$, there is a member of $S$ in the interval $(x, x + 1/n)$. Choose a member of $S$ that belongs to this interval and label it $s_n$. This defines a sequence $\{s_n\}$ that has the property that, $\forall n \in \mathbb{N}$, $|s_n - x| < 1/n$. Since the sequence $\{1/n\}$ converges to 0, it follows from the Squeeze Theorem (2.9) that $\{s_n\}$ converges to $x$, and, by the above choice, $\{s_n\}$ is a sequence in $S$.
>
> It remains to prove the converse. Suppose that the set $S$ has the property that every number is the limit of a sequence in $S$. We will show that $S$ is dense in $\mathbb{R}$. Indeed, consider an interval $(a, b)$. We must show that this interval contains a point of $S$. Consider the midpoint $s = (a + b)/2$ of the interval. By assumption, there is a sequence $\{s_n\}$ of points in $S$ that converges to $s$. Define $\epsilon \equiv (b - a)/2$. Then $\epsilon > 0$. By the definition of a convergent sequence, there is an index $N$ such that $s_n$ belongs to $(s - \epsilon, s + \epsilon)$ for each index $n \geq N$. However, $(s - \epsilon, s + \epsilon) = (a, b)$. The point $s_N$ belongs to $S$ and also belongs to $(a, b)$. Thus, $S$ is dense in $\mathbb{R}$.

Theorem (Sequential Density of the Rationals) Every number (in $\mathbb{R}$, notably the irrationals) is the limit of a sequence of rational numbers.

> Proof: We know from the theorem above that the set of rational numbers is dense in $\mathbb{R}$. By the preceding proposition, every number is the limit of a sequence of rational numbers.

We turn now to open and closed sets, beginning with the latter. The following definition will be augmented with a second, equivalent definition below.

Definition: A subset $S$ of $\mathbb{R}$ is said to be *closed* provided that, if $\{a_n\}$ is a sequence in $S$ that converges to a number $a$, then the limit $a$ also belongs to $S$.

All intervals of the form $[a, b]$ for $a \leq b$ are closed. The interval $(0, 1]$ is not closed (with respect to the "ambient space" $X = \mathbb{R}$) because $\{1/n\}$ is a sequence in this interval that converges to a point (in the ambient space $X = \mathbb{R}$) that does not belong to the interval.

The set $\mathbb{Q}$ of rational numbers is not closed since, by the sequential density of the rationals, there is a sequence $\{r_n\}$ of rational numbers that converges to the number $\sqrt{2}$, and $\sqrt{2}$ is not rational.

Definition: For any $\mathbf{x} \in \mathbb{R}^n$ and $r \in \mathbb{R}_{>0}$, the *open ball of radius* $r$ *around* $\mathbf{x}$ is the subset $B_r(\mathbf{x}) \subset \mathbb{R}^n$ with $B_r(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{y}\| < r\}$ (note the strict inequality), where, recalling (1.24), $\|\mathbf{x}\|$ is the *norm* of $\mathbf{x}$. We will also use the calligraphic $B$, i.e., $\mathcal{B}_r(\mathbf{x})$.

Definition: A *neighborhood* of a point $\mathbf{x} \in \mathbb{R}^n$ is a subset $A \subset \mathbb{R}^n$ such that there exists an $\epsilon > 0$ with $B_\epsilon(\mathbf{x}) \subset A$.

Definition: If, for some $r \in \mathbb{R}_{>0}$, the set $A \subset \mathbb{R}^n$ is contained in the ball $B_r(\mathbf{0})$, then $A$ is said to be *bounded*.

Definition: The subset $U \subset \mathbb{R}^n$ is *open in* $\mathbb{R}^n$ if, for every point $\mathbf{x} \in U$,

$$\exists r > 0 \text{ such that } B_r(\mathbf{x}) \subset U. \tag{3.1}$$

Theorem: A set $E$ is open if and only if its complement $E^c$ is closed.

> Proof:
> ($\Rightarrow$) Suppose $E$ is open. If $x_1, x_2, \ldots$ is a sequence of points in $E^c$, and $x_n \to x$, we must show $x \in E^c$. Assume, on the contrary, that $x \in E$. Since $E$ is open, there is an open ball $B_r(x)$ entirely contained in $E$, and since $x_n \to x$, we have $x_n \in B_r(x)$ for all sufficiently large $n$. But this contradicts the assumption that all $x_n$ belong to $\mathbb{E}^c$.
>
> ($\Leftarrow$) Now assume $E^c$ closed but $E$ not open. To say that $E$ is open means that every point $x \in E$ has an open ball $B_r(x) \subseteq \mathbb{E}$, so $E$ not open means that not every point in $E$ has this property. If $x$ is a "bad point" of $E$, then no open ball $B_r(x)$ is a subset of $E$; thus, for every $r > 0$, $B_r(x) \nsubseteq E$. This means we can find a point $y \in B_r(x)$ with $y \notin E$. Now if we take $r = 1/n$ and select $x_n \in B_r(x)$ with $x_n \notin E$, we have $d(x_n, x) < 1/n \to 0$, so $x_n \to x$. But $x_n \in E^c$, a closed set, so $x \in E^c$, contradicting the assumption that $x \in E$.

Note the contrapositive: $\{A \text{ open}\} \Leftrightarrow \{A^c \text{ closed}\} \iff \sim\{A \text{ open}\} \Leftrightarrow \sim\{A^c \text{ closed}\}$.

Based on the previous theorem, we can also take the following to be the definition of a closed set, which is indeed commonly seen in many real analysis books.

Definition: A set $C \subset \mathbb{R}^n$ is *closed* if its complement, $\mathbb{R}^n \setminus C$ is open.

We now turn to the famous Nested Interval Theorem. We base our proof from Fitzpatrick, Thm 2.29. See also Conway (2018, Theorem 1.6.8) and Duren (2012, p. 9) for fantastic presentations. Crucial in the theorem is that $I_n$ is closed. See Stoll, Example 3.3.4 for what happens if it is not closed.

Theorem (The Nested Interval Theorem): For each natural number $n$, let $a_n$ and $b_n$ be numbers such that $a_n < b_n$ and consider the interval $I_n \equiv [a_n, b_n]$. Assume that

$$I_{n+1} \subseteq I_n \quad \text{for every index } n. \tag{3.2}$$

Also assume that

$$\lim_{n\to\infty} [b_n - a_n] = 0. \tag{3.3}$$

Then there is exactly one point $x$ that belongs to the interval $I_n$ for all $n$, and both of the sequences $\{a_n\}$ and $\{b_n\}$ converge to this point.

> Proof: Assumption (3.2) means precisely that, for every index $n$,
>
> $$a_n \leq a_{n+1} < b_{n+1} \leq b_n.$$
>
> In particular, the sequence $\{a_n\}$ is a monotonically increasing sequence that is bounded above by $b_1$. The Monotone Convergence Theorem implies that the sequence $\{a_n\}$ converges to a number $a$ and that $a_n \leq a$ for every index $n$. A similar argument shows that the monotonically decreasing sequence $\{b_n\}$ converges to a number that we denote by $b$ such that $b \leq b_n$ for every index $n$. Thus,
>
> $$a_n \leq a \text{ and } b \leq b_n \text{ for every index } n. \tag{3.4}$$
>
> From assumption (3.3) and the difference property of convergent sequences, we conclude that
>
> $$0 = \lim_{n\to\infty} [b_n - a_n] = b - a.$$
>
> Thus, $a = b$. Setting $x = a = b$, it follows from (3.4) that the point $x$ belongs to $I_n$ for every natural number $n$. There can be only one such point since the existence of two such points would contradict the assumption (3.3) that the lengths of the intervals converge to 0.

Theorem: That $[0, 1]$ is uncountable follows from the Nested Intervals Theorem.

> Proof: Suppose $[0, 1]$ is countable. Then we can enumerate it as $[0, 1] = \{x_1, x_2, \ldots\}$. Let $I_1 \subset [0, 1]$ be a closed interval such that $x_1 \notin I_1$; let $I_2 \subset I_1$ be a closed interval such that $x_2 \notin I_2$; etc., i.e., for each $n \in \mathbb{N}$, let $I_n \subset I_{n-1}$ be a closed interval such that $x_n \notin I_n$. Observe that, $\forall n \in \mathbb{N}$, $I_n \subset [0, 1]$ is bounded. Thus, we obtain the sequence of nested closed and bound intervals $I_1 \supset I_2 \supset \cdots$, but by the nested intervals theorem, $\cap_{n=1}^{\infty} \neq \emptyset$. Thus, $\exists m \in \mathbb{N}$ such that $x_m \in I_n$ for every $n \in \mathbb{N}$. But, by construction, $x_m \notin I_n$ for all $n \geq m$. Therefore, by contradiction, $[0, 1]$ is not countable.

For the result (3.5) below, we need the definition of interior. We include also a few other definitions of interest. Section 5.1 will repeat these definitions, and present several others that we will require there.

Definition: The point $\mathbf{x} \in A \subset \mathbb{R}^n$ is an *interior point* of $A$ if $\exists r > 0$ such that $B_r(\mathbf{x}) \subset A$.

Definition: The *interior* of $A$ is the set of all interior points of $A$, denoted $A^o$ or $\mathrm{int}(A)$. Observe that the biggest open set contained in any set $A \subset \mathbb{R}^n$ is $A^o$.

Definition: The smallest closed set that contains $A$ is the *closure* of $A$, denoted $\overline{A}$; it is the set of $\mathbf{x} \in \mathbb{R}^n$ such that, $\forall r > 0$, $B_r(\mathbf{x}) \cap A \neq \emptyset$.

Definition: A *deleted (or punctured) neighborhood* of $\xi$ is an interval $(a, b)$ with the point $\xi$, $a < \xi < b$, removed.

Definition: A number $\xi$ is a *cluster, limit, or accumulation point* of a set $S \subset \mathbb{R}$ if each deleted neighborhood of $\xi$ contains a point of $S$.

Thus, $\xi$ is a cluster point of $S$ if for each $\varepsilon > 0$ there exists a point $x \in S$ such that $0 < |x - \xi| < \varepsilon$. Note that a finite set cannot have any cluster points. Also an unbounded infinite set need not have any cluster points; consider for example the set $\mathbb{N}$ of positive integers.

A cluster point of a set need not belong to the set. For instance, 0 is a cluster point of the set $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ but $0 \notin S$. Observe also that if $\xi$ is a cluster point of a set $S$, then every neighborhood of $\xi$ contains infinitely many points of $S$.

We now state some well-known results whose proofs are in all real analysis books.

Theorem (Distributive Laws): If $E_\alpha, \alpha \in A$, and $E$ are subsets of $a$ set $X$, then

$$E \bigcap \left( \bigcup_{\alpha \in A} E_\alpha \right) = \bigcup_{\alpha \in A} (E \cap E_\alpha), \quad E \bigcup \left( \bigcap_{\alpha \in A} E_\alpha \right) = \bigcap_{\alpha \in A} (E \cup E_\alpha).$$

Theorem (De Morgan's Laws): If $\{E_\alpha\}_{\alpha \in A}$ is a family of subsets of $X$, then

$$\left( \bigcup_{\alpha \in A} E_\alpha \right)^c = \bigcap_{\alpha \in A} E_\alpha^c, \quad \left( \bigcap_{\alpha \in A} E_\alpha \right)^c = \bigcup_{\alpha \in A} E_\alpha^c.$$

Theorem: Let $X$ be a set. Then

1. for any collection $\{O_\alpha\}_{\alpha \in A}$ of open subsets of $X$,
$$\bigcup_{\alpha \in A} O_\alpha \text{ is open,} \tag{3.5}$$
2. for any finite collection $\{O_1, \ldots, O_n\}$ of open subsets of $X$, $\bigcap_{j=1}^n O_j$ is open.

> Proof of (1): Let $A$ be any index set and let $O := \cup_{\alpha \in A} O_\alpha$ where, for each $\alpha \in A$, $O_\alpha$ is an open set of real numbers. If $x \in O$, then $x \in O_{\alpha_0}$ for some $\alpha_0 \in A$. As $O_{\alpha_0}$ is open, there is an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset O_{\alpha_0} \subset O$. Therefore, every point of the union is an interior point and thus, by definition, $O$ is open.

Theorem: Let $X$ be a set. Then

1. for any collection $\{F_\alpha\}_{\alpha \in A}$ of closed subsets of $X$, $\bigcap_{\alpha \in A} F_\alpha$ is closed, and
2. for any finite collection $\{F_1, \ldots, F_n\}$ of closed subsets of $X$, $\cup_{j=1}^n F_j$ is closed.

Proof of (1): Let $F_\alpha$ be a closed set of real numbers for each $\alpha$ in some index set $A$, and let $F = \cap_{\alpha \in A} F_\alpha$. Set $F$ is closed if $F^c$ is open. To see the latter, DeMorgan's rule implies $F^c = \cup_{\alpha \in A} F_\alpha^c$. This is open, as a union of open sets, so $F^c$ is open.

As examples of a countable intersection of open sets that is not open, and a countable union of closed sets that is not closed, consider

$$\{1\} = \cap_{j=1}^{\infty} (1 - 1/j, 1 + 1/j); \quad \text{and} \quad \cup_{j=2}^{\infty} [1/j, 1 - 1/j] = (0, 1).$$

Theorem (Characterization of the Open Subsets of $\mathbb{R}$): If $U$ is an open subset of $\mathbb{R}$, then there exists a finite or countable collection $\{I_n\}$ of pairwise disjoint open intervals such that $U = \bigcup_n I_n$.

Proofs can be found in most all real analysis books, e.g., Stoll (2021, p. 65) and Terrell (2019, p. 94).

## 3.2 Introduction to Metric Spaces

Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ be any points in $\mathbb{R}^n$, $n \in \mathbb{N}$. As in (1.24), let $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$ denote the Euclidean norm of $\mathbf{x}$. Recall the Cauchy-Schwarz inequality (1.22): For any $n \in \mathbb{N}$ and any points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$|x_1 y_1 + \cdots + x_n y_n| \le (x_1^2 + \cdots + x_n^2)^{1/2} \, (y_1^2 + \cdots + y_n^2)^{1/2}, \tag{3.6}$$

which can also be written as in (4.20), namely $|\mathbf{x} \cdot \mathbf{y}| \le \|\mathbf{x}\| \, \|\mathbf{y}\|$. Also recall the triangle inequality (1.23): $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$.

Definition: Let $p \ge 1$ be a real number and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. We define

$$\|\mathbf{x}\|_p := \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}. \tag{3.7}$$

For $p = 2$ we find

$$\|\mathbf{x}\|_2 = \left( \sum_{i=1}^{n} |x_i|^2 \right)^{1/2},$$

and therefore $\|\mathbf{x} - \mathbf{y}\|_2$ is the Euclidean distance of $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. We also have

$$\|\lambda \mathbf{x}\|_p = \left( \sum_{i=1}^{n} |\lambda x_i|^p \right)^{1/p} = |\lambda| \|\mathbf{x}\|_p, \quad \lambda \in \mathbb{R}, \tag{3.8}$$

and

$$\|\mathbf{x}\|_p \ge 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n \text{ and } \|\mathbf{x}\|_p = 0 \text{ if and only if } \mathbf{x} = \mathbf{0} \in \mathbb{R}^n. \tag{3.9}$$

We now prove three famous inequalities, referred to Young's, Hölder's, and Minkowski's, that are fundamental in analysis, notably in metric space theory and functional analysis. The latter two generalize the above Cauchy-Schwarz and triangle inequalities. Many books cover this material, notably with more general presentations that also apply to integral expressions. Our presentation is from Dzung Minh Ha, Functional Analysis: A Gentle Introduction, 2006 (reprinted with corrections 2023).

Lemma: Let $0 < \lambda < 1$. Then

$$t^\lambda \le 1 - \lambda + \lambda t \quad \text{for all } t \ge 0. \tag{3.10}$$

This inequality becomes an equality only when $t = 1$.

Proof: Define $f : [0, \infty) \to \mathbb{R}$ by $f(t) := 1 - \lambda + \lambda t - t^\lambda$. Then

$$f'(t) = \lambda - \lambda t^{\lambda - 1} = \lambda\left(1 - \frac{1}{t^{1-\lambda}}\right)$$

Thus,

$$f'(t) \begin{cases} > 0, & \text{if } t > 1, \\ < 0, & \text{if } 0 < t < 1. \end{cases}$$

Consequently, $0 = f(1)$ is the minimum value of $f$. Therefore, $f(t) \ge 0$ for all $t \ge 0$, with equality if and only if $t = 1$. Hence,

$$\begin{aligned} t^\lambda &\le 1 - \lambda + \lambda t \quad && \text{for all } t \ge 0, \\ t^\lambda &= 1 - \lambda + \lambda t && \text{if and only if } t = 1. \end{aligned}$$

Definition (Conjugate exponent): Positive real numbers $p, q$ such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

are called *conjugate exponents*. The pair $1, \infty$ is also considered to be a pair of conjugate exponents, since $p \to 1$ implies $q \to \infty$.

If $p, q$ are integers, the only pair of conjugate exponents is $p = q = 2$.

We begin with Young's inequality, which we have already seen and proved in (2.152).

Theorem (Young's inequality): Let $p, q$ be conjugate exponents with $1 < q < \infty$. Then

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \quad \text{for all } a, b \ge 0. \tag{3.11}$$

Equality holds if and only if $a^p = b^q$.

Proof: Assume $p, q$ are as above and $a, b \ge 0$. If $b = 0$, there is nothing to prove. Assume $b > 0$. In (3.10), substitute $\frac{1}{p} < 1$ for $\lambda$ and $a^p b^{-q}$ for $t$ to obtain

$$ab^{-\frac{q}{p}} = \left(a^p b^{-q}\right)^{\frac{1}{p}} \le 1 - \frac{1}{p} + \frac{1}{p} a^p b^{-q}.$$

So

$$ab^{-\frac{q}{p}} \le \frac{1}{p} a^p b^{-q} + \frac{1}{q}.$$

Therefore,

$$ab^{-\frac{q}{p}+q} \le \frac{a^p}{p} + \frac{b^q}{q}.$$

But $-\frac{q}{p} + q = 1$, so

$$ab \le \frac{a^p}{p} + \frac{b^q}{q},$$

with equality if and only if $1 = t = a^p b^{-q}$, i.e., $a^p = b^q$.

Theorem (Hölder's inequality): Let $p, q$ be conjugate exponents with $1 < q < \infty$. For any integer $n \geq 1$, assume that $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are nonnegative. Then

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}. \tag{3.12}$$

For $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, and using (3.7), this can also be expressed as

$$\sum_{i=1}^{n} |x_i y_i| \leq \|\mathbf{x}\|_p \, \|\mathbf{y}\|_q. \tag{3.13}$$

Proof: Let $p, q, n, a_1, \ldots, a_n, b_1, \ldots, b_n$ be as above. Let

$$A := \left(\sum_{k=1}^{n} a_k^p\right)^{1/p}, \quad B := \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}.$$

If $AB = 0$, clearly (3.12) is satisfied, so assume $AB > 0$. Observe that

$$\sum_{k=1}^{n} \frac{a_k^p}{A^p} = 1 = \sum_{k=1}^{n} \frac{b_k^q}{B^q}$$

Next, apply Young's inequality (3.11) to get

$$\frac{a_k}{A}\frac{b_k}{B} \leq \frac{a_k^p}{pA^p} + \frac{b_k^q}{qB^q} \quad \text{for all } k = 1, \ldots, n.$$

Thus we have

$$\sum_{k=1}^{n} \frac{a_k b_k}{AB} \leq \frac{1}{p} + \frac{1}{q} = 1,$$

and, hence,

$$\sum_{k=1}^{n} a_k b_k \leq AB = \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}.$$

Theorem (Minkowski's inequality): Let $p \geq 1$ and $n \in \mathbb{N}$. One expression is to let $a_1, \ldots, a_n \geq 0$ and $b_1, \ldots, b_n \geq 0$. Then

$$\left(\sum_{k=1}^{n} (a_k + b_k)^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}. \tag{3.14}$$

Another (more common) expression is for $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$. Then

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p}, \tag{3.15}$$

or

$$\|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p. \tag{3.16}$$

Proof: First, for $p = 1$ and using the notation in (3.16), the triangle inequality implies

$$\|\mathbf{x}+\mathbf{y}\|_1 = \sum_{i=1}^{n} |x_i + y_i| \le \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1.$$

Now assume that $p > 1$. It suffices to prove (3.14), because, from the triangle inequality, $|x_k + y_k| \le |x_k| + |y_k|$, so that

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1/p} \le \left(\sum_{k=1}^{n} (|x_k| + |y_k|)^p\right)^{1/p}.$$

We can also assume that at least one of the $a_j$ or at least one of the $b_j$ is nonzero. Consequently, we assume $\sum_{k=1}^{n} (a_k + b_k)^p \ne 0$. Let $q$ be the conjugate exponent of $p$. Then by Hölder's inequality (3.12),

$$\begin{aligned}
&\sum_{k=1}^{n} (a_k + b_k)^p = \sum_{k=1}^{n} (a_k + b_k)(a_k + b_k)^{p-1} \\
&= \sum_{k=1}^{n} a_k (a_k + b_k)^{p-1} + \sum_{k=1}^{n} b_k (a_k + b_k)^{p-1} \\
&\underbrace{\le}_{\text{Hölder}} \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} (a_k + b_k)^{(p-1)q}\right)^{\frac{1}{q}} + \left(\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} (a_k + b_k)^{(p-1)q}\right)^{\frac{1}{q}}.
\end{aligned}$$

But we also have $(p-1)q = p$. Thus,

$$\sum_{k=1}^{n} (a_k + b_k)^p \le \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} (a_k + b_k)^p\right)^{\frac{1}{q}} + \left(\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} (a_k + b_k)^p\right)^{\frac{1}{q}}.$$

Dividing by $\left(\sum_{k=1}^{n} (a_k + b_k)^p\right)^{\frac{1}{q}} \ne 0$, we get

$$\left(\sum_{k=1}^{n} (a_k + b_k)^p\right)^{1-\frac{1}{q}} \le \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}},$$

i.e.,

$$\left(\sum_{k=1}^{n} (a_k + b_k)^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}}.$$

Definition: Let $X$ be a nonempty set. A real valued function $d$ defined on $X \times X$ satisfying

1. $d(x, y) \ge 0$ for all $x, y \in X$,
2. $d(x, y) = 0 \iff x = y$,
3. $d(x, y) = d(y, x)$,
4. $d(x, y) \le d(x, z) + d(z, y)$ for all $x, y, z \in X$,

is called a metric on $X$. The set $X$ with metric $d$ is called a metric space, and is denoted by $(X, d)$. When the context is clear, sometimes one refers just to metric space $X$, with distance measure $d$ understood, or its choice is not relevant. The canonical example is $X = \mathbb{R}$ with the metric $d(x, y) := |x - y|$.

Remarks:

- Interestingly, the first property above, namely $d(x, y) \geq 0$ for all $x, y \in X$, is redundant. Note that, for $x, y \in X$, $0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y)$, implying $d(x, y) \geq 0$.

- Let (the "ambient set" be) $X = (0, 1)$. Then, as the empty set is (vacuously) open, its complement, $X$, is closed. This does not violate the definition with sequence $a_n = 1/n$, $n \in \mathbb{N}$, because the definition (given above in §3.1) says, "if $\{a_n\}$ is a sequence in $S$ *that converges to a number* $a$". In this case, $\{a_n\}$ in (metric) space $X$ does not converge. In general, for a metric space $X$, the set $X$ is closed.

- Let $(X, d)$ be a metric space and $\emptyset \neq Y \subset X$ be a subset. Define $d_Y : Y \times Y \to \mathbb{R}$, with $(x, y) \mapsto d_Y(x, y) := d(x, y)$. Then $(Y, d_Y)$ is again a metric space. The interpretation is that if we can define the distance for every pair of points in $X$, we can of course restrict this distance to subsets and the restriction defines a distance on this subset.

- Recall the discussion of denseness in §3.1. Various equivalent definitions exist for a set to be dense, and some refer to metric spaces. From, e.g., Lindstrøm, Definition 4.8.1 (or 3.7.1): Let $(X, d)$ be a metric space and assume that $A$ is a subset of $X$. We say that $A$ is dense in $X$ if, for each $x \in X$, there is a sequence from $A$ converging to $x$.

  Equivalently, Lindstrøm, p. 63: A subset $D$ of a metric space $X$ is dense if, for all $x \in X$ and all $\epsilon > 0$, there is an element $y \in D$ such that $d(x, y) < \epsilon$. Or his p. 111: A subset $A$ of a metric space $(X, d)$ is dense if and only if all open balls $B(a, r)$, $a \in X$, $r > 0$, contain elements from $A$.

  Finally: A subset $D$ of a metric space $X$ is dense in $X$ if the closure $\bar{D} = X$. See, e.g., Stoll (2021, p. 65) or Heil (2019, Introduction to Real Analysis, p. 17).

Lemma (Metric Spaces Are Hausdorff): If $X$ is a metric space and $x \neq y$ are two distinct elements of $X$, then there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $y \in V$.

> Proof: Suppose that $x \neq y$, and let $r = d(x, y)/2$. If $z \in B_r(x) \cap B_r(y)$, then, by the Triangle Inequality,
>
> $$d(x, y) \leq d(x, z) + d(z, y) < 2r = d(x, y),$$
>
> which is a contradiction. Therefore $B_r(x) \cap B_r(y) = \varnothing$. Since open balls are open sets, the proof is finished by taking $U = B_r(x)$ and $V = B_r(y)$.

We will see below other useful metrics than the canonical $X = \mathbb{R}$ and $d(x, y) := |x - y|$, or its extension to $\mathbb{R}^n$ and use of the Euclidean norm. Now we discuss a metric that is not usually used in practice, but serves to help understand the theory and forces us to not think always in terms of the Euclidean distance. This is the "trivial", or "discrete" metric, namely: Let $X$ be any nonempty set. For $p, q \in X$, $d(p, q) = 1$ for $p \neq q$, and $d(p, q) = 0$ if $p = q$. Let $E \subset X$ be an arbitrary subset of $X$. For $x_0 \in E$ and $0 < r \leq 1$, $B_r(x_0) = \{x_0\} \subset E$. So, by definition, every element of $E$ is an interior point of $E$.

To show $\partial E = \emptyset$, assume $b \in \partial E$ and $0 < r \le 1$. Then $B_r(b) = \{b\}$, and either $b \in E$ or $b \in E^c$, and thus $b$ cannot be a boundary point. As every element of $E$ is an interior point of $E$, and $\partial E = \emptyset$, $x \in E^c$ must be an exterior point of $E$.

We next show that all subsets of $X$ are both open and closed. Recall that $E$ is open if every point of $E$ is an interior point of $E$. Equivalently (see, e.g., Tao II, Def 1.2.12), subset $E \subset X$ is open if it contains none of its boundary points. Both definitions, along with the previous results, show that $E$ is open. As $E$ was any subset of $X$, all subsets of $X$ are open. As $E^c \in X$, $E^c$ is open, which by definition implies $E$ is closed. Thus, all subsets of $X$ are both open and closed.

We can also argue as follows. For $x_0 \in E$ and $0 < r \le 1$, $B_r(x_0) = \{x_0\}$. Thus, $E$ has no limit points. (All its points are *isolated points.*) Recall that a subset $F$ of a metric space $X$ is closed if and only if $F$ contains all its limit points. With the discrete metric, this is vacuously fulfilled, and thus $E$ is closed. As $E^c$ is also a subset of $X$ and is thus closed, $E$ is open, so that, as above, all subsets of $X$ are both open and closed.

The remainder of our presentation is based on Jacob and Evans, 2016, A Course in Analysis, Vol. II, chapter 1.

**Example 3.1** *Let $\varphi : \mathbb{R} \to \mathbb{R}_+$be an even function such that $\varphi(x) = 0$ if and only if $x = 0$. Further we assume that $\varphi$ is sub-additive, i.e. for all $x, y \in \mathbb{R}$ we have*

$$\varphi(x + y) \le \varphi(x) + \varphi(y). \tag{3.17}$$

*If we define*

$$d_\varphi(x, y) := \varphi(x - y), \tag{3.18}$$

*then $d_\varphi$ is a metric on $\mathbb{R}$, as now shown. From our assumptions it follows immediately that $d_\varphi(x, y) \ge 0$ and $d_\varphi(x, y) = 0$ if and only if $x = y$, as well as (recall $\varphi$ is even)*

$$d_\varphi(x, y) = \varphi(x - y) = \varphi(-(y - x)) = \varphi(y - x) = d_\varphi(y, x).$$

*Using sub-additivity we find also the triangle inequality*

$$\begin{aligned} d_\varphi(x, z) &= \varphi(x - z) = \varphi(x - y + y - z) \\ &\le \varphi(x - y) + \varphi(y - z) = d_\varphi(x, y) + d_\varphi(y, z). \end{aligned}$$

*Thus, $d_\varphi$ is a metric on $\mathbb{R}$.* ■

**Example 3.2** *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$be a monotone increasing function such that $\psi(x) = 0$ if and only if $x = 0$. Moreover assume that $\psi$ has a continuous derivative on $\mathbb{R}_+$ ($\psi'(0)$ is considered as one-sided derivative) which is monotone decreasing. For $0 \le x < y$ we get, with $s = t - x$,*

$$\psi(x + y) - \psi(x) = \int_x^{x+y} \psi'(t)dt = \int_0^y \psi'(s + x)ds \le \int_0^y \psi'(s)ds = \psi(y),$$

*or $\psi(x + y) \le \psi(x) + \psi(y)$. We now show that $d_\psi(x, y) := \psi(|x - y|)$ is a metric on $\mathbb{R}$. Clearly $d_\psi(x, y) \ge 0$ and $d_\psi(x, y) = 0$ if and only if $x = y$, as well as $d_\psi(x, y) = d_\psi(y, x)$. The triangle inequality follows because*

$$\begin{aligned} d_\psi(x, z) &= \psi(|x - z|) = \psi(|x - y + y - z|) \\ &\le \psi(|x - y| + |y - z|) \le \psi(|x - y|) + \psi(|y - z|) \\ &= d_\psi(x, y) + d_\psi(y, z). \end{aligned}$$

*Examples are* $\psi_1(t) = \arctan t$ *or* $\psi_2(s) = \ln(1+s)$*, noting that*

$$\psi_1'(t) = \frac{1}{1+t^2} \quad \textit{and} \quad \psi_2'(s) = \frac{1}{1+s}. \qquad \blacksquare$$

Lemma: For a metric $d$ on $X$ we have for all $x, y, z \in X$ that

$$|d(x,z) - d(z,y)| \le d(x,y). \tag{3.19}$$

> Proof: The triangle inequality yields together with the symmetry of $d$
>
> $$d(x,z) \le d(x,y) + d(y,z)$$
>
> or
>
> $$d(x,z) - d(z,y) \le d(x,y) \tag{3.20}$$
>
> as well as
>
> $$d(z,y) \le d(z,x) + d(x,y)$$
>
> or
>
> $$-(d(x,z) - d(z,y)) \le d(x,y),$$
>
> which together with (3.20) gives (3.19).

Definition: Let $V$ be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A norm $\|\cdot\|$ on $V$ is a mapping $\|\cdot\| : V \to \mathbb{R}$, denoted $x \mapsto \|x\|$, with the properties

1. $\|x\| = 0$ if and only if $x = 0$;
2. $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in V$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$);
3. $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in V$.

If $\|\cdot\|$ is a norm on $V$, we call $(V, \|\cdot\|)$ a normed (vector) space.

Since $0 = \|x - x\| \le \|x\| + \|x\| = 2\|x\|$, it follows that $\|x\| \ge 0$ for all $x \in V$. Observe that a norm is defined on a vector space $V$. A subset $Y$ of a vector space need not be a vector space. So, we can restrict a norm to $Y$, but the restriction will not, in general, be a norm on $Y$. Condition (2) is called the homogeneity of the norm and (3) is also referred to as the triangle inequality.

Proposition: Let $(V, \|\cdot\|)$ be a normed space. Then $d(x,y) := \|x - y\|$ defines a metric on $V$.

> Proof: Obviously we have $\|x - y\| = d(x,y) \ge 0$ and $0 = d(x,y) = \|x - y\|$ if and only if $x = y$. Moreover we find
>
> $$d(x,y) = \|x - y\| = \|-(y - x)\| = |-1|\|y - x\| = d(y,x).$$
>
> Furthermore, for $x, y, z \in V$ we have
>
> $$\begin{aligned} d(x,y) &= \|x - y\| = \|x - z + z - y\| \\ &\le \|x - z\| + \|z - y\| = d(x,z) + d(z,y). \end{aligned}$$

Proposition: For $1 \le p < \infty$, $\|\mathbf{x}\|_p$, as given in (3.7), is a norm on $\mathbb{R}^n$.

Proof: This follows from (3.8), (3.9), and (3.16).

Definition: For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define

$$\|\mathbf{x}\|_\infty := \max_{1 \le j \le n} \{|x_j|\}. \tag{3.21}$$

Proposition: $\|\mathbf{x}\|_\infty$ is a norm on $\mathbb{R}^n$.

Proof: Indeed $\|\mathbf{x}\|_\infty = 0$ holds if and only if $x_j = 0$ for $j = 1, \ldots, n$, i.e., $\mathbf{x} = \mathbf{0}$. To show homogeneity,

$$\|\lambda \mathbf{x}\| = \max_{1 \le j \le n} \{|\lambda x_j|\} = \max_{1 \le j \le n} \{|\lambda|\,|x_j|\} = |\lambda| \max_{1 \le j \le n} \{|x_j|\} = |\lambda| \|\mathbf{x}\|_\infty.$$

Finally, for the triangle inequality, with $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$,

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\|_\infty &= \max_{1 \le j \le n} \{|x_j + y_j|\} \le \max_{1 \le j \le n} \{|x_j| + |y_j|\} \\ &\le \max_{1 \le j \le n} \{|x_j|\} + \max_{1 \le j \le n} \{|y_j|\} = \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty. \end{aligned}$$

Proposition: Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then

$$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 = \left( \sum_{j=1}^n |x_j|^2 \right)^{1/2} \le \sqrt{n} \|\mathbf{x}\|_\infty, \tag{3.22}$$

$$\|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1 = \sum_{j=1}^n |x_j| \le \left( \sum_{j=1}^n 1 \right)^{1/2} \left( \sum_{j=1}^n |x_j|^2 \right)^{1/2} \le \sqrt{n} \|\mathbf{x}\|_2, \tag{3.23}$$

and

$$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_1 \le n \|\mathbf{x}\|_\infty. \tag{3.24}$$

Proof: (3.22) is easy to see. For the first inequality in (3.23), note that $a^2 + b^2 \le (|a| + |b|)^2$; and for the second inequality, this follows from the Cauchy-Schwarz inequality (3.6). For (3.24), the first inequality comes from combining (3.22) and (3.23); and the second inequality is because $\sum_{j=1}^n |x_j| \le n \max_{1 \le j \le n} \{|x_j|\}$.

## 3.3 Lim Inf and Lim Sup (for Sequences and Sets)

We begin with the lim inf and lim sup for sequences of real numbers, and then cover them *for sets.*

Let $\{s_n\}$ be a sequence. For each $k \in \mathbb{N}$, define the two sequences $a_k = \inf\{s_n : n \ge k\}$ and $b_k = \sup\{s_n : n \ge k\}$. A bit of thought confirms that $a_k$ is increasing and $b_k$ is decreasing. Thus, if they are bounded, then they converge to a value in $\mathbb{R}$, and if they are not bounded, then they diverge to plus or minus $\infty$. Either way, the two sequences have limits in $\mathbb{X}$.

Definition: The *limit supremum* (or *limit superior*) of $\{s_n\}$, denoted $\limsup s_n$, is

$$\limsup_{n \to \infty} s_n = \lim_{k \to \infty} b_k = \lim_{k \to \infty} \left( \sup_{n \ge k} s_n \right), \tag{3.25}$$

and the *limit infimum* (or *limit inferior*), denoted $\liminf s_n$, is

$$\liminf_{n\to\infty} s_n = \lim_{k\to\infty} a_k = \lim_{k\to\infty} \left( \inf_{n\geq k} s_n \right). \tag{3.26}$$

Because $a_k$ is increasing, and $b_k$ is decreasing, it follows that

$$\liminf_{n\to\infty} s_n = \sup_{k\in\mathbb{N}} \inf_{n\geq k} s_n \quad \text{and} \quad \limsup_{n\to\infty} s_n = \inf_{k\in\mathbb{N}} \sup_{n\geq k} s_n.$$

Theorem: Let $\{s_n\}$ be a sequence of real numbers. Then

$$u = \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n = v. \tag{3.27}$$

Proof: For each $k \in \mathbb{N}$, $a_k = \inf\{s_n : n \geq k\} \leq \sup\{s_n : n \geq k\} = b_k$. The result now follows from definitions (3.25), (3.26), and limit result (2.14).

Theorem: Let $\{s_n\}$ be a sequence. Then, for lim sup:

1. $\limsup s_n = -\infty$ iff $\lim_{n\to\infty} s_n = -\infty$.
2. $\limsup s_n = \infty$ iff, $\forall M \in \mathbb{R}$ and $n \in \mathbb{N}$, $\exists k \in \mathbb{N}$ with $k \geq n$ such that $s_k \geq M$.[25]
3. Suppose $\limsup s_n \in \mathbb{R}$ (i.e., it is finite). Then, $\forall \epsilon > 0$, $U = \limsup s_n$ iff

$$\exists N \in \mathbb{N} \text{ such that, } \forall n \geq N,\ s_n < U + \epsilon, \tag{3.28}$$

and

$$\text{Given } n \in \mathbb{N},\ \exists k \in \mathbb{N} \text{ with } k \geq n \text{ such that } s_k > U - \epsilon. \tag{3.29}$$

Theorem: Let $\{s_n\}$ be a sequence. Then, for lim inf:

1. $\liminf s_n = \infty$ iff $\lim_{n\to\infty} s_n = \infty$.
2. $\liminf s_n = -\infty$ iff, $\forall M \in \mathbb{R}$ and $n \in \mathbb{N}$, $\exists k \in \mathbb{N}$ with $k \geq n$ such that $s_k \leq M$.
3. Suppose $\liminf s_n \in \mathbb{R}$. Then, given any $\epsilon > 0$, $L = \liminf s_n$ iff

$$\exists N \in \mathbb{N} \text{ such that, } \forall n \geq N,\ s_n > L - \epsilon, \tag{3.30}$$

and

$$\text{Given } n \in \mathbb{N},\ \exists k \in \mathbb{N} \text{ with } k \geq n \text{ such that } s_k < L + \epsilon.$$

We have the following two intuitively plausible results, being converses of each other.

Theorem:

$$\textbf{If } \ell = \lim_{n\to\infty} s_n \textbf{ for } \ell \in \mathbb{X}\textbf{, then } \liminf s_n = \limsup s_n = \ell\textbf{.} \tag{3.31}$$

[25] Note the difference to saying that $\lim_{n\to\infty} s_n = \infty$: If $s_n \to \infty$, then $\limsup s_n = \infty$, but the converse need not be true.

Proof: (Heil, p. 36.) Assume that $w = \lim_{n\to\infty} s_n$ exists and is a finite real number. Since $(s_n)_{n\in\mathbb{N}}$ is a bounded sequence, both

$$u = \liminf_{n\to\infty} s_n \quad \text{and} \quad v = \limsup_{n\to\infty} s_n$$

are finite, and, from (3.27), $u \leq v$. For each $n$, set

$$y_n = \inf_{m\geq n} s_m \quad \text{and} \quad z_n = \sup_{m\geq n} s_m.$$

Observe that $y_1 \leq y_2 \leq \cdots$, while $z_1 \geq z_2 \geq \cdots$. Since $s_n \to w$, if we fix a number $\varepsilon > 0$, then there exists some $N > 0$ such that

$$w - \varepsilon \leq s_m \leq w + \varepsilon, \qquad \text{for all } m \geq N.$$

Hence, for each $n \geq N$ we have that

$$w - \varepsilon \leq \inf_{m\geq n} s_m = y_n \leq z_n = \sup_{m\geq n} s_m \leq w + \varepsilon.$$

Consequently, since the $y_n$ are increasing,

$$u = \sup_{n\geq 1} y_n = \sup_{n\geq N} y_n \geq w - \varepsilon.$$

Likewise, since the $z_n$ are decreasing,

$$v = \inf_{n\geq 1} z_n = \inf_{n\geq N} z_n \leq w + \varepsilon.$$

This is true for every $\varepsilon > 0$, so $u \geq w$ and $v \leq w$. Hence $w \leq u \leq v \leq w$, and therefore $w = u = v$.

Theorem:

$$\textbf{If } \liminf s_n = \limsup s_n = \ell \textbf{ for } \ell \in \mathbb{X}\textbf{, then } \ell = \lim_{n\to\infty} s_n\textbf{.} \tag{3.32}$$

Proof: To prove (3.32) for $\ell \in \mathbb{R}$, note that, for a given $\epsilon > 0$, results (3.28) and (3.30) imply that $\exists\, n_1, n_2 \in \mathbb{N}$ such that, $\forall n \geq n_1$, $s_n < \ell + \epsilon$ and, $\forall n \geq n_2$, $s_n > \ell - \epsilon$. Thus, with $N = \max\{n_1, n_2\}$, $\ell - \epsilon < s_n < \ell + \epsilon$ for all $n \geq N$, or $\lim_{n\to\infty} s_n = \ell$.

Recall from §2.6.1 the result: If $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are two sequences of real numbers that are each bounded above, then $\sup_{n\in\mathbb{N}} \{a_n + b_n\} \leq \sup_{n\in\mathbb{N}} a_n + \sup_{n\in\mathbb{N}} b_n$. Here is the related result for lim sup and lim inf.

Theorem: Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two bounded real sequences. Then we have

$$\limsup\,(a_n + b_n) \leq \limsup\,(a_n) + \limsup\,(b_n)\,, \tag{3.33}$$

and

$$\liminf\,(a_n) + \liminf\,(b_n) \leq \liminf\,(a_n + b_n)\,. \tag{3.34}$$

Proof: For every $k \geq 1$ we can write:

$$\inf_{n\geq k} a_n + \inf_{n\geq k} b_n \leq a_j + b_j \leq \sup_{n\geq k} a_n + \sup_{n\geq k} b_n, \quad \forall j \geq k.$$

The first inequality implies:

$$\inf_{n\geq k} a_n + \inf_{n\geq k} b_n \leq \inf_{n\geq k} (a_n + b_n),$$

while the second one gives:

$$\sup_{n\geq k} (a_n + b_n) \leq \sup_{n\geq k} a_n + \sup_{n\geq k} b_n.$$

Now take $k \to \infty$.

Theorem: Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two bounded real sequences such that $b_n$ converges to $b$. Then

$$\limsup (a_n + b_n) = \limsup (a_n) + b, \tag{3.35}$$

and

$$\liminf (a_n + b_n) = \liminf (a_n) + b. \tag{3.36}$$

Proof: We only prove the first identity, by showing a double inequality. From (3.33),

$$\limsup (a_n + b_n) \leq \limsup (a_n) + \limsup (b_n) = \limsup (a_n) + b,$$

where the second equality follows from (3.31). By writing $a_n = (a_n + b_n) + (-b_n)$ and again using (3.33),

$$\limsup (a_n) \leq \limsup (a_n + b_n) + \limsup (-b_n) = \limsup (a_n + b_n) - b,$$

where we used that $-b_n$ converges to $-b$. Hence $\limsup (a_n) + b \leq \limsup (a_n + b_n)$.

Theorem: Let $\{a_n\}, \{b_n\} \in \mathbb{R}_{\geq 0}$. Then

$$\limsup\{a_n b_n\} \leq \limsup\{a_n\} \limsup\{b_n\}. \tag{3.37}$$

Proof: Recall (2.235), i.e., $\sup_{k\geq n} a_k b_k \leq \sup_{k\geq n} a_k \sup_{k\geq n} b_k$. Let $c_k = a_k b_k$ and $\bar{c}_n = \sup_{k\geq n} c_k$, $\bar{a}_n = \sup_{k\geq n} a_k$, and $\bar{b}_n = \sup_{k\geq n} b_k$. Then (2.235) reads, for sequences $\{\bar{c}_n\}$, $\{\bar{a}_n\}$, and $\{\bar{b}_n\}$,

$$\forall n \in \mathbb{N}, \quad \bar{c}_n \leq \bar{a}_n \bar{b}_n. \tag{3.38}$$

Then, from (3.38); (2.14) (which says, for $x_n$ and $y_n$ sequences such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, if $x_n \leq y_n$ for all $n$ sufficiently large, then $x \leq y$); and that the limit of a product is the product of the limits,

$$\begin{aligned}\limsup\{a_n b_n\} = \lim_{n\to\infty} \sup_{k\geq n} a_k b_k = \lim_{n\to\infty} \bar{c}_n \leq \lim_{n\to\infty} \bar{a}_n \bar{b}_n \\ = \lim_{n\to\infty} \sup_{k\geq n} a_k \sup_{k\geq n} b_k = \limsup\{a_n\} \limsup\{b_n\},\end{aligned}$$

which is (3.37).

Relation (3.37) can be strict. As an example, take $a_n = ((-1)^{n+1} + 1)/2$, i.e., $\{a_n\} = \{1, 0, 1, 0, \ldots\}$; and $b_n = ((-1)^n + 1)/2$, i.e., $\{b_n\} = \{0, 1, 0, 1, \ldots\}$. This results in a strict inequality.

Theorem: Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two bounded real sequences such that $b_n$ converges to $b \geq 0$. Then

$$\limsup (a_n b_n) = b \limsup (a_n), \tag{3.39}$$

and

$$\liminf (a_n b_n) = b \liminf (a_n). \tag{3.40}$$

Proof: We only prove the first identity. Recall (2.8), namely, if $s_n$ is a bounded sequence and $t_n$ converges to zero, then $s_n t_n$ converges to zero. This covers the case $b = 0$, thus we can assume $b > 0$. We have $a_n b_n = a_n b + a_n (b_n - b)$. Since $a_n (b_n - b)$ converges to zero, we can use (3.35) to obtain $\limsup (a_n b_n) = \limsup (a_n b)$. Because $b > 0$ we have $\sup_{n\geq k} (a_n b) = b \sup_{n\geq k} (a_n)$, and similarly,

$$\inf_{k\geq 1} \left( b \sup_{n\geq k} (a_n) \right) = b \inf_{k\geq 1} \sup_{n\geq k} (a_n).$$

Thus $\limsup (a_n b) = b \limsup (a_n)$ and we are done.

Theorem: If $\{x_n\}_{n\in\mathbb{N}}$ is any sequence of real numbers, then there exist subsequences $\{x_{n_k}\}_{k\in\mathbb{N}}$ and $\{x_{m_j}\}_{j\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n \quad \text{and} \quad \lim_{j\to\infty} x_{m_j} = \liminf_{n\to\infty} x_n. \tag{3.41}$$

A proof can be found in, among other sources, Heil, Measure Theory for Scientists and Engineers, 2025 (forthcoming), p. 37; and Giv, Mathematical Analysis and Its Inherent Nature, Theorem 2.70. See also the set of notes https://people.math.aau.dk/~cornean/analyse2_F14/limsup-liminf.pdf, which gives the proof in the case of $\{x_n\}_{n\geq 1}$ bounded, as stated next.

We can state the previous theorem a bit more concretely if we assume that $\{x_n\}_{n\geq 1}$ is a bounded real sequence, i.e., there exists $M > 0$ such that $-M \leq x_n \leq M$ for all $n \geq 1$.

Theorem: Assume $\{x_n\}_{n\geq 1}$ is a bounded real sequence. Let $S$ denote the set of all real numbers for which there exists at least one subsequence $\{x_{n_j}\}_{j\geq 1}$ such that $x_{n_j}$ converges to $x$ when $j \to \infty$. Clearly, $S$ is a subset of $[-M, M]$. We have $\max(S) = \limsup x_n$ and $\min(S) = \liminf x_n$.

Lemma: Let $\{r_n\} \subset \mathbb{R}_{>0}$. Then

$$\liminf \frac{r_{n+1}}{r_n} \leq \liminf r_n^{1/n} \leq \limsup r_n^{1/n} \leq \limsup \frac{r_{n+1}}{r_n}. \tag{3.42}$$

Proof: The middle inequality follows from (3.27). Consider the last inequality. Let $\beta = \limsup_{n\to\infty} r_{n+1}/r_n$. If this is infinite, the result clearly holds. Assume $\beta$ is finite. Then, from (3.28), for each $\varepsilon > 0$, there is a number $N$ such that $0 < r_{n+1}/r_n < \beta + \varepsilon$ for all $n \geq N$. For $n > N$ it follows that

$$r_n = r_N \cdot \frac{r_{N+1}}{r_N} \cdot \frac{r_{N+2}}{r_{N+1}} \cdots \frac{r_n}{r_{n-1}} < r_N (\beta + \varepsilon)^{n-N}.$$

Recall (2.123), i.e., $\sqrt[n]{a} \to 1$ for each fixed $a > 0$. Thus, for all $n$ sufficiently large,

$$\sqrt[n]{r_n} < \sqrt[n]{r_N(\beta+\varepsilon)^{-N}}(\beta+\varepsilon) < \beta + 2\varepsilon.$$

Thus, for each $\varepsilon > 0$, $\limsup_{n\to\infty} \sqrt[n]{r_n} \leq \beta + 2\varepsilon$, which implies that $\limsup_{n\to\infty} \sqrt[n]{r_n} \leq \beta$. The proof of the first inequality is similar.

We end our discussion of lim inf and lim sup of sequences of numbers by revisting the root test (2.251), and augmenting it with a better version.

Definition: The *exponential growth rate* of the series $\sum_{k=1}^{\infty} f_k$ is given by

$$L = \limsup |f_k|^{1/k}. \tag{3.43}$$

Theorem: In the root test (2.251), the assumption that $\lim_{k\to\infty} |f_k|^{1/k}$ exists can be relaxed by working with the exponential growth rate of the series $\sum_{k=1}^{\infty} f_k$, which always exists (in $\mathbb{X}$).

Proof: This is similar to the proof of the root test. If $L < 1$, then $\exists \epsilon > 0$ such that $L+\epsilon < 1$, and, from (3.28), $\exists K \in \mathbb{N}$ such that, $\forall k \geq K$, $|f_k|^{1/k} < L+\epsilon$, or $|f_k| < (L+\epsilon)^k$. The comparison test is then used as before. A similar argument using (3.29) shows that the series diverges if $L > 1$.

**Example 3.3** *(Stoll, 2001, p. 289) Let $a_n = 2^{-k}$ if $n = 2k$ and $a_n = 3^{-k}$ if $n = 2k+1$, so that*

$$S = \sum_{n=1}^{\infty} a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots,$$

*and $a_n^{1/n} = 2^{-1/2}$ for $n = 2k$ and $a_n^{1/n} = 3^{-k/(2k+1)}$ for $n = 2k+1$. As $\lim_{k\to\infty} 3^{-k/(2k+1)} = 3^{-1/2}$ and $\max\left(2^{-1/2}, 3^{-1/2}\right) = 2^{-1/2}$, we have $\limsup a_n^{1/n} = 2^{-1/2} < 1$ and, thus, by the root test, $S$ converges.* ■

**Remarks**:

**(a)** The root test is "more powerful" than the ratio test, in the sense that, if the ratio test proves convergence, or divergence, then so does the root test, but the converse is not true. (Our comfortable statistics language is not actually used; one says that the root test has a *strictly wider scope* than the ratio test.) The reason for this is (3.42). This is no reason to eliminate the ratio test from our toolbox: In various problems, the ratio test can be far easier to apply than the root test.

**(b)** For the series $S = \sum_{k=1}^{\infty} f_k$, if $L = \limsup |f_k|^{1/k} < 1$, then $S$ is absolutely convergent, and hence also convergent. If $L > 1$, then $\sum_{k=1}^{\infty} |f_k|$ diverges, but could it be the case that $S$ converges?

To answer this, recall from (3.29) that, if $L > 1$, then there are infinitely many $k$ such that $|f_k| > 1$. Thus, whatever the sign of the terms, $\{f_k\}$ is not converging to zero, which implies that $S$ diverges. ■

We now consider lim inf and lim sup for sets. In the following, we do not define measurable spaces or measure spaces, so just ignore this terminology for now.

Definition: Let $\{E_k\}_{k=1}^{\infty}$ be a countable family of measurable subsets of measurable space $(X, \mathcal{A})$. We define

$$\liminf_{k\to\infty} E_k := \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} E_k, \quad \limsup_{k\to\infty} E_k := \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k. \tag{3.44}$$

Definition: Equivalent definitions are given by

$$\begin{aligned} \liminf_{k\to\infty} E_k &:= \big\{x \in X : x \in E_k \text{ for all but finitely many } k\big\}, \\ \limsup_{k\to\infty} E_k &:= \big\{x \in X : x \in E_k \text{ for infinitely many } k\big\} \end{aligned} \tag{3.45}$$

For $\liminf E_k$, "for all but finitely many $k$" means, $\exists k_0 \in \mathbb{N}$ such that, $\forall k \geq k_0$, $x \in E_k$. From (3.45), it is apparent that

$$\liminf E_k \subset \limsup E_k. \tag{3.46}$$

Theorem: The two formulations (3.44) and (3.45) are equivalent.

Proof: We begin with ($\Rightarrow$) and ($\Leftarrow$) for lim sup.
($\Rightarrow$) Let $x \in \limsup$ in (3.44), so that

$$x \in \left(\bigcup_{k=1}^{\infty} E_k\right) \cap \left(\bigcup_{k=2}^{\infty} E_k\right) \cap \left(\bigcup_{k=3}^{\infty} E_k\right) \cap \cdots. \tag{3.47}$$

Suppose $x \notin \big\{x \in X : x \in E_k \text{ for infinitely many } k\big\}$. Then $\exists k_0 \in \mathbb{N}$ such that, $\forall k \geq k_0$, $x \notin E_k$, which implies $x \notin \bigcup_{k=k_0}^{\infty} E_k$, which contradicts (3.47).

($\Leftarrow$) Let $x \in \limsup$ in (3.45). Then $x$ never stops reappearing in $\{E_k\}$ $\Leftrightarrow$ $x \in \bigcup_{k=1}^{\infty} E_k$, $x \in \bigcup_{k=2}^{\infty} E_k$, etc. $\Leftrightarrow$

$$x \in \left(\bigcup_{k=1}^{\infty} E_k\right) \cap \left(\bigcup_{k=2}^{\infty} E_k\right) \cap \cdots \Leftrightarrow x \in \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k.$$

Now we do ($\Rightarrow$) and ($\Leftarrow$) for lim inf.
($\Rightarrow$) Let $x \in \liminf$ in (3.44), so that

$$x \in \left(\bigcap_{k=1}^{\infty} E_k\right) \cup \left(\bigcap_{k=2}^{\infty} E_k\right) \cup \left(\bigcap_{k=3}^{\infty} E_k\right) \cup \cdots \tag{3.48}$$

Suppose $x \notin \big\{x \in X : x \in E_k \text{ for all but finitely many } k\big\}$. That means, $\forall n \in \mathbb{N}$, $\exists j > n$ such that $x \notin \bigcap_{k=j}^{\infty} E_k$, which contradicts (3.48).

($\Leftarrow$) Let $x \in \liminf$ in (3.45). That means $\exists j_x$ such that

$$x \in \bigcap_{k=j_x}^{\infty} E_k \iff x \in \left(\bigcap_{k=1}^{\infty} E_k\right) \cup \left(\bigcap_{k=2}^{\infty} E_k\right) \cup \left(\bigcap_{k=3}^{\infty} E_k\right) \cup \cdots = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} E_k.$$

## 3.4 Boxes, Covers, Exterior Measure

Here we define a few more concepts and prove some basic results that will prove useful to know in preparation for learning measure theory and the Lebesgue integral, and also studying measure-theoretic probability theory.

We turn first to the idea of *nonoverlapping covers* (defined below) of the real line, and, ultimately, regions in $\mathbb{R}^n$. Let $Q_j \subset \mathbb{R}$ refer to a closed, finite (bounded), nonempty interval on the real line, e.g., $[a, b]$, for $a, b \in \mathbb{R}$, with $a < b$. A *box* generalizes this to closed hyper-rectangles in $\mathbb{R}^d$, but for $d = 1$, a box is just the closed, bounded interval.

Definition: Let $J$ be an at most countable set. Common examples include $J = \{1, 2, \ldots, N\}$ for $N \in \mathbb{N}$; $J = \mathbb{N}$; and $J = \mathbb{Z}$. The collection $\{Q_k\}_{k \in J}$ is said to be *nonoverlapping* if their interiors are disjoint, i.e., $\forall j, k \in J,\ j \neq k \Rightarrow Q_j^\circ \cap Q_k^\circ = \emptyset$.

As an example, with $Q_k = [k, k+1]$, the elements of $\{Q_k\}_{k\in\mathbb{Z}}$ are nonoverlapping, though not disjoint. Note that $\{Q_k\}_{k\in\mathbb{Z}} = \mathbb{R}$.

The following comes from Heil, Measure Theory for Scientists and Engineers, forthcoming, Exercise 2.1.7. (The author kindly confirmed my solutions, and added one of his.)

1. Show that $(0, 1] = \bigcup_{k=0}^{\infty} \left[2^{-k-1}, 2^{-k}\right]$.

> We need to show both directions of set inclusion.
>
> To show $(0, 1] \supset \cup_{k=0}^{\infty}[2^{-k-1}, 2^{-k}]$, note that $\forall k \in (\{0\} \cup \mathbb{N}), [2^{-k-1}, 2^{-k}] \subset (0, 1]$.
>
> To show $(0, 1] \subset \cup_{k=0}^{\infty}[2^{-k-1}, 2^{-k}]$, observe that:
>
> • $\lim_{k\to\infty} 2^{-k-1} = \lim_{k\to\infty} 1/2^k = 0$, but $\forall k \in \mathbb{N}$, limit point $0 \notin [2^{-k-1}, 2^{-k}]$.
>
> • For $k = 0$, $1 \in [2^{-k-1}, 2^{-k}] = [1/2, 1]$. Thus, $1 \in \cup_{k=0}^{\infty}[2^{-k-1}, 2^{-k}]$.
>
> • Recall the Archimedean property: $\forall x > 0, \exists n \in \mathbb{N}$ such that $1/n < x$. Both $\{1/n\}$ and $\{1/2^{k+1}\}$ are positive, strictly monotone decreasing null sequences; recall (2.62). Thus, the Archimedean property also applies to the latter, namely: $\forall x > 0, \exists k \in \mathbb{N}$ such that $1/2^{k+1} < x$.
>
> • Let $x \in (0, 1)$. The set $A_x := \{k \in \mathbb{N} : 1/2^{k+1} \leq x\}$ is thus nonempty, and it is bounded below. From the Well-Ordering principle, it has a smallest element, $k_x$. By construction, $k_x - 1 \notin A_x$, and thus $1/2^{k_x+1} \leq x < 1/2^{k_x}$, i.e.,
>
> $$x \in [1/2^{k_x+1} \leq x < 1/2^{k_x}) \subset [1/2^{k_x+1} \leq x < 1/2^{k_x}] \subset \cup_{k=0}^{\infty}[2^{-k-1}, 2^{-k}],$$
>
> showing the desired inclusion.

2. Show $[1, \infty) = \bigcup_{k=0}^{\infty} \left[2^k, 2^{k+1}\right]$.

> We need to show both directions of set inclusion.
>
> To show $[1, \infty) \supset \cup_{k=0}^{\infty}[2^k, 2^{k+1}]$, note that $\forall k \in (\{0\} \cup \mathbb{N}),\ [2^k, 2^{k+1}] \subset [1, \infty)$.
>
> To show $[1, \infty) \subset \cup_{k=0}^{\infty}[2^k, 2^{k+1}]$, observe that:
>
> • For $k = 0$, $1 \in [2^k, 2^{k+1}] = [1, 2]$. Thus $1 \in \cup_{k=0}^{\infty}[2^k, 2^{k+1}]$.
>
> • Let $x > 1$. To see that $x \in \cup_{k=0}^{\infty}[2^k, 2^{k+1}]$, note that the $[2^k, 2^{k+1}]$ are nonoverlapping, but adjacent and not disjoint. Their union (starting from $k = 0$) clearly covers $(1, \infty)$, because $\lim_{k\to\infty} 2^{k+1} = \lim_{k\to\infty} 2^k = \infty$.

3. Show: Every finite open interval $(a, b)$ is a union of countably many nonoverlapping intervals $\{Q_k\}_{k\in\mathbb{N}}$.

> • (Marc) First consider the situation when $b - a > 2$. Observe that $(a, b)$ can be written as the union of *overlapping* closed intervals $(a, b) = \cup_{n=1}^{\infty} H_k$, where $H_k = [a + 1/k, b - 1/k]$. This motivates defining
>
> $$Q_0 = [a + 1, b - 1],\ \ Q_{a,k} = [a + 1/(k + 1), a + 1/k],\ \ Q_{b,k} = [b - 1/k, b - 1/(k + 1)],$$
>
> so that $(a, b) = Q_0 \cup (\cup_{k=1}^{\infty} Q_{a,k}) \cup (\cup_{k=1}^{\infty} Q_{b,k})$, which, being a finite union of countable unions, is itself a countable union, of nonoverlapping, closed, bounded, nonempty intervals.
>
> Now consider the case when $b - a \le 2$. Scale $(a, b)$ to have length greater than two, e.g., let $a^* = \kappa a/(b - a)$ and $b^* = \kappa b/(b - a)$, where $\kappa > 2$. Then the previous result is applicable to $(a^*, b^*)$, producing the desired nonoverlapping set of intervals $Q_0^*$, $\{Q_{a,k}^*\}_{k\in\mathbb{N}}$, and $\{Q_{b,k}^*\}_{k\in\mathbb{N}}$. Scale these by multiplying each by $(b - a)/\kappa$ to get the result.
>
> NOTE: For fun, observe $[a, b] = \cap_{n=1}^{\infty} (a - 1/n, b + 1/n)$.
>
> • (Christopher Heil) Part (a) shows how to cover $(0, 1]$ by nonoverlapping boxes, and a symmetric construction tells us how to cover $[1, 2)$. The union of these two covers gives us a cover of $(0, 2)$. Rescaling and translating gives us a cover of any finite interval $(a, b)$.
>
> • (Ralf) Let
>
> $$Q_k = \left[a + \frac{b-a}{2} 2^{-k-1}, a + \frac{b-a}{2} 2^{-k}\right], \quad Q_j = \left[b - \frac{b-a}{2} 2^{-k}, b - \frac{b-a}{2} 2^{-k-1}\right],$$
>
> $\forall k, j \in \mathbb{N}$. Then $\cup Q_k = (a, \frac{b+a}{2}]$ and $\cup Q_j = [\frac{b+a}{2}, b)$. The exercise requires finding a countable collection of nonoverlapping closed intervals. A finite union of countable sets is countable.

4. Show: Every infinite open interval is a union of countably many nonoverlapping intervals $\{Q_k\}_{k\in\mathbb{N}}$. Hint: The question does not say that $Q_k$ needs to be bounded.

> Assume the interval is of the form $(b, \infty)$, with the case of $(-\infty, a)$ being similar. Then we can write $(b, \infty) = [b + 1/2, \infty) \cup (\cup_{k=1}^{\infty} [b + 1/2^{k+1}, b + 1/2^k])$.

5. Show: If $U \subset \mathbb{R}$ is open, then there exists a countable collection of nonoverlapping intervals $\{Q_k\}_{k\in\mathbb{N}}$ such that $U = \cup Q_k$.

> We require the Characterization of the Open Subsets of $\mathbb{R}$. It might be required that one or two of the $I_n$ are infinite. The result follows from (i) this characterization; (ii) parts (3) and (4), whereby each $I_n$ can be represented as union of countably many nonoverlapping intervals; (iii) the $\{I_n\}$ are disjoint; and (iv) that a countable union of countable unions is itself a countable union.

We next provide the definition of what is often used to ultimately define the Lebesgue measure of a set. It makes use of boxes, from above, but not necessarily overlapping.

Definition: The *exterior Lebesgue measure* (or *outer Lebesgue measure*) of a set $E \subseteq \mathbb{R}$ is

$$|E|_e = \inf \left\{ \sum_k \text{length}\,(Q_k) \right\}, \tag{3.49}$$

where the infimum is taken over all possible countable collections of boxes $\{Q_k\}_k$ such that $E \subseteq \cup Q_k$.

For simplicity, we often abbreviate "exterior Lebesgue measure" as "exterior measure". Every subset $E$ of $\mathbb{R}$ has an exterior measure $|E|_e$. The exterior measure of $E$ is nonnegative but it could be infinite, so it lies in the range $0 \leq |E|_e \leq \infty$.

Lemma: Let $E$ be any subset of $\mathbb{R}$.

(a) If $\{Q_k\}_k$ is any countable cover of $E$ by boxes, then

$$|E|_e \leq \sum_k \text{length}\,(Q_k) \tag{3.50}$$

(b) If $\varepsilon > 0$, then there exists some (that is, at least one) countable cover $\{Q_k\}_k$ of $E$ by boxes such that

$$|E|_e \leq \sum_k \text{length}\,(Q_k) \leq |E|_e + \varepsilon. \tag{3.51}$$

> Proof:
>
> (a) By definition, $|E|_e$ is the greatest lower bound of the set $S$ of all quantities $\sum$ length $(Q_k)$, where $\{Q_k\}_k$ is any covering of $E$ by countably many boxes. Consequently, $|E|_e$ is a lower bound to this set, which means that $|E|_e \leq \sum$ length $(Q_k)$ for every covering of $E$ by countably many boxes.
>
> (b) If $|E|_e = \infty$, let $Q_k = [k, k+1]$ for $k \in \mathbb{Z}$. Then $\{Q_k\}_{k\in\mathbb{Z}}$ is one covering of $E$ by by countably many boxes that has the required properties, because we have $|E|_e = \infty$, $\sum \text{length}\,(Q_k) = \infty$, and $|E|_e + \varepsilon = \infty$.
>
> So, consider the case where $|E|_e$ is finite. By definition, $|E|_e$ is the greatest lower bound of the set $S$ of all numbers $\sum \text{length}\,(Q_k)$, where $\{Q_k\}_k$ is a covering of $E$ by countably many boxes. If $\varepsilon > 0$ then $|E|_e + \varepsilon$ is strictly greater than $|E|_e$. Therefore $|E|_e + \varepsilon$ cannot be a lower bound of $S$. That is, it is ***not true*** that
>
> $$|E|_e + \varepsilon \leq \sum_k \text{length}\,(Q_k) \;\text{ for every covering of } E \text{ by boxes } \{Q_k\}_k\,.$$
>
> Consequently, there must exist at least one covering of $E$ by countably many boxes $\{Q_k\}_k$ such that
>
> $$|E|_e + \varepsilon > \sum_k \text{length}\,(Q_k)\,.$$

Lemma: (Bounded Sets Have Finite Exterior Measure). If $E$ is a bounded subset of $\mathbb{R}$, then $|E|_e < \infty$.

> Proof: By definition, if $E$ is a bounded subset of $\mathbb{R}$ then there is some real number $M$ such that $|x| \leq M$ for every $x \in E$. Let $Q = [-M, M]$. Then $\{Q\}$ is a collection of one box that covers $E$. Part (a) of Lemma 2.2.3 therefore implies that
>
> $$|E|_e \leq \text{length}(Q) = 2M < \infty.$$

This does not imply that unbounded sets have infinite measure. An example is $E = \bigcup_{k=1}^{\infty} \left[k, k+2^{-k}\right]$.

Lemma: Exterior Lebesgue measure is monotonic. That is, if $A, B \subseteq \mathbb{R}$, then

$$A \subseteq B \Longrightarrow |A|_e \leq |B|_e. \tag{3.52}$$

Proof: Suppose that $A \subseteq B$, and let $\{Q_k\}_k$ be any countable cover of $B$ by boxes. Then $A \subseteq B \subseteq \cup Q_k$, so $\{Q_k\}_k$ is a countable cover of $A$ by boxes. Result (3.50) therefore implies that

$$|A|_e \leq \sum_k \text{length } (Q_k)$$

This is true for every possible covering of $B$, so we conclude that

$$|A|_e \leq \inf \left\{ \sum_k \text{length } (Q_k) : \text{ all covers of } B \text{ by boxes } \right\} = |B|_e.$$

Lemma: The empty set has exterior measure zero, i.e.,

$$|\varnothing|_e = 0. \tag{3.53}$$

Proof: If $\varepsilon > 0$, then the single box $Q = [0, \varepsilon]$ covers $\varnothing$. Therefore

$$0 \leq |\varnothing|_e \leq \text{length}(Q) = \varepsilon.$$

Since $\varepsilon$ can be any positive real number, this implies that $|\varnothing|_e = 0$.

Lemma: If $E$ is a countable subset of $\mathbb{R}$, then $|E|_e = 0$.

Proof: Let $E = \{x_k\}_{k \in J}$ be a countable subset of $\mathbb{R}$. In this case, we can assume that either $J = \{1, \ldots, N\}$ for some positive integer $N$, or that $J = \mathbb{N} = \{1, 2, \ldots\}$. For each $k \in J$, let $Q_k = \left[x_k, x_k + 2^{-k}\varepsilon\right]$. Then $Q_k$ is a box and $x_k$ belongs to $Q_k$, so $\{Q_k\}_{k \in J}$ is a covering of $E$ by boxes. Further, length $(Q_k) = 2^{-k}\varepsilon$. Using (3.50), we have

$$|E|_e \leq \sum_{k \in J} \text{length} \left(Q_k\right) \leq \sum_{k \in J} 2^{-k}\varepsilon \leq \varepsilon \sum_{k=1}^{\infty} 2^{-k} = \varepsilon.$$

Since $\varepsilon$ can be any positive real number, $|E|_e = 0$.

Theorem (Countable Subadditivity): If $E_1, E_2, \ldots$ are subsets of the real line, then

$$\left| \bigcup_{k=1}^{\infty} E_k \right|_e \leq \sum_{k=1}^{\infty} |E_k|_e\,. \tag{3.54}$$

Proof: If any particular set $E_k$ has infinite exterior measure then the righthand side of equation (2.14) is $\infty$, so we are done in this case. Therefore, we need only consider the case where $|E_k|_e < \infty$ for every $k$.

Fix any number $\varepsilon > 0$. By (3.50), for each $k \in \mathbb{N}$ there exists a covering $\left\{Q_j^{(k)}\right\}_j$ of $E_k$ by countably many boxes that satisfies

$$\sum_j \text{length}\left(Q_j^{(k)}\right) \leq |E_k|_e + \frac{\varepsilon}{2^k}.$$

Combining all of these coverings together gives us the countable collection of boxes $\left\{Q_j^{(k)}\right\}_{j,k}$. This combined collection covers $\cup_k E_k$, so

$$\left|\bigcup_{k=1}^{\infty} E_k\right|_e \leq \sum_{k=1}^{\infty} \sum_j \text{length}\left(Q_j^{(k)}\right) \leq \sum_{k=1}^{\infty} \left(|E_k|_e + \frac{\varepsilon}{2^k}\right) = \left(\sum_{k=1}^{\infty} |E_k|_e\right) + \varepsilon.$$

Countable subadditivity (3.54) follows because $\varepsilon$ can be any positive real number.

Subadditivity need not hold for uncountable collections of sets. For example, the real line is the union of uncountably many singletons: $\mathbb{R} = \bigcup_{x \in \mathbb{R}} \{x\}$. The exterior measure of each singleton $\{x\}$ is zero, yet (though we have not shown it) $|\mathbb{R}|_e = \infty$.

We now present an important result that comes up in measure-theoretic probability theory, and its proof uses many of the previous results (along with other results in the document). We state it in two versions, the first being in terms of exterior measure, while the second formulation is the one usually seen, in terms of Lebesgue measure, which, for a set $E_k$, is denoted $m\left(E_k\right)$ or $|E_k|$. The proof of the first formulation follows from that of the second.

We refer to some terms not yet defined in the proof of the second one, e.g., the (Lebesgue) measurability of a set, and the reader can gloss over this for now. All sets (in $\mathbb{R}^n$) have an exterior measure, but not all sets have Lebesgue measure. When a set has the latter, it coincides with the former. Thus, when we show that a set has Lebesgue measure zero, it means it is measurable, and that the exterior measure is also zero. We also invoke a standard result that, if each $E_k$ is a measurable subset of $\mathbb{R}^d$, then countable unions and intersections of $E_k$ are also measurable.

Finally, observe that, if we show that $\limsup E_k$ has (exterior) measure zero, then it also follows for $\liminf E_k$, from (3.46).

Theorem (Borel-Cantelli, with exterior measure): Suppose that $E_1, E_2, \ldots$ are countably many subsets of $\mathbb{R}$ that satisfy $\sum |E_k|_e < \infty$. Then

$$\liminf E_k \text{ and } \limsup E_k \text{ each have exterior measure zero.} \tag{3.55}$$

Theorem (Borel-Cantelli, with Lebesgue measure): Let $\{E_k\}_{k=1}^{\infty}$ be a countable family of measurable subsets of $\mathbb{R}^d$ such that $\sum_{k=1}^{\infty} m\left(E_k\right) < \infty$. Then $\limsup_{k \to \infty} \left(E_k\right)$ is measurable and has measure zero.

Proof: Let

$$E = \limsup_{k \to \infty} \left(E_k\right) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k.$$

Since each $E_k$ is a measurable subset of $\mathbb{R}^d$, $\bigcup_{k=n}^{\infty} E_k$ is measurable for each $n \in \mathbb{N}$, and so $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{n} E_k$ is measurable as well. Therefore, $E$ is measurable. Suppose that

$m(E) > \epsilon > 0$. Then

$$0 < \epsilon < m(E) = m\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right).$$

Since for all $n \in \mathbb{N}$, $E \subset \bigcup_{k=n}^{\infty} E_k$, by the monotonicity property (3.52),

$$m(E) \leq m\left(\bigcup_{k=n}^{\infty} E_k\right).$$

By the countable subadditivity property (3.54), for all $n \in \mathbb{N}$,

$$m(E) = m\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \sum_{k=n}^{\infty} m\left(E_k\right).$$

By assumption, $\sum_{k=1}^{\infty} E_k < \infty$. It follows from (2.238) that the tail of the series can be made arbitrarily small. In other words, for any $\delta > 0$, there is an $N' \in \mathbb{N}$ such that $\sum_{k=N'}^{\infty} m\left(E_k\right) < \delta$. However, if we choose $\delta = \epsilon/2$, we have

$$0 < \epsilon < m(E) \leq \sum_{k=N'}^{\infty} m\left(E_k\right) \leq \frac{\epsilon}{2},$$

which is not possible. Thus, by contradiction, the assumption that $m(E) > \epsilon > 0$ cannot hold, and thus, as $m(\cdot) \geq 0$, it must be the case that $m(E) = 0$.

**Example 3.4** *(Heil, Measure Theory for Scientists and Engineers, p. 166, Problem 3.4.11) Let $E \subseteq \mathbb{R}$ be measurable, and assume that $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of measurable functions on $E$ such that*

$$\sum_{n=1}^{\infty} \left|\left\{|f_n| > \frac{1}{n^2}\right\}\right| < \infty.$$

*We wish to prove that $\sum_{n=1}^{\infty} f_n(x)$ converges for almost every $x \in E$.*

*For each $n \in \mathbb{N}$, let*

$$A_n = \left\{\left|n^2 f_n\right| > 1\right\} = \left\{|f_n| > 1/n^2\right\}.$$

*By hypothesis, $\sum |A_n| < \infty$. The Borel-Cantelli Lemma therefore implies that*

$$A = \limsup_{k \to \infty} = \bigcap_{j=1}^{\infty}\left(\bigcup_{k=j}^{\infty} A_k\right)$$

*has measure zero. But $A$ is the set of all $x$ that belong to infinitely many $A_n$. Therefore, if $x \notin A$ then $x$ belongs to finitely many $A_n$. Let $N$ be the largest value of $n$ for which we have $x \in A_n$. Then $x \notin A_n$ for all $n > N$. Hence $|f_n(x)| < 1/n^2$ for all $n > N$. Consequently, from the Comparison Test (2.249), the series $\sum f_n(x)$ converges absolutely for all $x \notin A$, which is almost every $x \in E$.* ■

## 3.5 Compactness

> Compactness is the single most important concept in real analysis. It is what reduces the infinite to the finite.
>
> (Charles Pugh, Real Mathematical Analysis, 2nd edition, 2015, p. 79)

We begin with the simple but powerful notion of subsequences, and use this to develop the concept of sequential compactness. We then cover topological compactness, and show the equivalence of the two notions. The topological characterization of continuity is then discussed.

Definition: Consider a sequence $\{a_n\}$. Let $\{n_k\}$ be a sequence of natural numbers that is strictly increasing; that is, $n_1 < n_2 < n_3 < \cdots$. Then the sequence $\{b_k\}$ defined by $b_k = a_{n_k}$, for every index $k$, is called a *subsequence* of the sequence $\{a_n\}$. Often a subsequence of $\{a_n\}$ is simply denoted by $\{a_{n_k}\}$, it being implicitly understood that $\{n_k\}$ is a strictly increasing sequence of natural numbers and that the $k$ th term of the sequence $\{a_{n_k}\}$ is $a_{n_k}$.

Proposition: Let the sequence $\{a_n\}$ converge to the limit $a$. Then every subsequence of $\{a_n\}$ also converges to the same limit $a$.

> Proof: Let $\epsilon > 0$. We need to find an index $N$ such that
>
> $$|a_{n_k} - a| < \epsilon \quad \text{for all indices } k \geq N. \tag{3.56}$$
>
> Since the whole sequence $\{a_n\}$ converges to $a$, we can choose an index $N$ such that
>
> $$|a_n - a| < \epsilon \quad \text{for all indices } n \geq N. \tag{3.57}$$
>
> But observe that since $\{n_k\}$ is a strictly increasing sequence of natural numbers, $n_k \geq k$ for every index $k$. Thus, the required inequality (3.56) follows from inequality (3.57).

Corollary: The sequence $\{a_{n+1}\}$ is a subsequence of $\{a_n\}$. Hence, if $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} a_{n+1} = a$ as well.

Theorem: If $(b_k)$ is an increasing sequence and if some subsequence $(b_{n_k})$ of $(b_k)$ converges and $\lim_{k\to\infty} b_{n_k} = b$, then $(b_k)$ itself converges to the same limit, $\lim_{k\to\infty} b_k = b$.

> Proof: (From Terrell, A Passage to Modern Analysis, 2019, p. 42) Since the subsequence $(b_{n_k})$ is increasing, the convergence assumption implies that
>
> $$b = \lim_{k\to\infty} b_{n_k} = \sup_{k\in\mathbb{N}} \{b_{n_k}\}.$$
>
> Thus, for every $\epsilon > 0$, there is an $N = N(\epsilon)$ such that
>
> $$b - \epsilon < b_{n_{N(\epsilon)}} \leq b_{n_k} \leq b$$
>
> for all $k > N(\epsilon)$. Since $n_k \geq k$ for all $k \in \mathbb{N}$, we have $b_{n_k} \geq b_k$ for all $k \in \mathbb{N}$. Hence, $b_k \leq b$ for all $k$, and $b_{n_{N(\epsilon)}} \leq b_k \leq b_{n_k} \leq b$ for all $k > n_{N(\epsilon)}$. Therefore
>
> $$|b_k - b| < \epsilon \quad \text{for all } k > n_{N(\epsilon)}.$$
>
> Since $\epsilon$ is arbitrary, this shows that $\lim_{k\to\infty} b_k = b$.

Theorem: Every sequence has a monotone subsequence.

Proof: Consider a sequence $\{a_n\}$. We call an index $m$ a peak index for the sequence $\{a_n\}$ provided that $a_n \leq a_m$ for all indices $n \geq m$. Either there are only finitely many peak indices for the sequence $\{a_n\}$ or there are infinitely many such indices.

Case 1: There are only finitely many peak indices. Then we can choose an index $N$ such that there are no peak indices greater than $N$. We will recursively define a monotonically increasing subsequence of $\{a_n\}$. Indeed, define $n_1 = N + 1$. Now suppose that $k$ is an index such that positive integers $n_1 < n_2 < \cdots < n_k$ have been chosen such that $a_{n_1} < a_{n_2} < \cdots < a_{n_k}$. Since $n_k > N$, the index $n_k$ is not a peak index. Hence there is an index $n_{k+1} > n_k$ such that $a_{n_{k+1}} > a_{n_k}$. Thus, we recursively define a strictly increasing sequence of positive integers $\{n_k\}$ having the property that the subsequence $\{a_{n_k}\}$ is strictly increasing.

Case 2: There are infinitely many peak indices. For each natural number $k$, let $n_k$ be the $k$ th peak index. Directly from the definition of peak index it follows that the subsequence $\{a_{n_k}\}$ is monotonically decreasing.

The previous theorem can be described in terms of hotels: Imagine that there is an infinite chain of hotels along a line, where the $n$th hotel has height $a_n$, and at the horizon, there is a sea. A hotel is said to have the seaview property if it is higher than all hotels following it (so that from the roof of the hotel, one can view the sea). Now there are only two possibilities, namely the two above cases. This is described, with graphics, in Sasane, The How and Why of One Variable Calculus, pp. 71-2, where three references are given.

Theorem (Bolzano-Weierstrass):

$$\textbf{Every bounded sequence has a convergent subsequence.} \tag{3.58}$$

We provide two proofs. The first one is short and uses previous results. The second proof is more “brute force” and less elegant, but easier to understand, and very important to know.

Proof I: Let $\{a_n\}$ be a bounded sequence. According to the preceding theorem, we can choose a monotone subsequence $\{a_{n_k}\}$. Since $\{a_n\}$ is bounded, so is its subsequence $\{a_{n_k}\}$. Hence $\{a_{n_k}\}$ is a bounded monotone sequence. According to the Monotone Convergence Theorem, $\{a_{n_k}\}$ converges.

Proof II: We give the presentation from Pons (Real Analysis for the Undergraduate: With an Invitation to Functional Analysis, 2014, Thm 2.3.7). See also Duren (Invitation to Classical Analysis, 2012, pp. 7-8).

Let $(a_n)$ be a bounded sequence of real numbers. Using this hypothesis, we can find $M > 0$ such that $a_n \in [-M, M]$ for all $n \in \mathbb{N}$. To construct a sequence of closed, bounded, nested intervals with lengths decreasing to zero, we will successively cut the interval $[-M, M]$ into halves. For $I_1$, we bisect $[-M, M]$ into the closed subintervals $[-M, 0]$ and $[0, M]$. The fact that sequences must have countably many terms guarantees us that at least one of these two subintervals must contain countably many terms of the sequence $(a_n)$. Choose $I_1$ to be one such subinterval (if both $[-M, 0]$ and $[0, M]$ contain infinitely many terms of the sequence, then the choice is completely arbitrary). Note that the length of $I_1$ is $M$. See Figure 13.

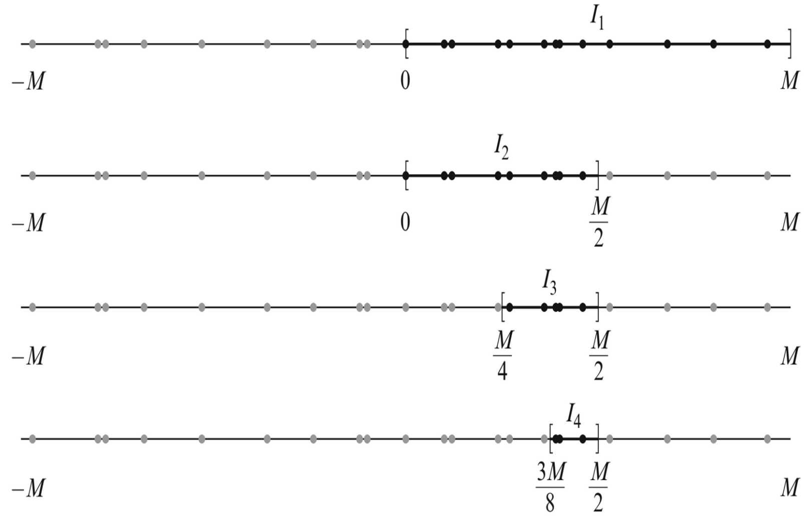



Figure 13: From Pons, p. 78.

To select $I_2$, we bisect $I_1$ into closed subintervals of length $M/2 = M/2^1$. Again, at least one of these subintervals must contain countably many terms of the sequence $(a_n)$ and we choose $I_2$ to be one such closed interval. Notice also that $I_2 \subseteq I_1$. Continuing inductively, we can construct a collection of closed, bounded, nested intervals $\{I_j\}$ with the property that each interval contains countably many terms of the sequence $(a_n)$ and the length of $I_j$ is $M/2^{j-1}$. Sequence $M/2^{j-1}$ converges to zero as $j \to \infty$. From the nested intervals theorem, it follows that the intersection $\cap_{j=1}^{\infty} I_j$ contains a unique element which we shall denote by $a$. The convergence of the sequence $M/2^{j-1}$ will also be key in the final component of the proof.

At this point we are ready to define a subsequence; this is also done in an inductive manner. Choose $n_1 \in \mathbb{N}$ so that $a_{n_1} \in I_1$. Next, choose $n_2 \in \mathbb{N}$ so that $n_2 > n_1$ and $a_{n_2} \in I_2$, which is permissible since $I_2$ contains countably many terms of the sequence $(a_n)$. Continuing, for each $j \in \mathbb{N}$, we choose $n_j \in \mathbb{N}$ so that $n_j > n_{j-1}$ and $a_{n_j} \in I_j$.

We claim now that the subsequence $\left(a_{n_j}\right)$ converges to $a$. Let $\epsilon > 0$ and choose $J \in \mathbb{N}$ such that $M/2^J < \epsilon$. Then if $j \geq J+1$, we have that $j-1 \geq J$ and $M/2^{j-1} \leq M/2^J < \epsilon$. In other words, for $j \geq J+1$, the length of the interval $I_j$ is less than $\varepsilon$. Finally, if we consider a term $a_{n_j}$ with $j \geq J+1$, then both $a_{n_j}$ and $a$ are in $I_j$ and, using the restriction on the length of this interval, it is clear that

$$\left|a_{n_j} - a\right| \leq \frac{M}{2^{j-1}} \leq \frac{M}{2^J} < \epsilon.$$

Thus we conclude that $\left(a_{n_j}\right) \to a$ as desired.

Theorem: The Bolzano-Weierstrass Theorem implies the Nested Intervals Theorem.

Proof: Recall: Bolzano-Weierstrass states: Every bounded sequence in $\mathbb{R}$ has a convergent subsequence. The Nested Intervals Theorem states: If $\{I_n\}_{n=1}^{\infty}$ is a sequence of closed and bounded intervals in $\mathbb{R}$ with $I_n \supset I_{n+1}$ for all $n \in \mathbb{N}$, then $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Let $I_n = [a_n, b_n]$ for each $n \in \mathbb{N}$, such that $I_n \supset I_{n+1}$. Then, for each $n \in \mathbb{N}$, $a_1 < b_n \leq b_1$, so $\{b_n\}$ is bounded. The Bolzano-Weierstrass theorem implies it contains a convergent subsequence $b_{n_k} \to b$.

First show that $b \le b_n$ for each $n \in \mathbb{N}$: Both $\{b_n\}$ and $\{b_{n_k}\}$ are decreasing, because $I_n \supset I_{n+1}$. If $b > b_m$ for some $m \in \mathbb{N}$, then, $\forall k \ge m$, $n_k \ge n_m \ge m$, which implies $b_{n_k} \le b_{n_m} \le b_m < b$ and

$$|b_{n_k} - b| = b - b_{n_k} \ge b - b_m.$$

But $b - b_m$ is a fixed positive number and $b_{n_k} \to b$, so by contradiction, $b \le b_m$.

Next show $b \ge a_n$ for each $n \in \mathbb{N}$: For each $n \in \mathbb{N}$, $b_{n_k} \ge a_n$. Taking the limit in $k$ and using (2.14) gives $b \ge a_n$.

Thus, $\forall n \in \mathbb{N}$, $a_n \le b \le b_n$, so, $\forall n \in \mathbb{N}$, $b \in I_n$, or $b \in \cap_{i=1}^{\infty} I_n$, implying the intersection is non-empty. As the choice of nested $\{I_n\}$ was arbitrary, the Nested Interval Theorem is proven.

Definition: A set of real numbers $S$ is said to be *sequentially compact* provided that every sequence $\{a_n\}$ in $S$ has a subsequence that converges to a point that belongs to $S$.

For example, define $S \equiv [0, \infty)$. Then $S$ is not sequentially compact. Indeed, for each index $n$, set $a_n = n$. Then $\{a_n\}$ is a sequence in $S$. However, every subsequence of $\{a_n\}$ is unbounded and therefore fails to converge. Thus, the set $S$ is not sequentially compact. Now define $S \equiv (0, 2]$. Then $S$ is not sequentially compact. Indeed, $\{1/n\}$ is a sequence in $S$. This sequence converges to 0, and hence every subsequence also converges to 0. But 0 does not belong to $S$. Thus, there is no subsequence of $\{1/n\}$ that converges to a point in $S$. So the set $S$ is not sequentially compact.

Theorem (The Sequential Compactness Theorem for $\mathbb{R}$): Let $a$ and $b$ be numbers such that $a < b$. Then the interval $[a, b]$ is sequentially compact; that is, every sequence in $[a, b]$ has a subsequence that converges to a point in $[a, b]$.

Proof: There are two distinct parts to the proof. First, it is necessary to show that a sequence in $[a, b]$ has a convergent subsequence. Then it must be shown that the limit of this subsequence belongs to the interval $[a, b]$. Let $\{x_n\}$ be a sequence in $[a, b]$. Then $\{x_n\}$ is bounded. Hence, by the preceding theorem, there is a subsequence $\{x_{n_k}\}$ that converges. But the sequence $\{x_{n_k}\}$ is a sequence in $[a, b]$, and hence, from (2.14), its limit is also in $[a, b]$.

We will require a more general version, applicable to $\mathbb{R}^n$, which we state without proof. A proof can be found in Fitzpatrick, p. 300.

Theorem (The Sequential Compactness Theorem for $\mathbb{R}^n$): Let $S$ be a subset of $\mathbb{R}^n$.

$$S \text{ is sequentially compact} \iff S \text{ is bounded and closed in } \mathbb{R}^n. \tag{3.59}$$

Proposition: Let $S$ be a subset of $\mathbb{R}$ that is closed and bounded. Then $S$ is sequentially compact.

Proof: We simply retrace the proof of the Sequential Compactness Theorem. Let $\{x_n\}$ be a sequence in $S$. Then $\{x_n\}$ is bounded since $S$ is bounded. Hence, by (3.58), there is a subsequence $\{x_{n_k}\}$ that converges to a number $x$. But the sequence $\{x_{n_k}\}$ is a sequence in $S$ that converges to $x$, and hence, by the assumption that the set $S$ is closed, the limit $x$ also belongs to $S$. Therefore, the set $S$ is sequentially compact.

Earlier, in and above (2.17), we gave the definition of Cauchy sequence, and proved that every convergent sequence is Cauchy. We repeat these two items here.

Definition: Sequence $\{s_n\}$ is termed a *Cauchy sequence* if, for a given $\epsilon \in \mathbb{R}_{>0}$, $\exists\, N \in \mathbb{N}$ such that $\forall n, m \geq N$, $|s_m - s_n| < \epsilon$.

Proposition: Every convergent sequence is Cauchy.

> Proof: Suppose that $\{a_n\}$ is a sequence that converges to the number $a$. Let $\epsilon > 0$. We need to find an index $N$ such that $|a_n - a_m| < \epsilon$ if $n \geq N$ and $m \geq N$. But since $\{a_n\}$ converges to $a$, we can choose an index $N$ such that $|a_k - a| < \epsilon/2$ for every index $k \geq N$. Thus, if $n \geq N$ and $m \geq N$, setting $a_n - a_m = (a_n - a) + (a - a_m)$, by the Triangle Inequality,
>
> $$|a_n - a_m| = |(a_n - a) + (a - a_m)| \leq |a_n - a| + |a_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We now wish to show the converse of the previous proposition, which together then implies that $\{s_n\}$ converges $\Leftrightarrow$ $\{s_n\}$ is Cauchy. In the proof, one can invoke (subsequences and) the Bolzano-Weierstrass theorem; or use (subsequences and) a property of lim sup. This is why we did not give the proof earlier, in (2.17).

Theorem (The Cauchy Convergence Criterion for Sequences):

$$\textbf{A sequence } \{a_n\} \textbf{ converges} \iff \{a_n\} \textbf{ is a Cauchy sequence.} \tag{3.60}$$

> Proof: Having showed ($\Rightarrow$) already, we need to show its converse, ($\Leftarrow$). Recall from §2.1 the lemma that states every Cauchy sequence is bounded. Suppose that $\{a_n\}$ is a Cauchy sequence. The lemma asserts that $\{a_n\}$ is bounded.
>
> Step 1, method A: As $\{a_n\}$ is bounded, Bolzano-Weierstrass (3.58) implies $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ that converges to a number $a$.
>
> Step 1, method B: As $\{a_n\}$ is bounded, $L = \limsup_{n\to\infty} a_n$ is a finite real number. Further, (3.41) implies that there exists a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ that converges to $L$.
>
> Step 2 uses the fact from Step 1 (A or B) regarding the convergent subsequence. We claim that the whole sequence $\{a_n\}$ converges to $a$. Indeed, let $\epsilon > 0$. We need to find an index $N$ such that $|a_n - a| < \epsilon$ if $n \geq N$. Since $\{a_n\}$ is a Cauchy sequence, we can choose an index $N$ such that
>
> $$|a_n - a_m| < \frac{\epsilon}{2} \quad \text{if } n \geq N \text{ and } m \geq N. \tag{3.61}$$
>
> On the other hand, since the subsequence $\{a_{n_k}\}$ converges to $a$, there is an index $K$ such that
>
> $$|a_{n_k} - a| < \frac{\epsilon}{2} \quad \text{if } k \geq K. \tag{3.62}$$
>
> Now choose any index $k$ such that $k \geq K$ and $n_k \geq N$. Using the inequalities (3.61) and (3.62) together with the Triangle Inequality, it follows that, if $n \geq N$, then
>
> $$|a_n - a| = |(a_n - a_{n_k}) + (a_{n_k} - a)| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Theorem (Function limit analogue to Cauchy's criterion) (As in Laczkovich and Sós, Thm 10.34.) Let $f$ be defined on a punctured neighborhood of $\alpha$. Then $\lim_{x\to\alpha} f(x)$ exists, and is finite, iff $\forall\epsilon > 0$, $\exists\dot{U}$ such that

$$\forall x_1, x_2 \in \dot{U}_\alpha,\ \ |f(x_1) - f(x_2)| < \epsilon, \tag{3.63}$$

where $\dot{U}_\alpha$ is a punctured neighborhood of $\alpha$.

> Proof:
> ($\Longrightarrow$) Suppose that $\lim_{x\to\alpha} f(x) = b \in \mathbb{R}$, and let $\epsilon > 0$ be fixed. Then $\exists\dot{U}_\alpha$ such that $\forall x \in \dot{U}_\alpha$, $|f(x) - b| < \epsilon/2$. The triangle inequality shows that (3.63) holds for all $x_1, x_2 \in \dot{U}_\alpha$.
>
> ($\Longleftarrow$) Now suppose (3.63) holds. If $x_n \to \alpha$ and $x_n \neq \alpha$ for all $n \in \mathbb{N}$, then the sequence $f(x_n)$ satisfies the Cauchy criterion. Indeed, for a given $\epsilon$, choose a punctured neighborhood $\dot{U}_\alpha$ such that (3.63) holds for all $x_1, x_2 \in \dot{U}_\alpha$. Since $x_n \to \alpha$ and $x_n \neq \alpha$ for all $n$, $\exists N \in \mathbb{N}$ such that, $\forall n \geq N$, $x_n \in \dot{U}_\alpha$. If $n, m \geq N$, then by (3.63), we have $|f(x_n) - f(x_m)| < \epsilon$. From Cauchy's criterion for sequences (e.g., Stoll, Thm 3.6.2(a) and Thm 3.6.5), the sequence $\{f(x_n)\}$ is convergent.
>
> Fix a sequence $x_n \to \alpha$ that satisfies $x_n \neq \alpha$ for all $n$, and let $\lim_{n\to\infty} f(x_n) = b$. If $y_n \to \alpha$ is another sequence satisfying $y_n \neq \alpha$ for all $n$, then the combined sequence $(x_1, y_1, x_2, y_2, \ldots)$ also satisfies this assumption, and so the sequence of function values $s = (f(x_1), f(y_1), f(x_2), f(y_2), \ldots)$ is also convergent. Since $\{f(x_n)\}$ is a subsequence of this, the limit of $s$ can only be $b$. On the other hand, $\{f(y_n)\}$ is also a subsequence of $s$, so $f(y_n) \to b$. This holds for all sequences $y_n \to \alpha$ for which $y_n \neq \alpha$ for all $n$, so $\lim_{x\to a} f(x) = b$.

Theorem: Suppose $E$ is a subset of a metric space $X$ and $f : E \to \mathbb{R}$ is uniformly continuous. If $\{x_n\}$ is a Cauchy sequence in $E$, then $\{f(x_n)\}$ is a Cauchy sequence.

> Proof: Let $\epsilon > 0$. As $f$ is uniformly continuous on $E$, $\exists\delta_0$ such that, if $x, y \in E$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. As $\{x_n\}$ is a Cauchy sequence in $E$, $\exists N \in \mathbb{N}$ such that, if $m \geq n \geq N$, then $|x_m - x_n| < \delta$. It follows that if $m \geq n \geq N$, then $|f(x_m) - f(x_n)| < \epsilon$.

Definition: Suppose that for each natural number $n, S_n$ is a set of real numbers. Then we denote the collection of these sets by $\{S_n\}_{n=1}^{\infty}$. For a set $S$ of real numbers, we say that the collection of sets $\{S_n\}_{n=1}^{\infty}$ is a *cover* for the set $S$ provided that, for each point $x$ in $S$, there is an index $n$ such that $x$ belongs to $S_n$. That is, $S \subseteq \cup_{n=1}^{\infty} S_n$. If it is the case that there is an index $N$ such that $S \subseteq \cup_{n=1}^{N} S_n$, then the finite collection of sets $\{S_1, \ldots, S_N\}$ is called a *finite subcover* of $\{S_n\}_{n=1}^{\infty}$ for the set $S$.

A cover of a set may, or may not, have a finite subcover. As an example, let $S$ be the set $[0, \infty)$ of nonnegative real numbers. Define $I_n \equiv (-n, n)$, for every index $n$. Then $S \subseteq \cup_{n=1}^{\infty} I_n$, so the collection of open intervals $\{I_n\}_{n=1}^{\infty}$ covers $S$. It is not the case that any finite collection covers $S$, because, no matter what index $N$ is chosen, $\cup_{n=1}^{N} I_n$ does not contain the number $N + 1$.

As a next example, take an interval $[a, b]$ and remove a point $c$ in $(a, b)$ so $S \equiv \{x \mid a \leq x \leq b, x \neq c\}$. Define $I_n \equiv (c - n, c - 1/n)$, if the index $n$ is odd; and $I_n \equiv (c + 1/n, c + n)$, if the index $n$ is even. From the convergence of $\{1/n\}$ to 0, it follows that the collection of open intervals $\{I_n\}_{n=1}^{\infty}$ covers the whole set of real numbers not equal to $c$, so it certainly covers the set $S$. It is not the case that any finite subcollection $\{I_n\}_{n=1}^{\infty}$ covers $S$, since no matter what index $N$ is chosen, $\cup_{n=1}^{N} I_n$ does not contain the points in $S$ whose distance from $c$ is less than $1/N$.

Definition: A subset $S$ of $\mathbb{R}$ is said to be *(topologically) compact* provided that any cover of $S$ by a collection $\{I_n\}_{n=1}^{\infty}$ of open intervals has a finite subcover. That is, if for each index $n$, $I_n$ is an open interval and $S \subseteq \cup_{n=1}^{\infty} I_n$, then there is an index $N$ such that $S \subseteq \cup_{n=1}^{N} I_n$.

The previous definition is sometimes called countable compactness, as the index set was $\mathbb{N}$. The index set can be uncountable. See, e.g., Jakob and Evans, volume I, Def. 20.22.

As an example, let $S = (0, 1]$, and let $S_n = (0, 1 + \frac{1}{n})$. It is tempting to say: Any $S_i$ is a finite cover for $S$, and thus $S$ is (topologically) compact. This is incorrect. For compactness of $S$, we need that *any* cover of $S$ by a collection $\{I_n\}_{n=1}^{\infty}$ of open intervals has a finite subcover. Take $S_n = (\frac{1}{n}, 2)$. Clearly, $S \subseteq \cup_{n=1}^{\infty} S_n$. Thus, the infinite union of $S_n$ is an open cover of $S$, however, it will not have a finite subcover, because, for any index $N \in \mathbb{N}$, the numbers between 0 and $1/N$ will not be covered.

**Example 3.5** *Let $A = \{1/n : n \in \mathbb{N}\}$. We wish to show that $A$ is not compact. For each $n \in \mathbb{N}$, choose $0 < \epsilon_n < n^{-1} - (n+1)^{-1}$, in which case $N_{\epsilon_n}(1/n) \cap A = \{1/n\}$. This forms a countably infinite cover of $A$ such that no finite subcover can cover $A$, and thus $A$ is not compact.*

*Below we give the Heine-Borel-Bolzano-Weierstrass theorem, which, for now, states that, if $A$ is not closed, then it is not compact. Notice 0 is the only limit point of $A$ (and all points of $A$ are isolated points), and, as $0 \notin A$, $A$ cannot be closed, and thus not compact.*

*Now we show that $K = A \cup \{0\}$ is compact. Any cover must include an open set around zero. For any $\epsilon > 0$, $B_\epsilon(0)$ is an open interval that covers $\{0\}$. It also covers an infinite number of the $1/n$ sequence in $A$ (because 0 is a limit point of $A$). The finite rest of the elements of $A$ can be covered by a finite number of open sets. Thus, any cover of $K$ by a collection $\{I_n\}_{n=1}^{\infty}$ of open intervals has a finite subcover, and thus $K$ is compact.* ■

Proposition: The set $K = [0, 1] \cap \mathbb{Q}$ is not compact.

> Proof: Observe that the cover
>
> $$[0,1] \cap \mathbb{Q} \subseteq \left(-1, \frac{\sqrt{2}}{2}\right) \cup \bigcup_{n \in \mathbb{N}} \left(\frac{\sqrt{2}}{2} + \frac{1}{n}, 2\right)$$
>
> has no finite subcover. The reason is that a finite cover implies $\exists N \in \mathbb{N}$ being the upper limit of the rhs union. Between $\sqrt{2}/2$ and $\sqrt{2}/2 + 1/N$, there is a rational number (infinitely many in fact), because the set of rational numbers is dense in $\mathbb{R}$.

The proof of the next result uses a countable (finite) index set if compact set $K$ consists of a finite number of (necessarily isolated) points of $\mathbb{R}$. It requires an uncountable index set for the cover of compact set $K$, if set $K$ is uncountable. From the previous proposition, $K$ cannot be an infinite subset of the rational numbers.

Theorem: Let $K$ be a compact subset of a metric space $X$, and let $f$ be a real-valued function on $K$. Suppose that, $\forall x \in K$, $\exists \epsilon_x > 0$ such that $f$ is bounded on $N_{\epsilon_x}(x) \cap K$. Then $f$ is bounded on $K$.

Proof: By hypothesis, $\forall x \in K$, $\exists \epsilon_x > 0$ and $\exists M_x > 0$ such that, $\forall y \in N_{\epsilon_x}(x) \cap K$, $|f(y)| \leq M_x$. The collection $\{N_{\epsilon_x}(x)\}_{x \in K}$ is an open cover of $K$. As $K$ is compact, $\exists N \in \mathbb{N}$ and a set $S = \{x_1, \ldots, x_N\}$ such that $\{N_{\epsilon_x}(x)\}_{x \in S}$ covers $K$. Let $M_S = \max\{M_{x_1}, \ldots, M_{x_N}\}$. Thus, $\forall y \in K$, $|f(y)| \leq M_S$.

Proposition: Let $S$ be a compact subset of $\mathbb{R}$. Then

$$S \text{ is both closed and bounded.} \tag{3.64}$$

Proof:

($S$ is bounded) For each index $n$, define $I_n \equiv (-n, n)$. Then $\{I_n\}_{n=1}^{\infty}$ is a collection of open intervals that covers $\mathbb{R}$, so certainly it also covers the set $S$. Since $S$ is compact, there is an index $N$ such that $S \subseteq \cup_{n=1}^{N} I_n$, so that $|x| < N$ for all $x \in S$. Thus, the set $S$ is bounded.

($S$ is closed) (As in Fitzpatrick, Prop 2.40.) Let $\{a_n\}$ be a sequence in the set $S$ that converges to the number $a$. We must show that $a$ also belongs to $S$. We argue by contradiction. Indeed, suppose that $a$ does not belong to the set $S$. Define $J_n \equiv (a-n, a-1/n)$, if the index $n$ is odd; and $J_n \equiv (a+1/n, a+n)$, if the index $n$ is even. As $\{1/n\}$ converges to 0, we see that $\{J_n\}_{n=1}^{\infty}$ is a collection of open intervals that covers the whole set of real numbers not equal to $a$. But we have supposed that $a$ does not belong to $S$, and therefore $\{J_n\}_{n=1}^{\infty}$ is a cover of $S$ by a collection of open intervals. It is not the case that any finite subcollection $\{J_n\}_{n=1}^{\infty}$ covers $S$, since no matter what index $N$ is chosen, because $\{a_n\}$ is a sequence in $S$ that converges to $a$, there are points in $S$ whose distances from $a$ are less than $1/N$. This contradiction of the compactness property of $S$ shows that in fact $a$ does belong to $S$. Thus, $S$ is closed.

See also Jakob and Evans, volume I, Prop. 20.24, for a different proof. In particular, their proof of boundedness is the same, but to show closure, they show $S^C$ is open. To do this, they use an uncountable cover of $S$. The same type of proof is given in Heil, Metrics, pp. 76-7, and is repeated here.

($S$ is closed) (As in Heil, pp. 76-7.) If $K = X$ then we are done (recall $X$ is always closed), so assume that $K \neq X$. Fix any point $y$ in $K^{\mathrm{C}} = X \backslash K$. If $x$ is a point in $K$ then $x \neq y$, so by the Hausdorff property, there must exist disjoint open sets $U_x, V_x$ such that $x \in U_x$ and $y \in V_x$. The collection $\{U_x\}_{x \in K}$ is an open cover of $K$, so it must contain some finite subcover. That is, there must exist finitely many points $x_1, \ldots, x_N \in K$ such that

$$K \subseteq U_{x_1} \cup \cdots \cup U_{x_N}.$$

Each $V_{x_j}$ is disjoint from $U_{x_j}$, so it follows from equation (2.21) that the set

$$V = V_{x_1} \cap \cdots \cap V_{x_N}$$

is entirely contained in the complement of $K$. Hence $V$ is an open set that satisfies

$$y \in V \subseteq K^{\mathrm{C}}.$$

This shows that $K^{\mathrm{C}}$ is open (because $y$ was an arbitrary point from $K^{\mathrm{C}}$, and thus each point $y$ satisfies definition (3.1)), so we conclude that $K$ is closed.

Recall from the definition above that a set of real numbers $S$ is sequentially compact, provided that every sequence $\{a_n\}$ in $S$ has a subsequence that converges to a point that belongs to $S$. The following proposition shows that a sequentially compact set is (topologically) compact. The proof we show is from Fitzpatrick, p. 50. It is brilliant and not difficult, though not trivial.

Proposition: Let $S$ be a sequentially compact subset of $\mathbb{R}$. Then $S$ is compact.

Proof: Suppose that $\{I_n\}_{n=1}^{\infty}$ is a cover of $S$ by a collection of open intervals. We will show there is an index $N$ such that

$$S \subseteq \cup_{n=1}^{N} I_n. \tag{3.65}$$

Since $\{I_n\}_{n=1}^{\infty}$ covers $S$, for a point $x$ in $S$ we can define its cover index to be the smallest index $k$ such that $x$ belongs to $I_k$ and denote this cover index by *cover index* $(x)$. Observe that

$$\text{cover index } (x) \leq k \quad \text{ if and only if } x \text{ belongs to } \cup_{n=1}^{k} I_n,$$

and that (3.65) holds if and only if

$$\text{cover index } (x) \leq N \quad \text{ for all } x \text{ in } S. \tag{3.66}$$

Now observe that, given a point $x$ in $S$, $x$ belongs to $I_n$, where $n$ is the cover index of $x$. Since $I_n$ is an open interval, there is an open interval $J$ centered at $x$ such that $J \subseteq I_n$. It follows that every point in $S \cap J$ has a cover index of at most $n$, that is,

$$\text{cover index } (z) \leq \text{cover index } (x) \qquad \text{for all } z \text{ in } S \cap J. \tag{3.67}$$

To help see this, note that $J \subset I_n$, so any $z \in S \cap J$ is in $I_n$, so index$(z)$ is at most $n$.

If there is no natural number $N$ such that (3.66) holds, then for each natural number $n$, there is a point in $S$ whose cover index is greater than $n$; choose such a point and label it $x_n$. Thus, $\{x_n\}$ is a sequence in $S$ such that

$$\text{cover index } (x_n) > n \quad \text{ for every index } n. \tag{3.68}$$

But, by assumption, the set $S$ is sequentially compact. Thus, there is a subsequence $\{x_{n_k}\}$ that converges to a point $x_0$ that also belongs to $S$. As noted above, we can choose an open interval $J$ centered at $x_0$ such that (3.67) holds. However, $x_0$ is the limit of the sequence $\{x_{n_k}\}$, so there is an index $K$ such that

$$x_{n_k} \text{ belongs to } J \text{ for each index } k \geq K.$$

Thus,

$$\text{cover index } (x_{n_k}) \leq \text{ cover index } (x_0) \quad \text{ for each index } k \geq K.$$

This contradicts the property that, by (3.68),

$$\text{cover index } (x_{n_k}) \geq n_k \geq k \quad \text{ for all indices } k.$$

Therefore, the assumption that there was no finite subcover has led to a contradiction and hence we conclude that there is a finite subcover. Thus, the sequentially compact set $S$ is compact.

The above results lead to the following famous result.

Theorem (Heine-Borel-Bolzano-Weierstrass, HBBW) For a subset $S$ of $\mathbb{R}$, the following three assertions are equivalent to each other:

1. $S$ is closed and bounded.
2. $S$ is sequentially compact.
3. $S$ is compact.

> Proof: The three above propositions show 1 implies 2; 2 implies 3; and 3 implies 1.

Remark: The assertion that a closed bounded subset of $\mathbb{R}$ is compact is often referred to as the Heine-Borel Theorem. The assertion that a closed bounded subset of $\mathbb{R}$ is sequentially compact is often also referred to as the Bolzano-Weierstrass Theorem.

Remark: For a direct proof of 2 to 1, see, e.g., Kuttler, Calculus of One and Many Variables, Prop. 3.6.17. For a direct proof of 1 to 3, see, e.g., Jakob and Evans, volume I, Thm 20.26. We reproduce the latter here, as the next proposition, because it is instructional. To do so, we first need a proposition, also from Jakob and Evans. It uses the nested interval theorem, showcasing yet again its importance in analysis.

Proposition: Every bounded closed interval $[a, b] \subset \mathbb{R}$ is compact.

> Proof: We prove the proposition by contradiction. Suppose that there is an open covering $(A_\nu)_{\nu \in I}$ of $[a, b]$ which has no finite subcovering. For $m = \frac{a+b}{2}$ it follows that at least one of the intervals $[a, m]$ and $[m, b]$ cannot be covered by a finite subcovering of $(A_\nu)_{\nu \in I}$. Call this interval $I_1$. By induction we get a sequence of closed intervals $(I_j)_{j \in \mathbb{N}}$ with the following properties:
>
> (i) $[a, b] \supset I_1 \supset I_2 \supset \ldots$
>
> (ii) $I_j$ is not covered by a finite subcovering of $(A_\nu)_{\nu \in I}$
>
> (iii) for $x, y \in I_j$ it follows that $|x - y| < 2^{-j}(b - a)$.
>
> By the Nested Interval, Theorem, there is one point $x_0$ that lies in $\bigcap_{j \in \mathbb{N}} I_j$. Therefore, for some $j_0$ we have $x_0 \in A_{j_0}$. Since $A_{j_0}$ is open there is some $\varepsilon > 0$ such that $|y - x_0| < \varepsilon$ implies $y \in A_{j_0}$. Taking $n$ such that $2^{-n}(b-a) < \varepsilon$, then it follows from (iii) that $I_n \subset A_{j_0}$ which contradicts (ii).

Theorem: For a subset $S$ of $\mathbb{R}$, if $S$ is closed and bounded, then $S$ is (topologically) compact.

> Proof: Let $(A_\nu)_{\nu \in I}$ be an open covering of the closed and bounded set $K$. Since $K$ is bounded, there exists a closed interval $[a, b] \subset \mathbb{R}$ such that $K \subset [a, b]$. The family of open sets $(A_\nu)_{\nu \in I}$, together with $A_p := \mathbb{R} \backslash K$ form an open covering of $\mathbb{R}$, since $\bigcup_{j \in I} A_j \cup A_p \supset K \cup K^c = \mathbb{R}$. Therefore, $(A_\nu)_{\nu \in I \cup \{p\}}$ is also an open covering of $[a, b]$ and by the previous proposition, it contains a finite subcovering $\left(A_{\nu_j}\right)_{\nu_j \in I_N}$ where $I_N$ is a finite subset of $I \cup \{p\}$. If $p \in I_N$, then, since $K \cap A_p = \emptyset$, we can remove $A_p$ and we still have a finite covering of $K$.

**Example 3.6** *Let $A$ and $B$ be compact subsets of $\mathbb{R}$. Is $A \setminus B$ necessarily compact? It is not. For example, let $A = [0, 1]$ and $B = [1/2, 1]$. Then $A \setminus B = [0, 1/2)$, which is not closed, and therefore by Heine-Borel, is not compact.* ■

Theorem: Let $B \subset \mathbb{R}$ be bounded. If $B$ is infinite, then $B$ has a limit point.

Proof: Supposing $B$ is infinite, then $\exists \{b_n\} \in B$ such that $b_i \neq b_j$ when $i \neq j$. As $\{b_n\}$ is bounded, Bolzano-Weierstrass implies that it contains a convergent subsequence $\{b_{n_k}\}$, with limit, say, $L$. As the $b_{n_k}$ are all distinct, at most one term can equal $L$. Form the sub-subsequence $\{b_{n_{k_\ell}}\}$ from $\{b_{n_k}\}$ by deleting the term equal to $L$, if it exists. Thus, $\{b_{n_{k_\ell}}\}$ is a sequence contained in $B$ and converging to $L$, with $b_{n_{k_\ell}} \neq L$ for all $\ell$. Therefore, $L$ is a limit point of $B$, so $B$ does indeed have a limit point.

Theorem: If $K$ is sequentially compact and if $H$ is a closed subset of $K$, then

$$H \text{ is sequentially compact.} \tag{3.69}$$

Proof: Set $H$ is closed, and is also bounded, because it is a subset of $K$, which is compact and thus, from HBBW, $K$ must also be (closed and) bounded. Thus, $H$ is compact, from HBBW.

The result can also be proven without use of HBBW, and only with the concept of sequential compactness. Recall (i) A subset $S$ of $\mathbb{R}$ is closed provided that, if $\{a_n\}$ is a sequence in $S$ that converges to a number $a$, then $a \in S$; and (ii) a set of real numbers $S$ is said to be sequentially compact provided that every sequence $\{a_n\}$ in $S$ has a subsequence that converges to a point that belongs to $S$. Now let $\{x_n\} \subseteq H$. Then since $K$ is sequentially compact, from the definition, there is a subsequence, $\{x_{n_k}\}$ that converges to a point, $x \in K$. But these $x_{n_k}$ are in the closed set $H$ and so $x \in H$ from the definition of closed.

The result can also be proven only using topological compactness (and the facts that (i) the complement of a closed set is open; and (ii) the union of open sets is open). Let $H$ be a closed subset of the compact set $K$ and let $\{O_\alpha\}_{\alpha \in A}$ be an open cover of $H$. Then $\{O_\alpha\}_{\alpha \in A} \cup \{H^c\}$ is an open cover of $K$. Since $K$ is compact, a finite number of these will cover $K$, and hence also $H$.

Theorem: For compact $A, B \subset \mathbb{R}$, $A \cup B$ and $A \cap B$ are compact.

Proof: Both $A \cup B$ and $A \cap B$ are closed and bounded, and thus Heine-Borel applies.

More challenging is to prove the results without invoking Heine Borel. Consider the following ideas.

Proof of compactness of $A \cup B$ via Sequential Compactness:

Let $\{u_n\}$ be any sequence in $A \cup B$. There are only three, distinct, possibilities:

- Sequence $\{u_n\}$ eventually only contains elements of set $A$, i.e., there is an $N \in \mathbb{N}$ such that, for $n \geq N$, $u_n \in A$.
- Sequence $\{u_n\}$ eventually only contains elements of set $B$.
- Neither of the above cases.

In the first case, there exists a subsequence of $\{u_n\}$ such that all its elements are in $A$. As $A$ is compact, there is a subsequence of this subsequence that converges to a point in $A$. As $A \subset (A \cup B)$, this subsubsequence converges in $A \cup B$ as well, and thus, for this case, $A \cup B$ is sequentially compact.

The second case is similar. For the third case, there is a subsequence of $\{u_n\}$ such that all its elements are in $A$ (and likewise in $B$, but we only need one of these two cases). As previous, as $A$ is compact, there is a subsequence of this subsequence that converges to a point in $A$; and as $A \subset (A \cup B)$, $A \cup B$ is compact.

Proof of compactness of $A \cap B$ via Sequential Compactness:

Let $\{u_n\}$ be any sequence in $A \cap B$. As $(A \cap B) \subset A$, $\{u_n\}$ is obviously a sequence in $A$. As $A$ is compact, $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ that converges to a point, say $p_A$, in $A$. As all $u_{n_k} \in (A \cap B) \subset B$, and $B$ is compact, there is a subsequence of this subsequence that converges to a point in $B$, say $p_B$. But $\{u_{n_k}\}$ is a convergent (sub)sequence, and thus its subsequences converge to the same limit, i.e., $p := p_A = p_B$. As $p_A \in A$ and $p_B \in B$, $p \in (A \cap B)$. Thus, we have found a (sub)subsequence $A \cap B$ that converges to a point in $A \cap B$, showing that $A \cap B$ is (sequentially) compact.

Proof of compactness of $A \cup B$ via Topological Compactness: We give two proofs that differ only slightly. The first avoids the use of double subscripts, while the second results in a minimal subcover.

- Proof 1: Let $\cup_{n=1}^{\infty} I_n$ be any arbitrary cover of $A \cup B$ where for each $n$, $I_n$ is an open interval. Note that $A \subset A \cup B$ and $B \subset A \cup B$, therefore $\cup_{n=1}^{\infty} I_n$ is an open cover of $A$ and of $B$. By compactness of $A$ and $B$ there exist $N_1, N_2 \in \mathbb{N}$ such that
$$A \subset \cup_{n=1}^{N_1} I_n \text{ and } B \subset \cup_{n=1}^{N_2} I_n.$$
Take the union of the finite subcovers. This union is again finite and a cover of $A \cup B$:
$$A \cup B \subset \left[\cup_{n=1}^{N_1} I_n\right] \cup \left[\cup_{n=1}^{N_2} I_n\right].$$
We have found a finite subcover for $A \cup B$ for an arbitrary cover of open intervals. Hence, $A \cup B$ is compact.

- Proof 2: Let $\cup_{n=1}^{\infty} I_n$ be any arbitrary cover of $A \cup B$ where for each $n$, $I_n$ is an open interval. Note that $A \subset A \cup B$ and $B \subset A \cup B$, therefore $\cup_{n=1}^{\infty} I_n$ is an open cover of $A$ and of $B$. By compactness of $A$, there exists a $k_A \in \mathbb{N}$ and a set of natural numbers $S_A := \{n_1, \ldots, n_{k_A}\}$ such that $I_{n_k}$, $k = 1, \ldots, k_A$, is a finite subcover of $A$. Likewise, as $B$ is compact, similarly define $k_B$ and $S_B := \{m_1, \ldots, m_{k_B}\}$. Then $S := S_A \cup S_B$ forms a finite subcover of $A \cup B$.

Proof of compactness of $A \cap B$ via Topological Compactness: We give three correct proofs, and one that is wrong, but might entice someone because it initially seem appealing.

- Proof 0: (Intentionally Erroneous)
"A finite subcover of $A \cup B$ also covers $A \cap B$, and we just proved that $A \cup B$ is compact. QED" This is wrong because we need to start with ***any*** countable cover of $A \cap B$. Say, for example, $A = [-1, 2]$ and $B = [1, 3]$, and $A \cap B = [1, 2]$. The cover $\{I_n\}$, with $I_n = (0, n)$ clearly covers $A \cap B$, but it is not a cover for $A \cup B$. So, this "proof" fails.

- Proof 1: From §3.1 $A \cap B$ is closed, and clearly, $A \cap B$ is a subset of $A$, and $A$ is compact. Thus, from (3.69), $A \cap B$ is compact.

- Proof 2: Let $\{I_n\}$ be any cover of $A \cap B$ by open sets. Add the set $B^c$ to the collection. Then $\{I_n, B^c\}$ is an open cover of $A$, so it has a finite subcover. That subcover might include $B^c$, but if you remove that one set, then the remaining finitely many must cover $A \cap B$.

- Proof 3: Let $\cup_{n=1}^{\infty} I_n$ be any arbitrary cover of $A \cap B$ where, $\forall n$, $I_n$ is an open interval. Note that $(A \cap B)^c = \mathbb{R} \setminus (A \cap B)$. Further note that $\cup_{n=1}^{\infty} I_n \cup [\mathbb{R} \setminus (A \cap B)] = \mathbb{R}$. Therefore, $\cup_{n=1}^{\infty} I_n \cup [\mathbb{R} \setminus (A \cap B)]$ is a union of open intervals that covers $\mathbb{R}$ and thus a valid cover of $A \cup B$. From the Characterization of the Open Subsets of $\mathbb{R}$, we can rewrite $[\mathbb{R} \setminus (A \cap B)] = \cup_{n=1}^{\infty} J_n$ where each $J_n$ is an open interval. As $A \cup B$ is compact, $\exists N \in \mathbb{N}$ such that

$$A \cup B \subset \left[\cup_{n=1}^{N} I_n\right] \cup \left[\cup_{n=1}^{N} J_n\right].$$

Note $(A \cap B) \subset (A \cup B)$ and $[A \cap B] \cap [\mathbb{R} \setminus (A \cap B)] = \emptyset$. The finite subcover of $A \cup B$ we produced is also a finite subcover of $A \cap B$, but all intervals in $\cup_{n=1}^{N} J_n$ are disjoint from $A \cap B$. We can thus remove these intervals and still have a finite subcover of $A \cap B$. That is, $A \cap B \subset \cup_{n=1}^{N} I_n$. We have found a finite subcover for any arbitrary open cover of $A \cap B$ and thus conclude that $A \cap B$ is compact.

Theorem: Let $K \subset \mathbb{R}$ be nonempty and compact. Then

$$\sup K \text{ and } \inf K \text{ exist and are in } K. \tag{3.70}$$

Proof: From HBBW, $K$ is (closed and) bounded. Existence of $\alpha := \sup K \in \mathbb{R}$ follows because $K$ is nonempty and bounded. To show $\alpha \in K$, assume the contrary, $\alpha \in K^c$. As $K^c$ is open, $\exists \epsilon > 0$ such that $B_\epsilon(\alpha) \in K^c$. By definition of sup, $\exists a_n \in K$ such that, $\forall n \in \mathbb{N}$, $\alpha - 1/n < a_n \leq \alpha$. But for $n > 1/\epsilon$, $a_n \in B_\epsilon(\alpha) \in K^c$, which is a contradiction. The proof for the infimum is similar.

Definition: The distance between nonempty subsets $A, B \in X$ is defined as

$$d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}. \tag{3.71}$$

Note that $A$ and $B$ disjoint does not imply that the distance between them must be strictly positive. As an example, the distance between the intervals $A = (0, 1)$ and $B = (1, 2)$ is zero, even though $A \cap B = \varnothing$.

Theorem: Let $A, B \subset \mathbb{R}$ with $A$ compact, $B$ closed, and $A \cap B = \emptyset$. Then $d(A, B) > 0$.

Proof: Assume $d(A, B) = 0$. Then $\exists \{a_n\} \subset A$ and $\exists \{b_n\} \subset B$ such that $|a_n - b_n| \to 0$. The latter implies that, for any subsequences $\{a_{n_k}\}$ and $\{b_{n_k}\}$, $|a_{n_k} - b_{n_k}| \to 0$. As $A$ is compact, HBBW implies $\exists \{a_{n_k}\}$ such that $a_{n_k} \to a \in A$, so that

$$|a - b_{n_k}| \leq |a - a_{n_k}| + |a_{n_k} - b_{n_k}| \xrightarrow{k \to \infty} 0.$$

Thus, $a$ is a limit point of $\{b_n\}$, but as $B$ is closed, $a \in B$. But $a \in A$, contradicting that $A \cap B = \emptyset$. So, $d(A, B) \neq 0$, which means $d(A, B) > 0$.

An important case is: If $[a, b]$ and $[c, d]$ are disjoint bounded closed intervals, then the distance between them is strictly positive. We cannot relax the condition in the previous theorem that $A$ is compact. To demonstrate this, we just need to find an example of $A, B \in X$ with $A \cap B = \emptyset$, $B$ is closed, and $A$ is closed but not necessarily compact, and such that $d(A, B) = 0$. We give two such examples.

1. Let $A = \{n : n = 2, 3, 4, \ldots\}$ and $B = \{n + 1/n : n = 2, 3, 4, \ldots\}$. Both $A$ and $B$ consist entirely of isolated points (in each case, the distance between distinct elements is at least 1/2), so both $A$ and $B$ are closed. Fix $\epsilon > 0$. From the Archimedean Property, $\exists N \in \mathbb{N}$ such that $N > \max(1/\epsilon, 1)$. Now, $N \in A$ and $N + 1/N \in B$, and $|N - (N + 1/N)| = 1/N < \epsilon$. Such an $N$ exists for every $\epsilon > 0$, so $\nexists \epsilon > 0$ such that $\forall (a \in A, b \in B), |a - b| > \epsilon$.

2. Consider any $x = n_1 \in A$ and $y = (n_2 + 1/n_2) \in B$, $n_1, n_2 = 2, 3, \ldots$. If $n_1 \neq n_2$, then $|x - y| > 1$. If $n_1 = n_2 = n$, $n = 2, 3, \ldots$, then $|x - y| = n^{-1} > 0$ and $|x - y| \to 0$ as $n \to \infty$. Thus, $d(A, B) = 0$.

Having the tool of (sequential) compactness, we can now prove the Extreme Value Theorem (2.59). We begin with a lemma.

Lemma: The image of a continuous function on a closed bounded interval, $f : [a, b] \to \mathbb{R}$, is bounded above; that is, there is a number $M$ such that $f(x) \leq M$ for all $x \in [a, b]$.

> Proof: We will argue by contradiction. Assume that there is no such number $M$. Let $n$ be a natural number. Then it is not true that $f(x) \leq n$ for all $x \in [a, b]$. Thus, there is a point $x$ in $[a, b]$ at which $f(x) > n$. Choose such a point and label it $x_n$. This defines a sequence $\{x_n\}$ in $[a, b]$ with the property that $f(x_n) > n$ for every index $n$. We can employ the Sequential Compactness Theorem to choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to a point $x_0$ in $[a, b]$. Since the function $f : [a, b] \to \mathbb{R}$ is continuous at $x_0$, the image sequence $\{f(x_{n_k})\}$ converges to $f(x_0)$. But from (2.3), a convergent sequence is bounded, so the sequence $\{f(x_{n_k})\}$ is bounded. This contradicts the property that
>
> $$f(x_{n_k}) > n_k \geq k \quad \text{for all indices } k.$$
>
> This contradiction proves that the image of $f : [a, b] \to \mathbb{R}$ is bounded above.

Theorem (Extreme Value Theorem): A continuous function on a closed bounded interval, $f : [a, b] \to \mathbb{R}$,

$$\text{attains both a minimum and a maximum value.} \tag{3.72}$$

> Proof: We first need to show that the image $f(D)$ is bounded above. The is the content of the previous lemma. Next, we need to demonstrate that the number $\sup f(D)$ is a functional value.
>
> Define $S \equiv f([a, b])$. Then $S$ is a nonempty set of real numbers that, by the preceding lemma, is bounded above. According to the Completeness Axiom, $S$ has a supremum. Define $c \equiv \sup S$. It is necessary to find a point $x_0$ in $[a, b]$ at which $c = f(x_0)$.
>
> Let $n$ be a natural number. Then the number $c - 1/n$ is smaller than $c$ and is therefore not an upper bound for the set $S$. Thus, there is a point $x$ in $[a, b]$ at which $f(x) > c - 1/n$. Choose such a point and label it $x_n$. From this choice and from the fact that $c$ is an upper bound for $S$, we see that $c - 1/n < f(x_n) \leq c$ for every index $n$. Hence the sequence $\{f(x_n)\}$ converges to $c$.

The Sequential Compactness Theorem asserts that there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to a point $x_0$ in $[a,b]$. Since $f:[a,b]\to\mathbb{R}$ is continuous at $x_0$, $\{f(x_{n_k})\}$ converges to $f(x_0)$. But $\{f(x_{n_k})\}$ is a subsequence of the sequence $\{f(x_n)\}$ that converges to $c$, so $c=f(x_0)$. The point $x_0$ is a maximizer of the function $f:[a,b]\to\mathbb{R}$.

To complete the proof, we observe that the function $-f:[a,b]\to\mathbb{R}$ is also continuous. Consequently, using what we have just proven, we can select a point in $[a,b]$ at which $-f:[a,b]\to\mathbb{R}$ attains a maximum value, and at this point the function $f:[a,b]\to\mathbb{R}$ attains a minimum value.

## 3.6 Functions, Compactness, and Continuity

Before stating the topological characterization of continuity of a function, we first review some definitions of limits and continuity. We present these in the more general framework of metric spaces, say $(X,d)$, as discussed in §3.2. For our purposes, the reader can just take space $X$ to be $\mathbb{R}$ or $\mathbb{R}^n$, and distance measure $d$ to be the usual Euclidean distance, e.g., $|x-y|$ for $x,y\in\mathbb{R}$, and $\|\mathbf{x}-\mathbf{y}\|$ for $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, where, from (1.24), the norm of $\mathbf{x}=(x_1,\ldots,x_n)\in\mathbb{R}^n$ is $\|\mathbf{x}\|=\sqrt{x_1^2+\cdots+x_n^2}$.

Let $(X,d)$ be a metric space, $E$ be a subset of $X$, and $f:E\to\mathbb{R}$.

Definition: Suppose that $p$ is a limit point of $E$. The function $f$ has a limit at $p$ if there exists a number $L\in\mathbb{R}$ such that, given any $\epsilon>0$, $\exists\delta>0$ for which $|f(x)-L|<\epsilon$ for all points $x\in E$ satisfying $0<d(x,p)<\delta$. The constraint on $x$ can be written in terms of a "punctured neighborhood" of $p$ as (with $\ni$ being a shortcut for "such that")

$$\exists L\in\mathbb{R}\ni\forall\epsilon>0,\exists\delta>0\ni\forall x\in B_\delta(p)\cap(E\setminus\{p\}),|f(x)-L|<\epsilon. \tag{3.73}$$

We now state some equivalent definitions of continuity of a function that we developed in §2.1.

1. ($\epsilon$-$\delta$) A function $f:E\to\mathbb{R}$ is continuous at $p\in E$ if given $\epsilon>0$, $\exists\delta>0$ such that

$$\forall x\in N_\delta(p)\cap E,\quad f(x)\in N_\epsilon(f(p)).$$

2. ($\epsilon$-$\delta$) One can also write this in terms of distance measures as follows. Let $(X,d_X)$ and $(Y,d_Y)$ be metric spaces; and let $f:X\to Y$. Let $X=E$ and $Y=\mathbb{R}$. Function $f:E\to\mathbb{R}$ is continuous at $p\in E$ if

$$(\forall\epsilon>0)\,(\exists\delta>0)\,(\forall y\in E)\ \big[d_X(p,y)<\delta\Longrightarrow d_Y(f(p),f(y))<\epsilon\big]. \tag{3.74}$$

3. (sequential) A function $f:E\to\mathbb{R}$ is continuous at $p\in E$ if

$$\{x_n\}\text{ any sequence in }E\text{ such that }x_n\to p\Longrightarrow f(x_n)\to f(p).$$

4. If $p\in E$ is a limit point of $E$, then ($f$ is continuous at $p$) $\Leftrightarrow\lim_{x\to p}f(x)=f(\lim_{x\to p}x)$. This follows from (3.73) with $L=f(p)$.

Recall also that function $f$ is continuous on $E$ if $f$ is continuous at every point $p\in E$.

The following theorem can be found in most all books on real analysis. It is not obvious initially why characterising continuity in a topological sense is of great value. It plays a crucial role in higher mathematics such as metric space theory and functional analysis. It also serves as an excellent motivation for the definition of a measurable function in measure theory. In the statement of the theorem, there is a reference to a set, say $U$, being relatively open with respect to another set $E$, with $U \subset E$, commonly shortened to just saying "$U$ is open in $E$". To do things correctly, we would need to detail this concept. Instead, we provide the proof in a simpler case that anyway gives the core idea of the proof. A discussion of relatively open sets can be found in, e.g., Stoll, 2021, as well as the general proof of the next theorem (Stoll, Thm 4.2.6). I have in the meantime discovered a very detailed and outstanding presentation of relatively open sets (and the general proof of the below theorem) in Heil, 2025, forthcoming.

Theorem (Topological Characterization of Continuity): Let $X, Y$ be metric spaces, and let $E \subset X$. Then a function $f : X \to Y$ is continuous on $E$ if and only if $f^{-1}(V)$ is open in $E$ for every open subset $V \subset Y$.

> Proof: We prove this in the simpler case for $E = X$, as given in Ash, Thm 4.1.6. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces; and let $f : X \to Y$. We need to show: The function $f$ is continuous on $X$ if and only if for each open set $V \subset Y$ the pre-image $f^{-1}(V)$ is an open subset of $X$.
>
> Assume $f$ continuous. Let $x$ belong to $f^{-1}(V)$, where $V$ is open in $Y$. Then $f(x) \in V$, so for some $\epsilon > 0, B_\epsilon(f(x)) \subset V$. If $\delta > 0$ is as given in (3.74), then
>
> $$y \in B_\delta(x) \Longrightarrow f(y) \in B_\epsilon(f(x)); \text{ hence, } f(y) \in V.$$
>
> Thus, $y \in f^{-1}(V)$, proving that $B_\delta(x) \subseteq f^{-1}(V)$. Therefore $f^{-1}(V)$ is open.
>
> Conversely, assume $V$ open implies $f^{-1}(V)$ open. If $x \in X$, we show that $f$ is continuous at $x$. Given $\epsilon > 0, f(x) \in B_\epsilon(f(x))$, which is an open set $V$. Thus, $x \in f^{-1}(V)$, which is open by hypothesis, so $B_\delta(x) \subset f^{-1}(V)$ for some $\delta > 0$. Consequently,
>
> $$y \in B_\delta(x) \Longrightarrow y \in f^{-1}(V) \Longrightarrow f(y) \in V = B_\epsilon(f(x));$$
>
> in other words, $d_X(x, y) < \delta \Longrightarrow d_Y(f(x), f(y)) < \epsilon$. From (3.74), $f$ is continuous.

We can package the above result in the following theorem, and we add a third characterization.

Theorem Let $X$ and $Y$ be metric spaces and $f : X \to Y$. The following statements are equivalent.

(1) $f$ is continuous.
(2) For every open subset $U$ of $Y, f^{-1}(U)$ is an open subset of $X$.
(3) For every closed subset $F$ of $Y, f^{-1}(F)$ is a closed subset of $X$.

> Proof: The previous theorem shows the equivalence of (1) and (2). We need to show the equivalence of (2) and (3). Let $(X, d_X)$, $(Y, d_Y)$ be metric spaces, $E \subset X$, $f : E \to Y$. We know already that $f$ is continuous on $E \Longleftrightarrow \forall$ open subsets $V$ of $Y$, $f^{-1}(V)$ is open in $E$. First show that, $\forall F \subset Y$, $f^{-1}(Y \backslash F) = E \backslash f^{-1}(F)$. Take any $x \in E$. Then,
>
> $$x \in f^{-1}(Y \backslash F) \Longleftrightarrow f(x) \in Y \backslash F \Longleftrightarrow f(x) \notin F \Longleftrightarrow x \notin f^{-1}(F) \Longleftrightarrow x \in E \backslash f^{-1}(F).$$
>
> Next, note $F \subset Y$ closed in $Y \Longleftrightarrow Y \backslash F$ open in $Y$. As $f$ continuous on $E \Longleftrightarrow f^{-1}(Y \backslash F)$ open in $E$, we finally have the desired relation: The topological characterization of con-

tinuity in terms of open sets is equivalent to that in terms of closed sets:

$$F \subset Y \text{ closed in } Y \iff E \backslash f^{-1}(F) \text{ open in } E \iff f^{-1}(F) \text{ closed in } E.$$

We now give what amounts to the same proof, just formulated slightly differently. It comes from Ash, Thm 4.1.6. We must show that the preimage of each closed set is closed if and only if the preimage of each open set is open. Suppose that for each closed $C \subseteq Y$, $f^{-1}(C)$ is closed, and assume $V$ is an open subset of $Y$. Then $V^c$ is closed, so $f^{-1}(V^c)$ is closed. But by (1.16),

$$f^{-1}(V^c) = \left[f^{-1}(V)\right]^c.$$

Thus, $\left[f^{-1}(V)\right]^c$ is closed, so $f^{-1}(V)$ is open. Conversely, if the preimage of each open set is open and $C$ is a closed subset of $Y$, then $C^c$ is open, and hence $f^{-1}(C^c) = \left[f^{-1}(C)\right]^c$ is open. Therefore $f^{-1}(C)$ is closed.

As an example showing that the direct image of an open set under a continuous function need not be open, consider taking $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$, and $A = (0, 2\pi)$ open. Then $f(A) = [-1, 1]$. Under certain conditions, the result does hold, as shown subsequently, after proving a fundamental result we will require.

Theorem: Let $I$ be an interval and suppose that the function $f : I \to \mathbb{R}$ is continuous. Then its

$$\text{image } f(I) \text{ also is an interval.} \tag{3.75}$$

Proof: Let $y_1$ and $y_2$ be points in $f(I)$, with $y_1 < y_2$. We must show that the closed interval $[y_1, y_2]$ is also contained in $f(I)$. Indeed, let $y_1 < c < y_2$. Since $y_1$ and $y_2$ are in $f(I)$, there are points $x_1$ and $x_2$ in $I$ with $f(x_1) = y_1$ and $f(x_2) = y_2$. If we let $J$ be the closed interval having $x_1$ and $x_2$ as endpoints, then $J$ is contained in $I$ since, by assumption, the set $I$ is an interval. Thus, we can apply the Intermediate Value Theorem (2.60) to the function $f : J \to \mathbb{R}$ in order to conclude that there is a point $x_0$ in $J$ at which $f(x_0) = c$. Thus, $x_0$ belongs to $I$ and $f(x_0) = c$. Recall $y_1, y_2 \in f(I)$ and $c \in (y_1, y_2)$ was arbitrary. It follows that $[y_1, y_2]$ is contained in $f(I)$.

Theorem: Let $I \subset \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be strictly increasing and continuous on $I$. If $U \subset I$ is open, then $f(U)$ is open.

Proof: The proof is quite short, but requires accessing several results. We first give the proof, and then list all the required results. Let $U = (a, b) \subset I$, with $-\infty \le a < b \le \infty$. Then $f((a, b)) = (f(a+), f(b-))$ is open in $\mathbb{R}$. For any open set $U \subset I$, write $U = \cup_n I_n$, where $\{I_n\}$ is an at most countable collection of open intervals. Then $f(U) = f(\cup_n I_n) = \cup_n f(I_n)$ is open.

We list the results invoked in the above short proof.

1. From (1.14): Let $f : X \to Y$, and let $A$ be a nonempty set. If $\{E_\alpha\}_{\alpha \in A}$ is a family of subsets of $X$, then $f(\cup_{\alpha \in A} E_\alpha) = \cup_{\alpha \in A} f(E_\alpha)$.

2. From (3.5): For collection $\{O_\alpha\}_{\alpha \in A}$ of open subsets of $(X, d)$, $\cup_{\alpha \in A} O_\alpha$ is open.

3. Characterization of the Open Subsets of $\mathbb{R}$: If $U$ is an open subset of $\mathbb{R}$, then there exists a finite or countable collection $\{I_n\}$ of pairwise disjoint open intervals such that $U = \cup_n I_n$.

4. Let $I \subset \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be monotone increasing on $I$. Then $f(p+)$ and $f(p-)$ exists for every $p \in I$ and

$$\sup_{x<p} f(x) = f(p-) \leq f(p) \leq f(p+) = \inf_{p<x} f(x).$$

Furthermore, if $p < q$, for $p, q \in I$, then $f(p+) \leq f(q-)$. This can be found in, e.g., Stoll, Thm 4.4.7.

5. From (3.75): Let $I$ be an interval and suppose that the function $f : I \to \mathbb{R}$ is continuous. Then its image $f(I)$ also is an interval.

Theorem: Let $X$ and $Y$ be metric spaces with respective metrics $d_X$ and $d_Y$, and assume that $f : X \to Y$ is continuous. If $K$ is a compact subset of $X$, then

$$f(K) \text{ is a compact subset of } Y. \tag{3.76}$$

Proof: Let $\{V_i\}_{i\in J}$ be any open cover of $f(K)$. Each set $U_i = f^{-1}(V_i)$ is open, and $\{U_i\}_{i\in J}$ is an open cover of $K$. As $K$ is compact, this cover admits a finite subcover $\{U_{i_1}, \ldots, U_{i_N}\}$. But then $\{V_{i_1}, \ldots, V_{i_N}\}$ is a finite subcover of $f(K)$, so $f(K)$ is compact.

We repeat now the Extreme Value Theorem from (3.72), in more general terms.

Corollary (Extreme Value Theorem): Let $K$ be a nonempty compact subset of $\mathbb{R}$ and let $f : K \to \mathbb{R}$ be continuous. Then there exist $p, q \in K$ such that

$$f(q) \leq f(x) \leq f(p) \quad \text{for all} \quad x \in K. \tag{3.77}$$

Proof: Let $M = \sup\{f(x) : x \in K\}$. From (3.76), $f(K)$ is compact; and from (3.64), is closed and bounded. Thus, since $f(K)$ is bounded, $M < \infty$. Also, since $f(K)$ is closed, $M \in f(K)$. Thus there exists $p \in K$ such that $f(p) = M$. Similarly for $m = \inf\{f(x) : x \in K\}$.

The result also follows directly from (3.70) and (3.76).

We now revisit uniform continuity. Recall part of result (2.55), namely: Let $D \subseteq \mathbb{R}$. If $D$ is a closed and bounded set, and $f \in \mathcal{C}^0(D)$, then $f$ is uniformly continuous on $D$. This was proven there, and invoked the Bolzano-Weierstrass theorem. We wish to give here another proof of this result, explicitly using (topological) compactness. The proof we show is from Jakob and Evans, volume I, Thm 20.30. We will also need the reverse triangle inequality (1.21), namely, $\forall a, b \in \mathbb{R}$, $||a| - |b|| \leq |a + b|$ and $||a| - |b|| \leq |b - a|$.

Theorem: Let $f : K \to \mathbb{R}$ be a continuous function on a compact set $K \subset \mathbb{R}$. Then

$$f \text{ is uniformly continuous and bounded.} \tag{3.78}$$

Proof: Let $\varepsilon > 0$. Since $f$ is continuous for each $x \in K$ there is $\delta_{x,\varepsilon}$ such that $y \in K$ and $|x-y| < \delta_{x,\varepsilon}$ implies $|f(x)-f(y)| < \frac{\varepsilon}{2}$. Denote by $I(x)$ the interval $\left(x - \frac{\delta_{x,\varepsilon}}{2}, x + \frac{\delta_{x,\varepsilon}}{2}\right)$. Clearly $(I(x))_{x\in K}$ is an open covering of $K$. By compactness there is a finite subcovering

$$\left(x_l - \frac{\delta_{x_l,\varepsilon}}{2}, x_l + \frac{\delta_{x_l,\varepsilon}}{2}\right)_{l\in\{1,\ldots,N\}}.$$

Take $\delta := \frac{1}{2}\min\left(\delta_{x_1,\varepsilon}, \ldots, \delta_{x_N,\varepsilon}\right)$. For $|x-y|<\delta$ it follows that for some $1 \le j \le N$ we have

$$x \in \left(x_j - \frac{\delta_{x_j,\varepsilon}}{2}, x_j + \frac{\delta_{x_j,\varepsilon}}{2}\right)$$

and further

$$|x_j - y| \le |x-y| + |x-x_j| < \delta + \frac{\delta_{x_j,\varepsilon}}{2} < \delta_{x_j,\varepsilon},$$

and therefore

$$|f(y)-f(x)| \le |f(y) - f(x_j)| + |f(x) - f(x_j)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

proving that $f$ is uniformly continuous.

Next we prove that $f$ is bounded. Continuity of $f$ implies that, for $\epsilon = 1$ and $x \in K$, there exists $\delta_x > 0$ such that

$$y \in K, \quad |x-y| < \delta_x \implies |f(x) - f(y)| < 1. \tag{3.79}$$

The intervals $J(x) := (x-\delta_x, x+\delta_x)$, $x \in K$, form an open covering of $K$. Hence, since $K$ is compact, we can cover $K$ by finitely many of these intervals, say $J(x_1), \ldots, J(x_N))$. On $J(x_j)$ we have from (3.79) that $|f(y) - f(x_j)| < 1$. From this and the reverse triangle inequality (and that, trivially, $a \le |a|$),

$$|f(y)| - |f(x_j)| \le \big|\,|f(y)| - |f(x_j)|\,\big| \le |f(y) - f(x_j)| < 1,$$

i.e., $|f(y)| \le 1 + |f(x_j)|$. This implies $|f(y)| \le 1 + \max_{1\le j\le N}|f(x_j)|$ for all $y \in K$.

## 3.7 Bounded and Total Variation

The concepts of bounded variation and total variation arise in various mathematical contexts, including measure theory and stochastic calculus.

Definition: A function $f$ is said to be of *bounded variation* on $[a,b]$ if there is a number $K$ such that

$$\sum_{j=1}^{n} |f(a_j) - f(a_{j-1})| \le K, \qquad \text{for any } a = a_0 < a_1 < \cdots < a_n = b. \tag{3.80}$$

Definition: The smallest number $K$ satisfying (3.80) is the *total variation* of $f$ on $[a,b]$.

Equivalently, one can use the following.

Definition: The total variation of a real-valued (or more generally complex-valued) function $f$, defined on an interval $[a,b] \subset \mathbb{R}$ is the quantity

$$V_a^b(f) = \sup_{\mathcal{P}} \sum_{i=0}^{n_P - 1} |f(x_{i+1}) - f(x_i)|,$$

where the supremum runs over the set of all partitions

$$\mathcal{P} = \left\{P = \{x_0, \ldots, x_{n_P}\} \mid P \text{ is a partition of } [a,b]\right\}$$

of the given interval.

Theorem: The total variation of a differentiable function $f$, defined on an interval $[a,b] \subset \mathbb{R}$, has the following expression if $f'$ is Riemann integrable:

$$V_a^b(f) = \int_a^b |f'(x)|\, dx.$$

If $f$ is differentiable and monotonic, then the above simplifies to $V_a^b(f) = |f(a) - f(b)|$.

For any differentiable function $f$, we can decompose the domain interval $[a,b]$, into subintervals $[a, a_1], [a_1, a_2], \ldots, [a_N, b]$ (with $a < a_1 < a_2 < \cdots < a_N < b$) in which $f$ is locally monotonic, then the total variation of $f$ over $[a,b]$ can be written as the sum of local variations on those subintervals:

$$\begin{aligned} V_a^b(f) &= V_a^{a_1}(f) + V_{a_1}^{a_2}(f) + \cdots + V_{a_N}^b(f) \\ &= |f(a) - f(a_1)| + |f(a_1) - f(a_2)| + \cdots + |f(a_N) - f(b)|. \end{aligned}$$

The above theorem was taken from https://en.wikipedia.org/wiki/Total_variation), where references are given, presumably containing proofs.

The next two results were taken from Trench, p. 135, Exercise #7.

Theorem: If $f$ is of bounded variation on $[a,b]$, then $f$ is bounded on $[a,b]$.

> Proof: Let $V$ be the total variation of $f$ on $[a,b]$. Then, as
>
> $$f(x) = \frac{f(a)+f(b)}{2} + \frac{(f(x)-f(a)) + (f(x)-f(b))}{2}, \quad \forall a < x < b,$$
>
> $$|f(x)| \le \frac{|f(a)+f(b)|}{2} + \frac{|f(a)-f(x)| + |f(x)-f(b)|)}{2} \le \frac{|f(a)+f(b)|+V}{2}.$$

Theorem: If $f$ is of bounded variation on $[a,b]$, then $f$ is integrable on $[a,b]$.

Before giving the proof, we summarize relevant results, all of which we have stated previously, but using the notation from Trench.

1. Trench, p. 114: If $f$ is defined on $[a,b]$, then a sum

   $$\sigma = \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}), \qquad \text{where} \qquad x_{j-1} \le c_j \le x_j, \quad 1 \le j \le n,$$

   is a *Riemann sum of* $f$ *over the partition* $P = \{x_0, x_1, \ldots, x_n\}$. Note: As $c_j$ can be chosen arbitrarily in $[x_j, x_{j-1}]$, there are infinitely many Riemann sums for a given function $f$ over a given partition $P$.

2. Trench, Thm 3.1.4: Let $f$ be bounded on $[a,b]$, and let $P$ be a partition of $[a,b]$. Then

   (a) The upper sum $S(P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1})$ of $f$ over $P$ is the supremum of the set of all Riemann sums of $f$ over $P$.

   (b) The lower sum $s(P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1})$ of $f$ over $P$ is the infimum of the set of all Riemann sums of $f$ over $P$.

3. Trench, Thm 3.2.7: If $f$ is bounded on $[a, b]$, then $f$ is integrable on $[a, b]$ if and only if, for each $\epsilon > 0$, there is a partition $P$ of $[a, b]$ for which $S(P) - s(P) < \epsilon$.

Proof: Note by the previous theorem, as $f$ is a function of bounded variation, $f$ is bounded.

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ and $\epsilon > 0$. From [Trench, Thm 3.1.4], we can choose $c_1, \ldots, c_n$ and $c'_1, \ldots, c'_n$ so that $x_{j-1} \le c_j, c'_j \le x_j$, and

$$(A) \qquad \left| S(P) - \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) \right| < \epsilon/2,$$

$$(B) \qquad \left| s(P) - \sum_{j=1}^{n} f\left(c'_j\right)(x_j - x_{j-1}) \right| < \epsilon/2.$$

Now add and subtract the term

$$\sum_{j=1}^{n} \left(f(c_j) - f\left(c'_j\right)\right)(x_j - x_{j-1})$$

to $S(P) - s(P)$ gives

$$\begin{aligned} S(P) - s(P) = S(P) &- s(P) \\ &- \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) \\ &+ \sum_{j=1}^{n} \left(f(c_j) - f\left(c'_j\right)\right)(x_j - x_{j-1}) \\ &+ \sum_{j=1}^{n} f\left(c'_j\right)(x_j - x_{j-1}), \end{aligned}$$

Reordering the terms and applying the triangle inequality:

$$\begin{aligned} S(P) - s(P) \le & \underbrace{\left| S(P) - \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) \right|}_{A} \\ & + \left| \sum_{j=1}^{n} \left(f(c_j) - f\left(c'_j\right)\right)(x_j - x_{j-1}) \right| \\ & + \underbrace{\left| \sum_{j=1}^{n} f\left(c'_j\right)(x_j - x_{j-1}) - s(P) \right|}_{B}, \end{aligned}$$

Recall that $||P||$ is the norm of the partition, defined as

$$||P|| = \max\{\Delta x_j : j = 1, 2, \ldots, n\}.$$

Using (A) and (B), if $||P|| < \epsilon/K$, where $K$ is defined in (3.80),

$$S(P) - s(P) \le \epsilon + \sum_{j=1}^{n} \left| f(c_j) - f\left(c'_j\right) \right| (x_j - x_{j-1}) \le \epsilon + K||P|| < 2\epsilon.$$

[Trench, Thm 3.2.7] then implies that $f$ is integrable on $[a, b]$.

# 4 Some Relevant Linear Algebra

> Do not worry about your problems with mathematics, I assure you, mine are far greater. (Albert Einstein)

This section borrows heavily from five fantastic books, recent and old, namely Anthony and Harvey, Linear Algebra, 2012; Lang, Calculus of Several Variables, 3rd ed., 1987; Flanigan and Kazdan, Calculus Two: Linear and Nonlinear Functions, 2nd ed., 1990; Shifrin and Adams, Linear Algebra: A Geometric Approach, 2nd ed., 2011; and Olver and Shakiban, Applied Linear Algebra, 2nd ed., 2018. The reader will note some repetition and redundancy. Terseness does not help students; what is useful is seeing various good approaches and presentations, noting their commonalities, and their differences. Blue text are additional comments from me.

## 4.1 (Hyper-)planes, Vector-Parametric and Cartesian Equations

We use bold face to denote a point, or $n$-tuple, in $\mathbb{R}^n$, e.g., $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, and also for multivariate mappings, e.g., $\mathbf{f} : \mathbb{R} \to \mathbb{R}^m$, $m > 1$. We assume the reader has a basic familiarity with $n$-tuples, row and column vectors, and basic operations with them, e.g., addition, transpose (denoted $\mathbf{x}^{\mathrm{T}}$ or $\mathbf{x}'$) and the inner (dot) product. In particular, for the latter:

Definition: For all column vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, and for all $\alpha \in \mathbb{R}$, the inner product is

$$\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}'\mathbf{y} = \mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \tag{4.1}$$

and satisfies the following properties:
(i) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$,
(ii) $\alpha \langle \mathbf{x}, \mathbf{y} \rangle = \langle \alpha \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \alpha \mathbf{y} \rangle$,
(iii) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$,
(iv) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

**Example 4.1** *(Flanigan and Kazdan, p. 77)*

*(1) To prove that $\langle Z, X \rangle = 0$ for all $X \in \mathbb{R}^n$ implies $Z = 0$, let $X = Z$. Then $\langle Z, Z \rangle = 0$. But by property (iv), $Z = 0$.*

*(2) Prove that if $\langle Z_1, X \rangle = \langle Z_2, X \rangle$ for all $X \in \mathbb{R}^n$, then $Z_1 = Z_2$.*
*Proof: We have $\langle Z_1, X \rangle - \langle Z_2, X \rangle = 0$. From properties (ii) and (iii), for all $X$,*

$$0 = \langle Z_1, X \rangle - \langle Z_2, X \rangle = \langle Z_1, X \rangle + \langle -Z_2, X \rangle = \langle Z_1 - Z_2, X \rangle .$$

*By the first exercise, $Z_1 - Z_2 = 0$, and, thus, $Z_1 = Z_2$.* ■

Definition: The length, or norm, or magnitude, of vector $\mathbf{a}$ is

$$\|\mathbf{a}\| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle} = \langle \mathbf{a}, \mathbf{a} \rangle^{1/2} \geq 0. \tag{4.2}$$

Theorem (Polarization identity): For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4} \left( \|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2 \right), \tag{4.3}$$

expressing the standard inner product in terms of the norm.

Proof: Note that

$$\|\mathbf{x} \pm \mathbf{y}\|^2 = \langle \mathbf{x} \pm \mathbf{y}, \mathbf{x} \pm \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle \pm \langle \mathbf{x}, \mathbf{y} \rangle \pm \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \pm 2\langle \mathbf{x}, \mathbf{y} \rangle.$$

Let $A = (a_1, a_2)$ be a point in the plane $\mathbb{R}^2$. We associate this point with the vector $\mathbf{a} = (a_1, a_2)^{\mathrm{T}}$, as representing a *displacement from the origin*, $(0, 0)$, to the point $A$. In this context, $\mathbf{a}$ is the position vector of the point $A$. Graphically, this displacement is illustrated by an arrow, or directed line segment, with the initial point at the origin and the terminal point at $A$. Even if a displacement does not begin at the origin, two displacements of the same length and the same direction are considered to be equal.

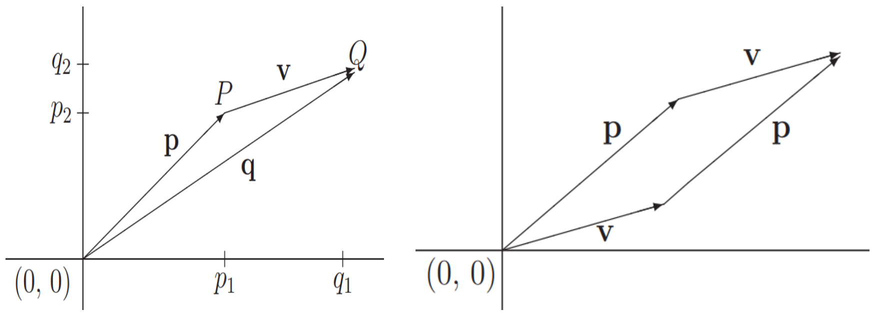



Figure 14: Left: Addition of two vectors. Right: The parallelogram law

If an object is displaced from the origin to a point $P$ by the displacement $\mathbf{p}$, and then displaced from $P$ to $Q$ by the displacement $\mathbf{v}$, then the total displacement is given by the vector from 0 to $Q$, which is the position vector $\mathbf{q}$. So we would expect vectors to satisfy $\mathbf{q} = \mathbf{p} + \mathbf{v}$, both geometrically (in the sense of a displacement) and algebraically (by the definition of vector addition). This is true generally, for vectors in $\mathbb{R}^n$, and shown in the left panel of Figure 14. The order of displacements does not matter (similar to how the order of vector addition does not matter), so $\mathbf{q} = \mathbf{v} + \mathbf{p}$. For this reason, the addition of vectors is said to follow the parallelogram law; see the right panel of Figure 14. From the equation $\mathbf{q} = \mathbf{p} + \mathbf{v}$, we have $\mathbf{v} = \mathbf{q} - \mathbf{p}$. This is the displacement from $P$ to $Q$. To help determine in which direction the vector $\mathbf{v}$ points, think of $\mathbf{v} = \mathbf{q} - \mathbf{p}$ as the vector that is added to the vector $\mathbf{p}$ in order to obtain the vector $\mathbf{q}$. The distance between points $P$ and $Q$ is (defined to be) $\|Q - P\| = \sqrt{(Q-P) \cdot (Q-P)} = \|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2}$.

Theorem (The Law of Cosines): For a triangle with sides $a, b, c$ and opposite angles $A, B, C$, respectively, as pictured in Figure 15,

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C, \\ b^2 &= a^2 + c^2 - 2ac\cos B, \\ a^2 &= b^2 + c^2 - 2bc\cos A. \end{aligned} \tag{4.4}$$

Proof: To prove (4.4), use of Pythagoras and a basic trigonometric identity gives

$$\begin{aligned} c^2 &= (b - a\cos C)^2 + (0 - a\sin C)^2 \\ &= b^2 - 2ab\cos C + a^2\cos^2 C + a^2\sin^2 C \\ &= b^2 - 2ab\cos C + a^2\left(\cos^2 C + \sin^2 C\right) = a^2 + b^2 - 2ab\cos C. \end{aligned}$$

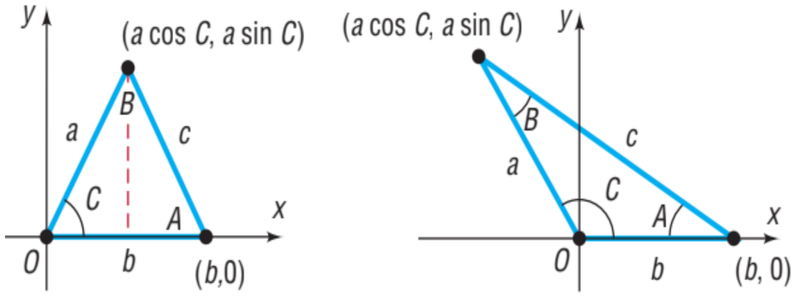



Figure 15: Triangles with C acute and obtuse, respectively. Taken from Sullivan, Trigonometry, 9th ed., 2012, p. 275

This can be used to provide the (very common) proof of the following crucial result:

Theorem: Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and let $\theta$ denote the angle between them. Then

$$\langle \mathbf{a}, \mathbf{b} \rangle = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta. \tag{4.5}$$

Proof: As in Anthony and Harvey, the law of cosines states that $c^2 = a^2 + b^2 - 2ab\cos\theta$, where $c = \|\mathbf{b} - \mathbf{a}\|$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$. That is, $\|\mathbf{b} - \mathbf{a}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$. Expanding the inner product and using its properties, we have

$$\|\mathbf{b} - \mathbf{a}\|^2 = \langle \mathbf{b} - \mathbf{a}, \mathbf{b} - \mathbf{a} \rangle = \langle \mathbf{b}, \mathbf{b} \rangle + \langle \mathbf{a}, \mathbf{a} \rangle - 2\langle \mathbf{a}, \mathbf{b} \rangle,$$

so that $\|\mathbf{b} - \mathbf{a}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\langle \mathbf{a}, \mathbf{b} \rangle$. Comparing the two expressions yields (4.5).

We now show a different proof of (4.5), from Flanigan and Kazdan, using Figure 16.

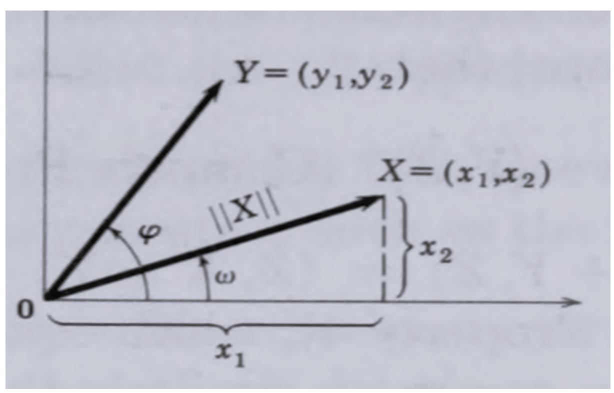



Figure 16: For the 2nd proof of (4.5). From Flanigan and Kazdan, p. 78.

Proof: Let $\theta$ be the angle between $X$ and $Y$, i.e., $\theta = \varphi - \omega$. We know $\langle X, Y \rangle = x_1y_1 + x_2y_2$, where $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ as usual. We will translate $x_1y_1 + x_2y_2$ into trigonometry. Let $\omega$ and $\varphi$ be the angles from the horizontal axis to $X$ and $Y$, respectively. Now we note that

$$\cos\omega = \frac{x_1}{\|X\|} \text{ and } \sin\omega = \frac{x_2}{\|X\|},$$

whence $x_1 = \|X\|\cos\omega$ and $x_2 = \|X\|\sin\omega$. Likewise $y_1 = \|Y\|\cos\varphi$ and $y_2 = \|Y\|\sin\varphi$. Thus

$$\langle X, Y \rangle = x_1y_1 + x_2y_2 = \|X\|\|Y\|(\cos\omega\cos\varphi + \sin\omega\sin\varphi).$$

Recall from (2.79) that $\cos\theta = \cos(\varphi - \omega) = \cos\omega\cos\varphi + \sin\omega\sin\varphi$. Thus $\langle X, Y\rangle = \|X\|\|Y\|\cos\theta$, as claimed.

Definition: The non-zero vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ are said to be *orthogonal*, or *perpendicular*, when the angle between them is $\theta = \pi/2$. As $\cos(\pi/2) = 0$, this is precisely when their inner product is zero. That is:

$$\textbf{The vectors a and b are orthogonal if and only if } \langle \mathbf{a}, \mathbf{b}\rangle = 0. \tag{4.6}$$

Definition: A *line in* $\mathbb{R}^n$ is given by a vector equation with one parameter of the form $\mathbf{x} = \mathbf{p} + t\mathbf{v}$, where $\mathbf{x}$ is the position vector of a point on the line, $\mathbf{p}$ is any particular point on the line, $\mathbf{v}$ is the direction of the line, and $t \in \mathbb{R}$.

If $\mathbf{p} = \mathbf{0}$, then line $\mathbf{x} = t\mathbf{v}$ goes through the origin, though note that a line with $\mathbf{p} \neq \mathbf{0}$ could still go through the origin. For $n = 3$,

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + t\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad t \in \mathbb{R}. \tag{4.7}$$

In terms of Cartesian equations, equating components in (4.7) gives

$$x = p_1 + tv_1, \quad y = p_2 + tv_2, \quad z = p_3 + tv_3.$$

provided $v_i \neq 0$, $i = 1, 2, 3$, solving for $t$ and equating,

$$\frac{x - p_1}{v_1} = \frac{y - p_2}{v_2} = \frac{z - p_3}{v_3}.$$

For example, to find Cartesian equations of the line

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}, \quad t \in \mathbb{R},$$

equate components $x = 1 - t$, $y = 2$, $z = 3 + 5t$, and then solve for $t$ in the first and third equation. The Cartesian equations are $1 - x = (z - 3)/5$ and $y = 2$, which is a line parallel to the $xz$-plane in $\mathbb{R}^3$.

Definition: A subset $\mathcal{S}$ of $\mathbb{R}^n$ is a *plane through the origin* (also called a two-dimensional linear subspace) if and only if it is the linear span of two vectors $X, Y$ that do not lie on the same line through the origin; that is, if and only if

$$\mathcal{S} = \{Z \in \mathbb{R}^n : Z = \alpha X + \beta Y \text{ with } \alpha, \beta \in \mathbb{R}\}.$$

Definition: Vectors that lie on a common line through the origin are said to be collinear with the origin.

Thus the requirement on $X$ and $Y$ in the preceding definition is that $X$ and $Y$ be non-collinear with the origin. Two vectors in $\mathbb{R}^n$ that are non-collinear with the origin span a plane in $\mathbb{R}^n$. Further, a plane $\mathcal{S}$ is spanned by any two vectors that are in $\mathcal{S}$, and that are non-collinear with the origin.

If $\mathcal{S}$ is a subset of $\mathbb{R}^n$ and $Z$ is a vector in $\mathbb{R}^n$, then we write

$$\mathcal{S} + Z = \{X' + Z : X' \in \mathcal{S}\}\,.$$

Definition: A subset $\mathcal{A}$ of $\mathbb{R}^n$ is called an *affine* subspace if and only if $\mathcal{A}$ is of the form $\mathcal{S} + Z$ for some linear subspace $\mathcal{S}$ and some vector $Z$ in $\mathbb{R}^n$. In this case, $\mathcal{A}$ and $\mathcal{S}$ are said to be parallel. We call $\mathcal{A}$ a line if $\mathcal{S}$ is a line through the origin, or a plane if $\mathcal{S}$ is a plane through the origin.

Thus, dropping the requirement that a plane goes through the origin results in an affine subspace. The general definition (Flanigan and Kazdan, p. 53) is typical within the language of linear algebra:

Definition: A subset $\mathcal{A}$ of $\mathbb{R}^3$ is a plane if and only if $\mathcal{A}$ is the affine subspace consisting of the solutions of a linear equation $a_1x_1 + a_2x_2 + a_3x_3 = b$ with at least one of the coefficients $a_1, a_2$, and $a_3$ different from zero. Moreover, in this case, $\mathcal{A}$ is parallel to the two-dimensional linear subspace $\mathcal{S}$ of solutions of the homogeneous equation $a_1x_1 + a_2x_2 + a_3x_3 = 0$, that is, $\mathcal{A} = \mathcal{S} + Z$, where $Z$ is any solution of $a_1x_1 + a_2x_2 + a_3x_3 = b$.

Definition: A plane in $\mathbb{R}^3$ is given by the *vector parametric equation*

$$\mathbf{x} = \mathbf{p} + s\mathbf{v} + t\mathbf{w}, \quad s, t, \in \mathbb{R}, \;\; \mathbf{p}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3,$$

provided that the vectors $\mathbf{v}$ and $\mathbf{w}$ are non-zero and are not parallel.

From (4.6), vector $\mathbf{x}$ is orthogonal to $\mathbf{n}$ if and only if $\langle \mathbf{n}, \mathbf{x} \rangle = 0$. This latter equation also characterizes the plane: If $\mathbf{n} = (a, b, c)^{\mathrm{T}}$ and $\mathbf{x} = (x, y, z)^{\mathrm{T}}$, then this equation can be written as

$$\langle \mathbf{n}, \mathbf{x} \rangle = \left\langle \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle = 0,$$

or

$$ax + by + cz = 0. \tag{4.8}$$

Definition: Form (4.8) is a *Cartesian equation* of a plane through the origin in $\mathbb{R}^3$.

Definition: The vector $\mathbf{n}$ is called a *normal vector to the plane.*

Any vector that is parallel to $\mathbf{n}$ will also be a normal vector and will lead to the same Cartesian equation. On the other hand, given any Cartesian equation of the form $ax + by + cz = 0$, this equation represents a plane through the origin in $\mathbb{R}^3$ with normal vector $\mathbf{n} = (a, b, c)^{\mathrm{T}}$.

To describe a plane that does not go through the origin, we choose a normal vector $\mathbf{n}$ and one point $P$ on the plane with position vector $\mathbf{p}$. We then consider all displacement vectors that lie in the plane with initial point at $P$. If $\mathbf{x}$ is the position vector of any point on the plane, then the displacement vector $\mathbf{x} - \mathbf{p}$ lies in the plane, and $\mathbf{x} - \mathbf{p}$ is orthogonal to $\mathbf{n}$. Conversely, if the position vector $\mathbf{x}$ of a point satisfies $\langle \mathbf{n}, \mathbf{x} - \mathbf{p} \rangle = 0$, then the vector $\mathbf{x} - \mathbf{p}$ lies in the plane, so the point (with position vector $\mathbf{x}$) is on the plane. This is illustrated in Figure 17, albeit with different notation.

The orthogonality condition (4.6) means that the position vector of any point on the plane is given by the equation $\langle \mathbf{n}, \mathbf{x} - \mathbf{p} \rangle = 0$. Using properties of the inner product, we can rewrite this as $\langle \mathbf{n}, \mathbf{x} \rangle = \langle \mathbf{n}, \mathbf{p} \rangle$, where $\langle \mathbf{n}, \mathbf{p} \rangle = d$ is a constant. If $\mathbf{n} = (a, b, c)^{\mathrm{T}}$ and $\mathbf{x} = (x, y, z)^{\mathrm{T}}$, then $ax + by + cz = d$ is a Cartesian equation of a plane in $\mathbb{R}^3$. The plane goes through the

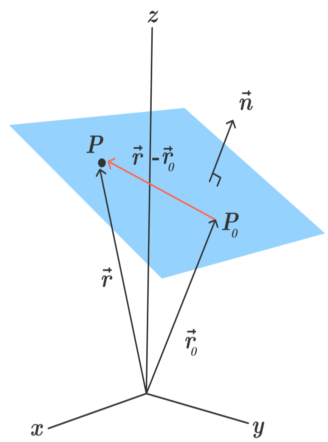



Figure 17: From https://brilliant.org/wiki/3d-coordinate-geometry-equation-of-a-plane/. Let $P_0$ be a point on the plane with position vector $\mathbf{r}_0$, and let $P$ be some other point on the plane with position vector $\mathbf{r}$ (in place of $\mathbf{x}$). Then observe that the vector $\mathbf{r} - \mathbf{r}_0$ is the vector originating at $P_0$ and ending at $P$, and thus lies in the plane. Indeed, from the figure, note that $\mathbf{r} = \mathbf{r}_0 + (\mathbf{r} - \mathbf{r}_0)$. It is orthogonal to normal vector $\mathbf{n}$ as indicated.

origin if and only if $d = 0$. For example, the equation $2x - 3y - 5z = 2$ represents a plane that does not go through the origin, as $(x, y, z) = (0, 0, 0)$ does not satisfy the equation. To find a point on the plane, choose any two of the coordinates, say $y = 0$ and $z = 0$, implying $x = 1$, and that the point $(1, 0, 0)$ is on this plane. The components of a normal to the plane can be read from this equation as the coefficients of $x, y, z : \mathbf{n} = (2, -3, -5)^{\mathrm{T}}$.

The next example indicates one of a handful of "typical" questions involving this material, and most all books have something like it. Indeed, forthcoming Example 4.7 is similar. There, more explanation is provided as to why we need to solve the system of two equations (4.9); namely, for the plane $\mathcal{S}$, there must be a line orthogonal to $\mathcal{S}$, so we need a vector $\mathbf{a} = (a_1, a_2, a_3)$ that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. It must satisfy (4.9).

**Example 4.2** *(Flanigan and Kazdan, p. 53). Let us find an equation representing the plane $\mathcal{S} + Z$, where $Z = (3, 2, 1)$ and $\mathcal{S}$ is the plane through the origin that contains $\mathbf{u} = (0, 1, -3)$ and $\mathbf{v} = (1, 4, -1)$. We first find an equation representing $\mathcal{S}$ by solving the system*

$$a_2 - 3a_3 = 0, \qquad a_1 + 4a_2 - a_3 = 0, \tag{4.9}$$

*for the unknowns $a_1, a_2, a_3$. Eliminate $a_2$ from the second equation to obtain the following solution set: $a_1 = -11a_3$, $a_2 = 3a_3$, and $a_3$ is arbitrary. We may choose $a_3 = 1$, which gives $a_1 = -11$ and $a_2 = 3$. Thus, $\mathcal{S}$ is the set of vectors $(x_1, x_2, x_3)$ such that $-11x_1 + 3x_2 + x_3 = 0$. Now we want to find a inhomogeneous equation with the same coefficients such that $Z$ is a solution. Plugging $Z = (z_1, z_2, z_3) = (3, 2, 1)$ into the left-hand side of the homogeneous equation, we get $(-11 \times 3) + (3 \times 2) + (1 \times 1) = -26$, so $\mathcal{S} + Z$ is the set of solutions of $-11x_1 + 3x_2 + x_3 = -26$.* ■

The two representations, *vector parametric equation*, and *Cartesian equation*, are easily related, as shown in the next example.

**Example 4.3** *(Anthony and Harvey, p. 42). Consider the plane*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 7 \end{pmatrix} = s\mathbf{v} + t\mathbf{w}, \quad s, t \in \mathbb{R}. \tag{4.10}$$

*To obtain a Cartesian equation in $x, y$ and $z$, we equate the components in this vector equation,*

$$x = s - 2t, \quad y = 2s + t, \quad z = -s + 7t,$$

*and eliminate the parameters $s$ and $t$. We begin by solving the first equation for $s$, and then substitute this into the second equation to solve for $t$ in terms of $x$ and $y$,*

$$s = x + 2t \Rightarrow y = 2(x + 2t) + t = 2x + 5t \Rightarrow 5t = y - 2x \Rightarrow t = \frac{y - 2x}{5}.$$

*We then substitute back into the first equation to obtain $s$ in terms of $x$ and $y$,*

$$s = x + 2\left(\frac{y - 2x}{5}\right) \Rightarrow 5s = 5x + 2y - 4x \Rightarrow s = \frac{x + 2y}{5}.$$

*Finally, we substitute for $s$ and $t$ in the third equation, $z = -s + 7t$, and simplify to obtain a Cartesian equation of the plane*

$$3x - y + z = 0.$$

*This Cartesian equation can be expressed as $\langle \mathbf{n}, \mathbf{x} \rangle = 0$, where*

$$\mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

*The vector $\mathbf{n}$ is a normal vector to the plane. We can check that $\mathbf{n}$ is, indeed, orthogonal to the plane by taking the inner product with the vectors $\mathbf{v}$ and $\mathbf{w}$, which lie in the plane.*

*Now consider displacing the plane so that it does not go through the origin, taking $\mathbf{p} = (3, 7, 2)'$, e.g.,*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 7 \end{pmatrix} = \mathbf{p} + s\mathbf{v} + t\mathbf{w}, \quad s, t \in \mathbb{R}, \tag{4.11}$$

*which passes through the point $(3, 7, 2)$. Since the two planes (4.10) and (4.11) are parallel, they will have the same normal vectors, and the Cartesian equation of this plane is of the form $3x - y + z = d$. Since $(3, 7, 2)$ is a point on the plane, it must satisfy the equation for the plane. Substituting into the equation we find $d = 3(3) - (7) + (2) = 4$ (which is equivalent to finding $d$ by using $d = \langle \mathbf{n}, \mathbf{p} \rangle$). So the Cartesian equation we obtain is $3x - y + z = 4$.*

*Conversely, starting with a Cartesian equation of a plane, we can obtain a vector equation. Consider the plane just discussed. We are looking for the position vector of a point on the plane whose components satisfy $3x - y + z = 4$, or, equivalently, $z = 4 - 3x + y$. (We can solve for any one of the variables $x, y$ or $z$, but we chose $z$ for simplicity.) So we are looking for all vectors $\mathbf{x}$ such that*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 4 - 3x + y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + x \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

*for any $x, y \in \mathbb{R}$. Therefore,*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad s, t \in \mathbb{R}$$

*is a vector equation of the same plane as (4.11). It is difficult to spot at a glance that these two different vector equations in fact describe the same plane. The planes represented by the two vector equations have the same normal vector* $\mathbf{n}$*, since the vectors* $(1, 0, -3)^{\mathrm{T}}$ *and* $(0, 1, 1)^{\mathrm{T}}$ *are also orthogonal to* $\mathbf{n}$*. So we know that the two vector equations represent parallel planes. They are the same plane if they have a point in common. It is far easier to find values of* $s$ *and* $t$ *for which* $\mathbf{p} = (3, 7, 2)^{\mathrm{T}}$ *satisfies the new vector equation*

$$\begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad s, t \in \mathbb{R}$$

*than the other way around (which is by showing that* $(0, 0, 4)$ *satisfies the original equation) because of the positions of the zeros and ones in these direction vectors.* ∎

**Example 4.4** *(Anthony and Harvey, p. 46). The planes*

$$x + 2y - 3z = 0 \quad \textit{and} \quad x - 2y + 5z = 4$$

*intersect in a line. The points of intersection are the points* $(x, y, z)$ *that satisfy both equations, so we solve the equations simultaneously. We begin by eliminating the variable* $x$ *from the second equation, by subtracting the first equation from the second. This will naturally lead us to a vector equation of the line of intersection:*

$$\left.\begin{aligned} x + 2y - 3z &= 0 \\ x - 2y + 5z &= 4 \end{aligned}\right\} \Rightarrow \begin{aligned} x + 2y - 3z &= 0 \\ -4y + 8z &= 4. \end{aligned} \tag{4.12}$$

*This last equation tells us that if* $z = t$ *is any real number, then* $y = -1 + 2t$*. Substituting these expressions into the first equation, we find* $x = 2 - t$*. Then a vector equation of the line of intersection is*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 - t \\ -1 + 2t \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} =: \mathbf{p} + t\mathbf{v}.$$

*This can be verified by showing that the point* $\mathbf{p} = (2, -1, 0)$ *satisfies both Cartesian equations, and that the vector* $\mathbf{v} = (-1, 2, 1)^{\mathrm{T}}$ *is orthogonal to the normal vectors of each of the planes (and therefore lies in both planes).* ∎

In the previous example, we can envision the line induced by the two planes from Figure 18. The caption also indicates another way to compute a vector parallel to the intersecting line, namely by use of of the cross product of the two normal plane vectors. The cross product will be introduced below in §4.4, in (4.78). For two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, it is given by

$$(x_2 y_3 - x_3 y_2, \;\; x_3 y_1 - x_1 y_3, \;\; x_1 y_2 - x_2 y_1). \tag{4.13}$$

Based on the two normal vectors $(1, 2, -3)$ and $(1, -2, 5)$, this yields $(10-6, -3-5, -2-2) = (4, -8, -4)$, which is indeed parallel to $\mathbf{v} = (-1, 2, 1)^{\mathrm{T}}$. To recover point $\mathbf{p}$, note from the latter set of equations in (4.12) that $x$ and $y$ are the (in the common linear algebra language for Gaussian elimination) "basic" variables, while $z$ is the "free" variable. Taking $z = 0$ (the simplest choice) implies $y = -1$ and $x = 2$, which is precisely point $\mathbf{p}$.

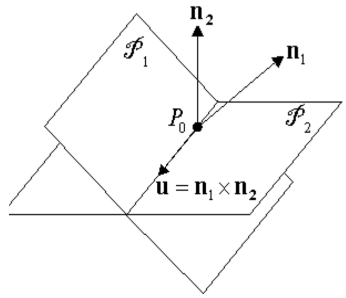



Figure 18: From https://math.stackexchange.com/questions/2387317. We have two non-parallel planes $\mathcal{P}_1$ and $\mathcal{P}_2$ with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively. Let $\mathcal{L}$ be the line of intersection of $\mathcal{P}_1$ and $\mathcal{P}_2$. Let $P_0$ be a point on $\mathcal{L}$ and suppose that $\mathbf{v}$ is a vector parallel to $\mathcal{L}$. Note that $\mathbf{v}$ is a vector in both $\mathcal{P}_1$ and $\mathcal{P}_2$. This means that $\mathbf{v}\cdot\mathbf{n}_1 = 0$ and $\mathbf{v}\cdot\mathbf{n}_2 = 0$. That is, $\mathbf{v}$ is a vector orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$. Hence $\mathbf{v}$ is parallel to the cross-product $\mathbf{u} = \mathbf{n}_1 \times \mathbf{n}_2$.

Definition: The set of all points $(x_1, x_2, \ldots, x_n)$ that satisfy one Cartesian equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = d$$

is called a *hyperplane in* $\mathbb{R}^n$. That is, in $\mathbb{R}^n$, a hyperplane is an affine subspace of dimension $n-1$.

In $\mathbb{R}^2$, a hyperplane is a line, and in $\mathbb{R}^3$ it is a plane, but for $n > 3$ we simply use the term hyperplane. The column vector $\mathbf{n} = (a_1, \ldots, a_n)'$ is a normal vector to the hyperplane. Writing the Cartesian equation in vector form, a hyperplane is the set of all vectors, $\mathbf{x} \in \mathbb{R}^n$ such that $\langle \mathbf{n}, \mathbf{x} - \mathbf{p}\rangle = 0$, where the normal vector $\mathbf{n}$ and the position vector $\mathbf{p}$ of a point on the hyperplane are given.

**Example 4.5** *Using the Gauss-Jordan method, we find that the solution set of the system*

$$\begin{aligned} x_1 - x_2 + x_3 - 2x_4 &= -1 \\ x_2 + 3x_3 &= 0 \end{aligned}$$

*is the set of vectors of the form*

$$(x_1, x_2, x_3, x_4) = (-1, 0, 0, 0) + x_3(-4, -3, 1, 0) + x_4(2, 0, 0, 1),$$

*with $x_3$ and $x_4$ arbitrary. (Here, $x_1$ and $x_2$ are the* basic variables*, and $x_3$ and $x_4$ are the* free variables*.) Thus, the solution set is the two-dimensional affine subspace $\mathcal{A}$ of $\mathbb{R}^4$ that contains the vector $(-1, 0, 0, 0)$ and is parallel to the plane $\mathcal{S}$ through the origin spanned by the two linearly independent vectors $(-4, -3, 1, 0)$ and $(2, 0, 0, 1)$.* ■

## 4.2 Projection

We give presentations taken from three different textbooks.

### 4.2.1 Shifrin and Adams, Linear Algebra: A Geometric Approach

Starting with a two-dimensional picture of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, where $\mathbf{y} \neq \mathbf{0}$, it suggests itself that we should be able to write $\mathbf{x}$ as the sum of a vector, $\mathbf{x}^{\parallel}$ (read "x-parallel"), that is a scalar multiple of $\mathbf{y}$ and a vector, $\mathbf{x}^{\perp}$ (read "x-perp"), that is orthogonal to $\mathbf{y}$. Let's suppose we have such an equation $\mathbf{x} = \mathbf{x}^{\parallel} + \mathbf{x}^{\perp}$, where $\mathbf{x}^{\parallel}$ is a scalar multiple of $\mathbf{y}$ and $\mathbf{x}^{\perp}$ is orthogonal to $\mathbf{y}$. To say that $\mathbf{x}^{\parallel}$ is a scalar multiple of $\mathbf{y}$ means that we can write $\mathbf{x}^{\parallel} = c\mathbf{y}$ for some scalar $c$. Now, assuming such an expression exists, we can determine $c$ by taking the dot product of both sides of the equation with $\mathbf{y}$:

$$\mathbf{x} \cdot \mathbf{y} = \left(\mathbf{x}^{\parallel} + \mathbf{x}^{\perp}\right) \cdot \mathbf{y} = \left(\mathbf{x}^{\parallel} \cdot \mathbf{y}\right) + \left(\mathbf{x}^{\perp} \cdot \mathbf{y}\right) = \mathbf{x}^{\parallel} \cdot \mathbf{y} = (c\mathbf{y}) \cdot \mathbf{y} = c\|\mathbf{y}\|^2.$$

This means that

$$c = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}, \quad \text{and so} \quad \mathbf{x}^{\parallel} = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y}.$$

The vector $\mathbf{x}^{\parallel}$ is called the projection of $\mathbf{x}$ onto $\mathbf{y}$, written $\operatorname{proj}_{\mathbf{y}} \mathbf{x}$.

The fastidious reader may be puzzled by the logic here. We have apparently assumed that we can write $\mathbf{x} = \mathbf{x}^{\parallel} + \mathbf{x}^{\perp}$ in order to prove that we can do so. Of course, as it stands, this is not fair. Here's how we fix it. We now *define*

$$\mathbf{x}^{\parallel} = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y}, \quad \mathbf{x}^{\perp} = \mathbf{x} - \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y}.$$

Obviously, $\mathbf{x}^{\parallel} + \mathbf{x}^{\perp} = \mathbf{x}$ and $\mathbf{x}^{\parallel}$ is a scalar multiple of $\mathbf{y}$. All we need to check is that $\mathbf{x}^{\perp}$ is in fact orthogonal to $\mathbf{y}$. Well,

$$\begin{aligned} \mathbf{x}^{\perp} \cdot \mathbf{y} &= \left(\mathbf{x} - \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y}\right) \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{y} - \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y} \cdot \mathbf{y} \\ &= \mathbf{x} \cdot \mathbf{y} - \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\|\mathbf{y}\|^2 = \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{y} = 0, \end{aligned}$$

as required. Note that by finding a formula for $c$ above, we have shown that $\mathbf{x}^{\parallel}$ is the unique multiple of $\mathbf{y}$ that satisfies the equation $\left(\mathbf{x} - \mathbf{x}^{\parallel}\right) \cdot \mathbf{y} = 0$.

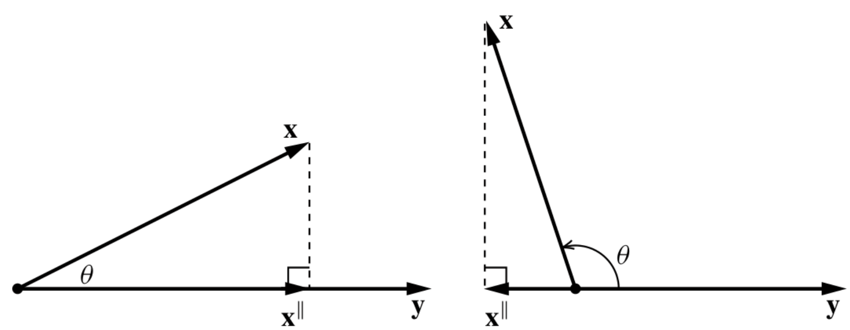



Figure 19: Projection and angle

Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. We shall see next that the formula for the projection of $\mathbf{x}$ onto $\mathbf{y}$ enables us to calculate the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$. Consider the right triangle in Figure 19. Let $\theta$ denote the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$. Remembering that the cosine of an angle is the ratio of the signed length of the adjacent side to the length of the hypotenuse, we see that

$$\cos\theta = \frac{\text{signed length of } \mathbf{x}^{\parallel}}{\text{length of } \mathbf{x}} = \frac{c\|\mathbf{y}\|}{\|\mathbf{x}\|} = \frac{\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}\|\mathbf{y}\|}{\|\mathbf{x}\|} = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}.$$

This, then, is the geometric interpretation of the dot product:

$$\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|\cos\theta. \tag{4.14}$$

Note that if the angle $\theta$ is obtuse, i.e., $\pi/2 < |\theta| < \pi$, then $c < 0$ (the signed length of $\mathbf{x}^{\parallel}$ is negative) and $\mathbf{x}\cdot\mathbf{y}$ is negative. Will this formula still make sense even when $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$? Geometrically, we simply restrict our attention to the plane spanned by $\mathbf{x}$ and $\mathbf{y}$ and measure the angle $\theta$ in that plane. This results in the following definition.

Definition: Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$. We define the angle between them to be the unique $\theta$ satisfying $0 \le \theta \le \pi$ so that

$$\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}.$$

**Example 4.6** *Consider the line $\ell_0$ through the origin in $\mathbb{R}^2$ with direction vector $\mathbf{v} = (1, -3)$. The points on this line are all of the form*

$$\mathbf{x} = t(1, -3), \quad t \in \mathbb{R}.$$

*Because $(3, 1)\cdot(1, -3) = 0$, we may take $\mathbf{a} = (3, 1)$ to be the normal vector to the line, and the Cartesian equation of $\ell_0$ is*

$$\mathbf{a}\cdot\mathbf{x} = 3x_1 + x_2 = 0.$$

*As a check, suppose we start with $3x_1 + x_2 = 0$. Then we can write $x_1 = -\frac{1}{3}x_2$, and so the solutions consist of vectors of the form*

$$\mathbf{x} = (x_1, x_2) = \left(-\frac{1}{3}x_2, x_2\right) = -\frac{1}{3}x_2(1, -3), \quad x_2 \in \mathbb{R}.$$

*Letting $t = -\frac{1}{3}x_2$, we recover the original parametric equation.*

*Now consider the line $\ell$ passing through $\mathbf{x}_0 = (2, 1)$ with direction vector $\mathbf{v} = (1, -3)$. Then the points on $\ell$ are all of the form*

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} = (2, 1) + t(1, -3), \quad t \in \mathbb{R}.$$

*As promised, we take the same vector $\mathbf{a} = (3, 1)$ and compute that*

$$3x_1 + x_2 = \mathbf{a}\cdot\mathbf{x} = \mathbf{a}\cdot(\mathbf{x}_0 + t\mathbf{v}) = \mathbf{a}\cdot\mathbf{x}_0 + t(\mathbf{a}\cdot\mathbf{v}) = \mathbf{a}\cdot\mathbf{x}_0 = (3, 1)\cdot(2, 1) = 7.$$

*This is the Cartesian equation of $\ell$.* ■

The next example is similar to Example 4.2.

**Example 4.7** *Consider the plane $\mathcal{P}_0$ passing through the origin spanned by $\mathbf{u} = (1, 0, 1)$ and $\mathbf{v} = (2, 1, 1)$. Our intuition suggests that there is a line orthogonal to $\mathcal{P}_0$, so we look for a vector $\mathbf{a} = (a_1, a_2, a_3)$ that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. It must satisfy the equations*

$$\begin{aligned} a_1 + a_3 &= 0 \\ 2a_1 + a_2 + a_3 &= 0. \end{aligned}$$

*Substituting $a_3 = -a_1$ into the second equation, we obtain $a_1 + a_2 = 0$, so $a_2 = -a_1$ as well. Thus, any candidate for $\mathbf{a}$ must be a scalar multiple of the vector $(1, -1, -1)$, and so we take $\mathbf{a} = (1, -1, -1)$ and try the equation*

$$\mathbf{a} \cdot \mathbf{x} = (1, -1, -1) \cdot \mathbf{x} = x_1 - x_2 - x_3 = 0$$

*for $\mathcal{P}_0$. Now, we know that $\mathbf{a} \cdot \mathbf{u} = \mathbf{a} \cdot \mathbf{v} = 0$. Does it follow that $\mathbf{a}$ is orthogonal to every linear combination of $\mathbf{u}$ and $\mathbf{v}$ ? We just compute: If $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$, then*

$$\begin{aligned} \mathbf{a} \cdot \mathbf{x} &= \mathbf{a} \cdot (s\mathbf{u} + t\mathbf{v}) \\ &= s(\mathbf{a} \cdot \mathbf{u}) + t(\mathbf{a} \cdot \mathbf{v}) = 0 \end{aligned}$$

*as desired. As before, if we want the equation of the plane $\mathcal{P}$ parallel to $\mathcal{P}_0$ and passing through $\mathbf{x}_0 = (2, 3, -2)$, we take*

$$\begin{aligned} x_1 - x_2 - x_3 = \mathbf{a} \cdot \mathbf{x} &= \mathbf{a} \cdot (\mathbf{x}_0 + s\mathbf{u} + t\mathbf{v}) \\ &= \mathbf{a} \cdot \mathbf{x}_0 + s(\mathbf{a} \cdot \mathbf{u}) + t(\mathbf{a} \cdot \mathbf{v}) \\ &= \mathbf{a} \cdot \mathbf{x}_0 = (1, -1, -1) \cdot (2, 3, -2) = 1. \end{aligned}$$

*As this example suggests, a point $\mathbf{x}_0$ and a normal vector $\mathbf{a}$ give rise to the Cartesian equation of a plane in $\mathbb{R}^3$ : $\mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$, or, equivalently, $\mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{x}_0$. Thus, every plane in $\mathbb{R}^3$ has an equation of the form $a_1x_1 + a_2x_2 + a_3x_3 = c$, where $\mathbf{a} = (a_1, a_2, a_3)$ is the normal vector and $c \in \mathbb{R}$.*

*Consider the set of points $\mathbf{x} = (x_1, x_2, x_3)$ defined by the equation $x_1 - 2x_2 + 5x_3 = 3$. Let's verify that this is, in fact, a plane in $\mathbb{R}^3$ according to our original parametric definition. If $\mathbf{x}$ satisfies this equation, then $x_1 = 3 + 2x_2 - 5x_3$ and so we may write*

$$\begin{aligned} \mathbf{x} = (x_1, x_2, x_3) &= (3 + 2x_2 - 5x_3, x_2, x_3) \\ &= (3, 0, 0) + x_2(2, 1, 0) + x_3(-5, 0, 1). \end{aligned}$$

*So, if we let $\mathbf{x}_0 = (3, 0, 0)$, $\mathbf{u} = (2, 1, 0)$, and $\mathbf{v} = (-5, 0, 1)$, we see that $\mathbf{x} = \mathbf{x}_0 + x_2\mathbf{u} + x_3\mathbf{v}$, where $x_2$ and $x_3$ are arbitrary scalars. This is in accordance with our original definition of a plane in $\mathbb{R}^3$.* ■

Finally, generalizing to $n$ dimensions:

Definition: If $\mathbf{a} \in \mathbb{R}^n$ is a nonzero vector and $c \in \mathbb{R}$, then the equation $\mathbf{a} \cdot \mathbf{x} = c$ defines a *hyperplane in* $\mathbb{R}^n$. This means that the solution set has "dimension" $n - 1$, i.e., 1 less than the dimension of the ambient space $\mathbb{R}^n$.

Let's write an explicit formula for the general vector $\mathbf{x}$ satisfying this equation: If $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $a_1 \neq 0$, then we rewrite the equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$ to solve for $x_1$:

$$x_1 = \frac{1}{a_1}\left(c - a_2x_2 - \cdots - a_nx_n\right),$$

and so the general solution is of the form

$$\begin{aligned}\mathbf{x} = (x_1, \ldots, x_n) &= \left(\frac{1}{a_1}\left(c - a_2x_2 - \cdots - a_nx_n\right), x_2, \ldots, x_n\right) \\ &= \left(\frac{c}{a_1}, 0, \ldots, 0\right) + x_2\left(-\frac{a_2}{a_1}, 1, 0, \ldots, 0\right) + x_3\left(-\frac{a_3}{a_1}, 0, 1, \ldots, 0\right) \\ &+ \cdots + x_n\left(-\frac{a_n}{a_1}, 0, \ldots, 0, 1\right).\end{aligned}$$

(We leave it to the reader to write down the formula in the event that $a_1 = 0$.)

**Example 4.8** *We give a parametric description of the line of intersection of the planes*

$$\begin{aligned}x_1 + 2x_2 - x_3 &= 2 \\ x_1 - x_2 + 2x_3 &= 5.\end{aligned}$$

*Subtracting the first equation from the second yields* $-3x_2+3x_3 = 3$*, or* $-x_2+x_3 = 1$*. Adding twice the latter equation to the first equation in the original system yields* $x_1 + x_3 = 4$*. Thus, we can determine both* $x_1$ *and* $x_2$ *in terms of* $x_3$*:*

$$x_1 = 4 - x_3, \quad x_2 = -1 + x_3.$$

*Then the general solution is of the form*

$$\mathbf{x} = (x_1, x_2, x_3) = (4 - x_3, -1 + x_3, x_3) = (4, -1, 0) + x_3(-1, 1, 1).$$

*The direction vector* $(-1, 1, 1)$ *is orthogonal to* $\mathbf{a} = (1, 2, -1)$ *and* $\mathbf{b} = (1, -1, 2)$*.* ■

### 4.2.2 Flanigan and Kazdan, Calculus Two: Linear and Non-linear Functions

Let $Y$ be a vector and $\mathcal{S}$ a linear subspace of $\mathbb{R}^n$. Projecting $Y$ onto $\mathcal{S}$ means writing $Y$ as $P + Q$, where $P$ is a vector in $\mathcal{S}$ and $Q$ is perpendicular to every vector in $\mathcal{S}$. We will show in the next theorem that this decomposition of $Y$ into the sum $P + Q$ is unique. The vector $P$ is called the orthogonal projection or simply the projection of $Y$ onto $\mathcal{S}$.

Theorem: Let $Y$ be a vector and $\mathcal{S}$ a linear subspace of $\mathbb{R}^n$. Suppose that $Y = P + Q$, where $P$ is a vector in $\mathcal{S}$ and $Q$ is orthogonal to every vector in $\mathcal{S}$. Then for any vector $Z$ in $\mathcal{S}$ other than $P$, $\|Y - P\| < \|Y - Z\|$. The vector $P$ is the unique vector in $\mathcal{S}$ such that $Y - P$ is orthogonal to every vector in $\mathcal{S}$.

> Proof: Let $Z$ be some vector in $\mathcal{S}$ other than $P$. Since $\mathcal{S}$ is a linear subspace and $P$ is in $\mathcal{S}$, the vector $P - Z$ is also in $\mathcal{S}$. Since the vector $Q = Y - P$ is orthogonal to every vector in $\mathcal{S}$, $Y - P$ and $P - Z$ are orthogonal. Thus, the Pythagorean relationship holds for $Y - P$ and $P - Z$:
>
> $$\|Y - P\|^2 + \|P - Z\|^2 = \|(Y - P) + (P - Z)\|^2 = \|Y - Z\|^2.$$
>
> Since $P$ is not equal to $Z$, $\|P - Z\|^2 > 0$. It follows that $\|Y - P\|^2 < \|Y - Z\|^2$, or equivalently, $\|Y - P\| < \|Y - Z\|$ for every vector $Z$ in $\mathcal{S}$ other than $P$. We have shown that if $P$ satisfies the hypothesis of the theorem, then $P$ is closer to $Y$ than any other vector in $\mathcal{S}$. Therefore, $P$ is the only vector that satisfies that hypothesis.

I interject here another proof of uniqueness. It is instructive, and more general, as it applies to any inner product. Recall the definition of the inner product, in (4.1), as, specifically for our context, the dot product for vectors in $\mathbb{R}^n$, and the properties listed there. The proof comes from Atanasiu and Mikusinski, Linear Algebra: Core Topics for the Second Course, 2023, p. 134.

Theorem: Let $\mathcal{U}$ be a subspace of an inner product space $\mathcal{V}$ and let $\mathbf{v} \in \mathcal{V}$. If an orthogonal projection of $\mathbf{v}$ on the subspace $\mathcal{U}$ exists, then it is unique.

> Proof: Assume that both $\mathbf{p}_1$ and $\mathbf{p}_2$ are orthogonal projections of $\mathbf{v}$ on the subspace $\mathcal{U}$, that is, $\langle \mathbf{v} - \mathbf{p}_1, \mathbf{u} \rangle = \langle \mathbf{v} - \mathbf{p}_2, \mathbf{u} \rangle = 0$ for every $\mathbf{u} \in \mathcal{U}$. Since $\mathbf{p}_1, \mathbf{p}_2 \in \mathcal{U}$, we have
>
> $$0 = \langle \mathbf{v} - \mathbf{p}_1, \mathbf{p}_2 \rangle = \langle \mathbf{v}, \mathbf{p}_2 \rangle - \langle \mathbf{p}_1, \mathbf{p}_2 \rangle$$
>
> and
> $$0 = \langle \mathbf{v} - \mathbf{p}_2, \mathbf{p}_2 \rangle = \langle \mathbf{v}, \mathbf{p}_2 \rangle - \langle \mathbf{p}_2, \mathbf{p}_2 \rangle = \langle \mathbf{v}, \mathbf{p}_2 \rangle - \|\mathbf{p}_2\|^2 .$$
> Consequently, $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = \|\mathbf{p}_2\|^2$. Similarly, we can show that $\langle \mathbf{p}_2, \mathbf{p}_1 \rangle = \|\mathbf{p}_1\|^2$. Hence
>
> $$\|\mathbf{p}_1 - \mathbf{p}_2\|^2 = \langle \mathbf{p}_1 - \mathbf{p}_2, \mathbf{p}_1 - \mathbf{p}_2 \rangle = \|\mathbf{p}_1\|^2 - \langle \mathbf{p}_1, \mathbf{p}_2 \rangle - \langle \mathbf{p}_2, \mathbf{p}_1 \rangle + \|\mathbf{p}_2\|^2 = 0,$$
>
> proving that $\mathbf{p}_1 = \mathbf{p}_2$.

The projection $P$ of $Y$ onto $\mathcal{S}$ is closer to $Y$ than any other vector in $\mathcal{S}$. As such, we have:

Definition: The quantity $\|Y - P\| = \|Q\|$ is called the *distance between the point $Y$ and linear subspace $\mathcal{S}$*.

**Example 4.9** *Let us find the projection of the vector $Y = (-2, -6, -17)$ onto the plane $\mathcal{S}$ spanned by $X_1 = (1, 1, -2)$ and $X_2 = (1, -5, -4)$.*

*This is conceptually illustrated in Figure 20, with the shown vectors not corresponding to the numeric values in $Y$, $X_1$, and $X_2$. It shows the projection of vector $\mathbf{v}$, denoted $P(\mathbf{v})$, onto the plane $\mathcal{S}$ spanned by $\mathbf{w}_1$ and $\mathbf{w}_2$.*

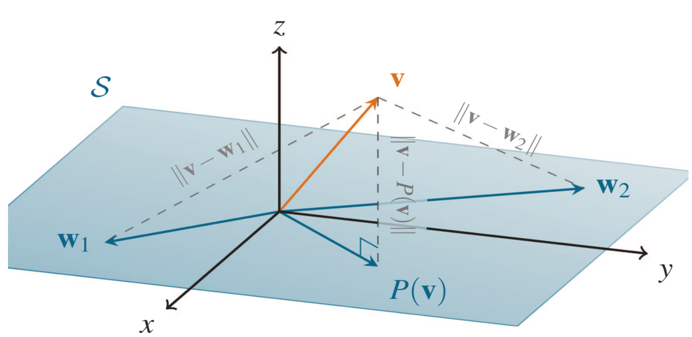



Figure 20: From Johnston, Advanced Linear and Matrix Algebra, 2021, p. 105.

*We are looking for a vector $P$ of the form $\alpha_1(1, 1, -2) + \alpha_2(1, -5, -4)$ such that the vector $Q = Y - P$ is orthogonal to every vector in $\mathcal{S}$. Note that if $Q$ is orthogonal to both $X_1$ and*

*$X_2$, then $Q$ is orthogonal to every linear combination of $X_1$ and $X_2$ (can you see why?), so $Q$ is orthogonal to every vector in $\mathcal{S}$. Thus, we only need $P$ to satisfy*

$$\langle Y - P, X_1 \rangle = 0 \quad \text{and} \quad \langle Y - P, X_2 \rangle = 0.$$

*Substituting in the values for $Y$, $X_1$, and $X_2$ and the expression for $P$, we obtain the following two equations in the unknowns $\alpha_1$ and $\alpha_2$:*

$$\langle(-2,-6,-17) - \alpha_1(1,1,-2) - \alpha_2(1,-5,-4), (1,1,-2)\rangle = 0,$$
$$\langle(-2,-6,-17) - \alpha_1(1,1,-2) - \alpha_2(1,-5,-4), (1,-5,-4)\rangle = 0,$$

*which, after the various inner products are computed, become*

$$6\alpha_1 + 4\alpha_2 = 26, \qquad 4\alpha_1 + 42\alpha_2 = 96.$$

*This system of two equations is easily solved using the Gauss-Jordan method, with solution $\alpha_1 = 3$, $\alpha_2 = 2$. The projection of $Y$ onto $\mathcal{S}$ is $P = 3(1,1,-2) + 2(1,-5,-4) = (5,-7,-14)$. The distance between $Y$ and $\mathcal{S}$ is the norm of the vector*

$$Q = Y - P = (-2,-6,-17) - (5,-7,-14) = (-7,1,-3),$$

*which is $\sqrt{59}$. You should verify that $Q$ is orthogonal to the vectors $P$, $X_1$, and $X_2$.*

*This illustrates the general method for projecting a vector $Y$ onto a plane $\mathcal{S}$ in $\mathbb{R}^n$. If $\mathcal{S}$ is spanned by $X_1$ and $X_2$, we look for a vector $P$ of the form $\alpha_1 X_1 + \alpha_2 X_2$ such that $Y - P$ is orthogonal to both $X_1$ and $X_2$. In terms of inner products,*

$$\langle Y - \alpha_1 X_1 - \alpha_2 X_2, X_1 \rangle = 0 \quad \text{and} \quad \langle Y - \alpha_1 X_1 - \alpha_2 X_2, X_2 \rangle = 0.$$

*Using the properties of inner products, these two equations may be rewritten as*

$$\begin{aligned} \alpha_1 \langle X_1, X_1 \rangle + \alpha_2 \langle X_1, X_2 \rangle &= \langle Y, X_1 \rangle \\ \alpha_1 \langle X_1, X_2 \rangle + \alpha_2 \langle X_2, X_2 \rangle &= \langle Y, X_2 \rangle. \end{aligned} \tag{4.15}$$

*Solve these two equations for the unknowns $\alpha_1$ and $\alpha_2$. Take the corresponding linear combination of $X_1$ and $X_2$ to obtain the projection $P$. The distance between $Y$ and $\mathcal{S}$ is $\|Y - P\|$. It is interesting to note that, even though the vectors $Y$, $X_1$, and $X_2$ are in $\mathbb{R}^n$, we always obtain two equations in two unknowns when finding the projection of a vector onto a plane, whatever the value of $n$.*

*The method just outlined extends naturally to projections onto $k$-dimensional linear subspaces. If $\mathcal{S}$ is spanned by the vectors $X_1, \ldots, X_k$, then we want the vector $P$ to satisfy $\langle Y - P, X_i \rangle = 0$ for $i = 1, \ldots, k$. Substitute $\alpha_1 X_1 + \cdots + \alpha_k X_k$ for $P$ to obtain $k$ equations in the $k$ unknowns $\alpha_1, \ldots, \alpha_k$. Solve the equations and then compute $P$. The projection of $Y$ onto $\mathcal{S}$ is $P$, and the distance between $Y$ and $\mathcal{S}$ is $\|Y - P\|$.* ■

Remark 1: We will later see a name for the $2 \times 2$ matrix implied in (4.15), and its $k \times k$ extension discussed in the previous paragraph: This is called the Gram matrix, as given in (4.91). Indeed, with $\mathbf{K}$ denoting the Gram matrix of inner products, $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_k)'$, and $\mathbf{c} = (c_1, \ldots, c_k)'$, with $c_i = \langle Y, X_i \rangle$, we can express (4.15) as $\mathbf{K}\boldsymbol{\alpha} = \mathbf{c}$. If $\mathbf{K}$ is full rank, which is equivalent to $\{X_i\}$ being linearly independent (with $k = 2$, this means, $X_1$ and $X_2$ are not collinear), then there is a unique solution for $\boldsymbol{\alpha}$. If the $\{X_i\}$ are orthogonal, or orthonormal, as induced by, e.g., Gram-Schmidt, then the solution is very easy to express. We address this in §4.5.3 below.

Remark 2: Another way of posing the task in Example 4.9 is to give a point on the plane $\mathcal{S}$ (for which we can take $X_1$, or $X_2$, or any nonzero linear combination of them) and the normal vector $N$, to the plane. Normal $N$ can be computed using the cross product from (4.13) based on $X_1$ and $X_2$, giving $(-14, 2, -6)$, which is parallel to $N = (-7, 1, -3)$.

We now change notation, because we wish to use the method in Example 4.13, which poses a similar question, but gives, as we just computed, a point on the plane, called $P$, and take $P$ to be $X_1 = (1, 1, -2)$; the point $Y$, which we rename to $Q = (-2, -6, -17)$; and the normal vector $N$, as just computed. Plugging these into (4.22), we get the distance from $Q$ to the plane:

$$\frac{|(Q-P)\cdot N|}{\|N\|} = \frac{|((-2,-6,-17)-(1,1,-2))\cdot(-7,1,-3)|}{\|(-7,1,-3)\|} = \frac{|59|}{\sqrt{59}} = \sqrt{59},$$

as before.

Suppose one is interested in finding the "projection" of a vector $Y$ onto an affine subspace $\mathcal{A}$ and the "distance" from $Y$ to $\mathcal{A}$. Here is the issue described more precisely. We want to find $P$ and $Q$ such that $Y = P + Q$, $P \in \mathcal{A}$, and $Q$ is perpendicular to the difference of any two members of $\mathcal{A}$. It develops that this can always be done, and in only one way. The vector $P$, thus uniquely determined, is the projection of $Y$ onto $\mathcal{A}$, and $\|Q\|$ is the distance from $Y$ to $\mathcal{A}$.

Write $\mathcal{A} = \mathcal{S} + Z$ for some $Z$. Write $Y - Z = R + Q$, where $R$ is the projection of $Y - Z$ onto the linear subspace $\mathcal{S}$. Then $Y = P + Q$, where $P = R + Z$. Now let us check that $P$ is the projection of $Y$ onto the affine subspace $\mathcal{A}$ and that, therefore, $\|Q\|$ is the distance from $Y$ to $\mathcal{A}$. It is clear that $P \in \mathcal{A}$ since it is the sum of $Z$ and a member of $\mathcal{S}$. That $Q$ is perpendicular to the difference of any two members of $\mathcal{A}$ follows from the facts that $Q$ is perpendicular to every member of $\mathcal{S}$ and that the difference of any two members of $\mathcal{A}$ is a member of $\mathcal{S}$. The drawing of an accurate picture for the following example can be of help in grasping these ideas.

**Example 4.10** *Let us calculate the projection of* $(3, 5)$ *onto the line* $\mathcal{S} + (-1, 3)$ *where* $\mathcal{S}$ *is the one-dimensional linear subspace spanned by* $(3, 4)$*. We subtract* $(-1, 3)$ *from* $(3, 5)$ *and find the projection of the difference* $(4, 2)$ *onto* $(3, 4)$*:*

$$\frac{\langle(3,4),(4,2)\rangle}{\|(3,4)\|^2}(3,4) = \left(\frac{12}{5}, \frac{16}{5}\right).$$

*Thus, the projection of* $(3, 5)$ *onto* $\mathcal{S} + (-1, 3)$ *is* $\left(\frac{12}{5}, \frac{16}{5}\right) + (-1, 3) = \left(\frac{7}{5}, \frac{31}{5}\right)$*. To get the distance from* $(3, 5)$ *to* $\mathcal{S} + (-1, 3)$ *we subtract its projection from it and take the norm:*

$$\left\|\left(3 - \frac{7}{5}, 5 - \frac{31}{5}\right)\right\| = \sqrt{4} = 2. \qquad \blacksquare$$

### 4.2.3 Lang, Calculus of Several Variables

We define a *located vector* to be an ordered pair of points that we write $\overrightarrow{AB}$. (This is not a product.) We visualize this as an arrow between $A$ and $B$. We call $A$ the beginning point and $B$ the end point of the located vector. The difference, $B - A$, is defined by writing $B = A + (B - A)$. Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be two located vectors. We shall say that they are equivalent if $B - A = D - C$. Every located vector $\overrightarrow{AB}$ is equivalent to one whose beginning point is the origin, because $\overrightarrow{AB}$ is equivalent to $\overrightarrow{O(B-A)}$ (see the left panel of Figure 21). Clearly this is the only located vector whose beginning point is the origin and which is equivalent to $\overrightarrow{AB}$. If you visualize the parallelogram law in the plane, then it is clear that equivalence of two located vectors can be interpreted geometrically by saying that the lengths of the line segments determined by the pair of points are equal, and that the "directions" in which they point are the same. Figure 21 shows the located vectors $\overrightarrow{O(B-A)}$, $\overrightarrow{AB}$, and $\overrightarrow{O(A-B)}$, $\overrightarrow{BA}$.

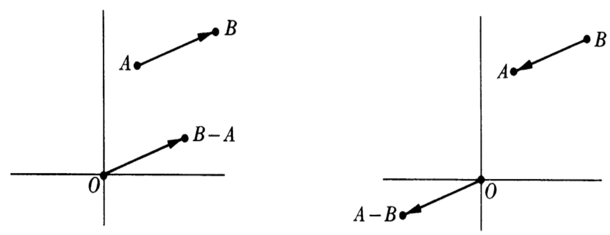



Figure 21: Various located vectors. From Lang, p. 12

Given a located vector $\overrightarrow{OC}$ whose beginning point is the origin, we shall say that it is located at the origin. Given any located vector $\overrightarrow{AB}$, we shall say that it is located at $A$. A located vector at the origin is entirely determined by its end point. In view of this, we shall call an $n$-tuple either a point or a vector, depending on the interpretation which we have in mind. Two located vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are said to be parallel if there is a number $c \neq 0$ such that $B - A = c(Q - P)$. They are said to have the same direction if there is a number $c > 0$ such that $B - A = c(Q - P)$, and have opposite direction if there is a number $c < 0$ such that

$$B - A = c(Q - P).$$

Instead of writing $A \cdot A$ for the scalar product of a vector with itself, it will be convenient to write also $A^2$. (This is the only instance when we allow ourselves such a notation. Thus $A^3$ has no meaning.) As an exercise, verify the following identities using the properties listed after (4.1):

$$(A + B)^2 = A^2 + 2A \cdot B + B^2, \qquad (A - B)^2 = A^2 - 2A \cdot B + B^2. \tag{4.16}$$

As previously stated, for $A$ and $B$ be two points in $\mathbb{R}^n$, we define the distance between them to be

$$\|A - B\| = \sqrt{(A - B) \cdot (A - B)}.$$

This definition coincides with our geometric intuition when $A, B$ are points in the plane; see the left panel of Figure 21. It is the same thing as the length of the located vector $\overrightarrow{AB}$ or the located vector $\overrightarrow{BA}$.

We shall say that a vector $E$ is a unit vector if $\|E\| = 1$. Given any vector $A$, let $a = \|A\|$. If $a \neq 0$, then $A/a$ is a unit vector, because

$$\left\| \frac{1}{a} A \right\| = \frac{1}{a} a = 1.$$

We say that two vectors $A, B$ (neither of which is $O$) have the same direction if there is a number $c > 0$ such that $cA = B$. In view of this definition, the vector $A/\|A\|$ is a unit vector in the direction of $A$ (provided $A \neq O$).

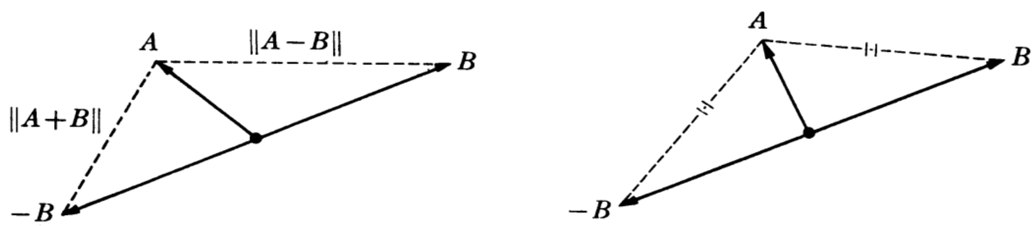



Figure 22: From Lang, p. 24

As stated above in (4.6), we define two (compatible) vectors $A$ and $B$ to be perpendicular, or orthogonal, if $A \cdot B = 0$. The method of proof from Lang is different than that above.

Given $A, B$ in the plane, the condition that $\|A + B\| = \|A - B\|$ (illustrated in Figure 22) coincides with the geometric property that $A$ should be perpendicular to $B$. To prove

$$\|A + B\| = \|A - B\| \text{ if and only if } A \cdot B = 0, \tag{4.17}$$

use (4.16) to write

$$\begin{aligned} \|A + B\| = \|A - B\| &\Leftrightarrow \|A + B\|^2 = \|A - B\|^2 \\ &\Leftrightarrow A^2 + 2A \cdot B + B^2 = A^2 - 2A \cdot B + B^2 \\ &\Leftrightarrow 4A \cdot B = 0 \Leftrightarrow A \cdot B = 0. \end{aligned}$$

Theorem (The General Pythagoras Theorem): If $A$ and $B$ are perpendicular, then

$$\|A + B\|^2 = \|A\|^2 + \|B\|^2. \tag{4.18}$$

> Proof: Use the definitions, namely
>
> $$\|A + B\|^2 = (A + B) \cdot (A + B) = A^2 + 2A \cdot B + B^2 = \|A\|^2 + \|B\|^2,$$
>
> because $A \cdot B = 0$, and $A \cdot A = \|A\|^2$, $B \cdot B = \|B\|^2$ by definition.

Note: If $A$ is perpendicular to $B$, and $x$ is any number, then $A$ is also perpendicular to $xB$ because $A \cdot xB = xA \cdot B = 0$.

We shall now use the notion of perpendicularity to derive the notion of projection. Let $A, B$ be two vectors and $B \neq O$. Let $P$ be the point on the line through $\overrightarrow{OB}$ such that $\overrightarrow{PA}$ is perpendicular to $\overrightarrow{OB}$, as shown in Figure 23. We can write $P = cB$ for some number $c$. We want to find this number $c$ explicitly in terms of $A$ and $B$. The condition $\overrightarrow{PA} \perp \overrightarrow{OB}$ means

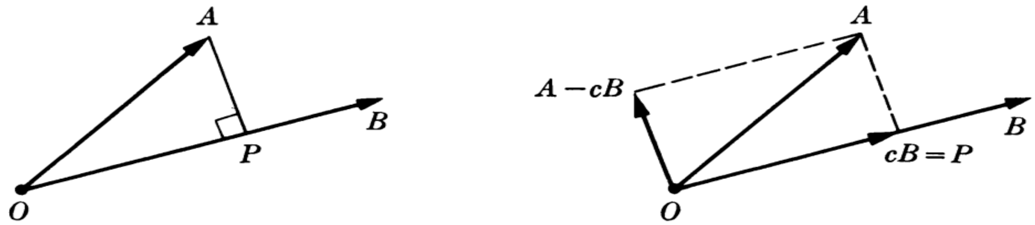



Figure 23: From Lang, p. 25

that $A - P$ is perpendicular to $B$, and since $P = cB$ this means that $(A - cB) \cdot B = 0$; in other words, $A \cdot B - cB \cdot B = 0$. We can solve for $c$, and we find $A \cdot B = cB \cdot B$, so that

$$c = \frac{A \cdot B}{B \cdot B}.$$

Conversely, if we take this value for $c$, and then use distributivity, dotting $A - cB$ with $B$ yields 0, so that $A - cB$ is perpendicular to $B$. Hence we have seen that there is a unique number $c$ such that $A - cB$ is perpendicular to $B$, and $c$ is given by the above formula. We define:

Definition: The *component of $A$ along $B$* is the number $c = (A \cdot B)/(B \cdot B)$. The *projection of $A$ along $B$* is the vector $cB$.

Our construction gives an immediate geometric interpretation for the scalar product. Namely, assume $A \neq O$ and consider the angle $\theta$ between $A$ and $B$, using the left panel of Figure 23. Then, from plane geometry, we see that

$$\cos\theta = \frac{c\|B\|}{\|A\|},$$

or, substituting the value for $c$ obtained above,

$$A \cdot B = \|A\|\|B\| \cos\theta \quad \text{and} \quad \cos\theta = \frac{A \cdot B}{\|A\|\|B\|}. \tag{4.19}$$

Using this construction, we will prove two fundamental inequalities. First observe: If $E_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the $i$th unit vector of $\mathbb{R}^n$, and $A = (a_1, \ldots, a_n)$, then $A \cdot E_i = a_i$ is the $i$th component of $A$, i.e., the component of $A$ along $E_i$. We have

$$|a_i| = \sqrt{a_i^2} \leqq \sqrt{a_1^2 + \cdots + a_n^2} = \|A\|,$$

so that the absolute value of each component of $A$ is at most equal to the length of $A$. More generally, let $E$ be any unit vector (a vector of norm 1). Let $c$ be the component of $A$ along $E$, which reduces to $c = A \cdot E$. Then $A - cE$ is perpendicular to $E$, $A = A - cE + cE$, and $A - cE$ is also perpendicular to $cE$. Thus, by Pythagoras,

$$\|A\|^2 = \|A - cE\|^2 + \|cE\|^2 = \|A - cE\|^2 + c^2,$$

and we have the inequality $c^2 \leqq \|A\|^2$, i.e., $|c| \leqq \|A\|$.

We now generalize this inequality to a dot product $A \cdot B$ when $B$ is not necessarily a unit vector, yielding the Cauchy-Schwarz inequality (1.22), where it was proven in a different way.

Theorem (The (Cauchy-)Schwarz Inequality): Let $A, B$ be two vectors in $\mathbb{R}^n$. Then

$$|A \cdot B| \leqq \|A\|\|B\|. \tag{4.20}$$

Proof: If $B = O$, then both sides of the inequality are equal to 0, and so our assertion is obvious. Suppose that $B \neq O$. Let $c$ be the component of $A$ along $B$, so $c = (A \cdot B)/(B \cdot B)$. We write $A = A - cB + cB$, and, by Pythagoras,

$$\|A\|^2 = \|A - cB\|^2 + \|cB\|^2 = \|A - cB\|^2 + c^2\|B\|^2.$$

Hence $c^2\|B\|^2 \leqq \|A\|^2$. But

$$c^2\|B\|^2 = \frac{(A \cdot B)^2}{(B \cdot B)^2}\|B\|^2 = \frac{|A \cdot B|^2}{\|B\|^4}\|B\|^2 = \frac{|A \cdot B|^2}{\|B\|^2},$$

or

$$\frac{|A \cdot B|^2}{\|B\|^2} \leqq \|A\|^2.$$

Multiply by $\|B\|^2 > 0$ and take the square root to conclude the proof.

In view of the (Cauchy-)Schwarz inequality, we see that, for vectors $A, B$ in $n$-space, the number $(A \cdot B)/(\|A\|\|B\|)$ has absolute value bounded by 1. Consequently,

$$-1 \leqq \frac{A \cdot B}{\|A\|\|B\|} \leqq 1,$$

and there exists a unique angle $\theta$ such that $0 \leqq \theta \leqq \pi$, and such that

$$\cos\theta = \frac{A \cdot B}{\|A\|\|B\|}.$$

Definition: We define this angle to be the angle between $A$ and $B$.

As in (1.23), we use Cauchy-Schwarz to prove:

Theorem (The Triangle Inequality): Let $A, B$ be vectors. Then

$$\|A + B\| \leqq \|A\| + \|B\|. \tag{4.21}$$

Proof: Both sides of this inequality are positive or 0. Hence it will suffice to prove that their squares satisfy the desired inequality, in other words,

$$(A + B) \cdot (A + B) \leqq (\|A\| + \|B\|)^2.$$

To do this, use $(A + B) \cdot (A + B) = A \cdot A + 2A \cdot B + B \cdot B$ from (4.16), which, as just demonstrated, satisfies the inequality $\leqq \|A\|^2 + 2\|A\|\|B\| + \|B\|^2$, the rhs of which is none other than $(\|A\| + \|B\|)^2$.

The name triangle inequality comes from the following: If we draw a triangle as in the left panel of Figure 24, then the triangle inequality expresses the fact that the length of one side is bounded by the sum of the lengths of the other two sides.

We now revisit the material on planes and their mathematical representations, giving further, different, and useful examples from Lang.

Let $P$ be a point in 3-space and consider a located vector $\overrightarrow{ON}$. We define the plane passing through $P$ perpendicular to $\overrightarrow{ON}$ to be the collection of all points $X$ such that the

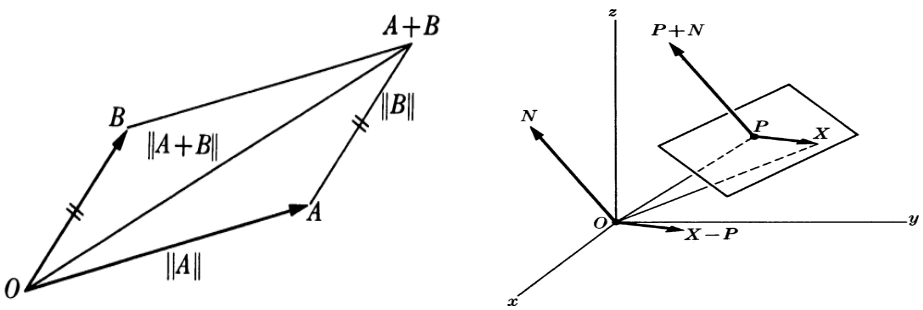



Figure 24: From Lang, p. 30; and 36

located vector $\overrightarrow{PX}$ is perpendicular to $\overrightarrow{ON}$. According to our definitions, this amounts to the condition $(X-P)\cdot N = 0$, which can also be written as $X \cdot N = P \cdot N$. We shall also say that this plane is the one perpendicular to $N$, and consists of all vectors $X$ such that $X - P$ is perpendicular to $N$. The right panel of Figure 24 shows a typical situation in 3-space.

Instead of saying that $N$ is perpendicular to the plane, one also says that $N$ is normal to the plane. Let $t$ be a number $\neq 0$. Then the set of points $X$ such that $(X-P)\cdot N = 0$ coincides with the set of points $X$ such that $(X-P)\cdot tN = 0$. Thus we may say that our plane is the plane passing through $P$ and perpendicular to the line in the direction of $N$. To find the equation of the plane, we could use any vector $tN$ (with $t \neq 0$) instead of $N$.

**Example 4.11** *Let $Q = (1,1,1)$, $P = (1,-1,2)$, and $N = (1,2,3)$. We wish to find the point of intersection of the line through $P$ in the direction of $N$, and the plane through $Q$ perpendicular to $N$. The parametric representation of the line through $P$ in the direction of $N$ is $X = P + tN$. The equation of the plane through $Q$ perpendicular to $N$ is $(X-Q)\cdot N = 0$; and is visualized in Figure 25.*

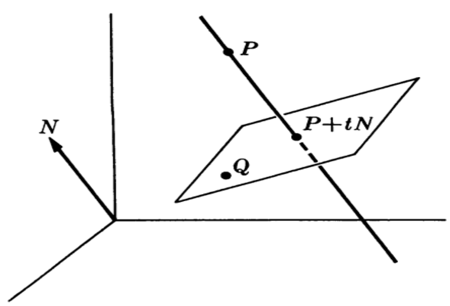



Figure 25: From Lang, p. 39

*We must find $t$ such that $X = P + tN$ also satisfies $(X-Q)\cdot N = 0$; that is, $(P + tN - Q)\cdot N = 0$, or, after using the rules of the dot product, $(P-Q)\cdot N + tN\cdot N = 0$. Solving for $t$ yields*

$$t = \frac{(Q-P)\cdot N}{N \cdot N} = \frac{1}{14}.$$

*Thus,*

$$P + tN = (1,-1,2) + \frac{1}{14}(1,2,3) = \left(\frac{15}{14}, -\frac{12}{14}, \frac{31}{14}\right)$$

*is the desired point of intersection.* ∎

**Example 4.12** *Find the equation of the plane passing through the three points*

$$P_1 = (1, 2, -1), \quad P_2 = (-1, 1, 4), \quad P_3 = (1, 3, -2).$$

*We find a vector $N$ perpendicular to $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$, or in other words, perpendicular to $P_2 - P_1$ and $P_3 - P_1$. We have $P_2 - P_1 = (-2, -1, +5)$ and $P_3 - P_1 = (0, 1, -1)$. Let $N = (a, b, c)$. We must solve $N \cdot (P_2 - P_1) = 0$ and $N \cdot (P_3 - P_1) = 0$, or*

$$-2a - b + 5c = 0, \qquad b - c = 0.$$

*We take $b = c = 1$ and solve for $a = 2$. Then $N = (2, 1, 1)$ satisfies our requirements. The plane perpendicular to $N$, passing through $P_1$ is the desired plane. Its equation is therefore $X \cdot N = P_1 \cdot N$, that is, $2x + y + z = 2 + 2 - 1 = 3$.* ■

**Example 4.13** *Consider a plane defined by the equation $(X - P) \cdot N = 0$, and let $Q$ be an arbitrary point. We wish to find a formula for the distance between $Q$ and the plane. By this we mean the length of the segment from $Q$ to the point of intersection of the perpendicular line to the plane through $Q$, as shown in Figure 26. We let $Q'$ be this point of intersection.*

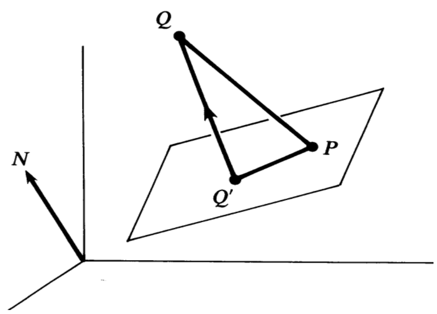



Figure 26: From Lang, p. 41

*From the geometry, we have: length of the segment $\overline{QQ'}$ = length of the projection of $\overline{QP}$ on $\overline{QQ'}$. We can express the length of this projection in terms of the dot product as follows. A unit vector in the direction of $N$, which is perpendicular to the plane, is given by $N/\|N\|$. Then*

$$\begin{aligned} & \textit{length of the projection of } \overline{QP} \textit{ on } \overline{QQ'} \\ = \;& \textit{norm of the projection of } Q - P \textit{ on } N/\|N\| \\ = \;& \left| (Q - P) \cdot \frac{N}{\|N\|} \right| = \frac{|(Q - P) \cdot N|}{\|N\|}, \end{aligned} \tag{4.22}$$

*this being the distance between $Q$ and the plane.* ■

See also Remark 2 of Example 4.9.

## 4.3 Matrix Determinants

For this and subsequent subsections, we assume the reader has a basic familiarity with some fundamental concepts from linear and matrix algebra, e.g., linear (in)dependence of vectors, subspaces, dimension, bases, addition and multiplication of matrices, the transpose of a matrix, the rank of a matrix, and a basic exposure to computing determinants. Our goal in this section is to develop the theory of matrix determinants more rigorously. We will require some of these results when we cover the cross product, e.g., to justify the first equality in (4.83); but arguably more importantly, determinants are essential for understanding the material in §5.6 and §6.6.

We begin by stating some definitions and results (without proofs) on the range and null space of a matrix, the rank of a matrix, and invertibility. These are all standard results that appear in all beginning linear algebra books.

Definition: The range, or column space, or image, of an $m \times n$ matrix $A$ is denoted $\mathrm{range}(A)$, $\mathrm{col}(A)$, or $\mathrm{img}(A)$. It is the subspace of $\mathbb{R}^m$ spanned by the columns of $A$. A vector $\mathbf{b} \in \mathbb{R}^m$ belongs to $\mathrm{range}(A)$ if can be written as a linear combination $\mathbf{b} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$ of the columns of $A = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$.

Definition: The kernel, or null, of $m \times n$ matrix $A$ is denoted $\ker(A)$ or $\mathrm{null}(A)$. It is the subspace of $\mathbb{R}^n$ consisting of all vectors that are annihilated by $A$, meaning $\ker(A) = \{\mathbf{z} \in \mathbb{R}^n \mid A\mathbf{z} = \mathbf{0}\} \subset \mathbb{R}^n$. Its dimension is $\mathrm{nullity}(A)$, with $0 \le \mathrm{nullity}(A) \le n$.

Theorem: Suppose $A$ is an $n \times n$ matrix. The following are equivalent:

$$\begin{array}{ll} \text{(a)} & A \text{ is invertible.} \\ \text{(b)} & \mathrm{range}(A) = \mathbb{R}^n. \\ \text{(c)} & \mathrm{null}(A) = \{\mathbf{0}\}. \\ \text{(d)} & \text{The columns of } A \text{ form a basis of } \mathbb{R}^n \text{ (and, thus, are linearly independent).} \\ \text{(e)} & \text{The rows of } A \text{ form a basis of } \mathbb{R}^n. \end{array} \tag{4.23}$$

Definition: The rank of matrix $A$, denoted by $\mathrm{rank}(A)$, is the dimension of its range.

Theorem: If $A$ is $m \times n$, then

$$\mathrm{rank}(A) \le \min\{m, n\}. \tag{4.24}$$

Definition: We say that $m \times n$ matrix $A$ has full rank when $\mathrm{rank}(A) = \min\{m, n\}$.

From (4.23) and (4.24), an $n \times n$ matrix $A$ is invertible if and only if its range is $\mathbb{R}^n$, and $A$ is invertible if and only if its rank is $n$. Since $\mathrm{rank}(A) \le n$ for all $n \times n$ matrices $A$, invertible matrices are exactly those with the largest possible rank. Thus an $n \times n$ matrix is full rank if and only if it is invertible.

Theorem: Let $A$ be an $n \times n$ matrix. The following are equivalent:

$$\begin{array}{ll} \text{(a)} & A \text{ is invertible.} \\ \text{(b)} & \mathrm{rank}(A) = n. \\ \text{(c)} & \mathrm{nullity}(A) = 0. \end{array} \tag{4.25}$$

The remainder of this section is based on one of the best presentations I have seen on the subject of determinants, namely combining the two relevant chapters in Lang, Introduction to Linear Algebra, 2nd ed., 1986; and (the book I had as a student around 1988) Lang, Linear Algebra, 3rd ed., 1987.

### 4.3.1 Fundamental Determinant Results

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a $2 \times 2$ matrix. We define its determinant to be $ad - bc$. Thus the determinant is a number. One (of several) ways of denoting it is by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{4.26}$$

The determinant can be viewed as a function of the matrix $A$. It can also be viewed as a function of its two columns. Denote these as $A^1$ and $A^2$. Then we write the determinant as

$$D(A), \quad \mathrm{Det}(A), \quad \text{or} \quad D\left(A^1, A^2\right).$$

Property 1: As a function of the column vectors, the determinant is linear, meaning, it has the additivity and homogeneity properties: Suppose $A^1 = C + C'$ is a sum of two columns. Then

$$D\left(C + C', A^2\right) = D\left(C, A^2\right) + D\left(C', A^2\right);$$

and, if $x$ is a number, then $D\left(xA^1, A^2\right) = xD\left(A^1, A^2\right)$. A similar formula holds with respect to the second variable.

> Proof: Let $b'$, $d'$ be two numbers. Then, from (4.26),
>
> $$\begin{aligned} \mathrm{Det}\begin{pmatrix} a & b+b' \\ c & d+d' \end{pmatrix} &= a\left(d+d'\right) - c\left(b+b'\right) \\ &= ad + ad' - cb - cb' = ad - bc + ad' - b'c \\ &= \mathrm{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \mathrm{Det}\begin{pmatrix} a & b' \\ c & d' \end{pmatrix}. \end{aligned}$$
>
> Furthermore, if $x$ is a number, then
>
> $$\mathrm{Det}\begin{pmatrix} xa & b \\ xc & d \end{pmatrix} = xad - xbc = x(ad - bc) = x\,\mathrm{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Property 2: If the two columns are equal, then the determinant is equal to 0.

> Proof: This is immediate, since by hypothesis, the determinant is $ab - ab = 0$.

Property 3: If $I$ is the unit matrix, $I = \left(E^1, E^2\right)$, then $D(I) = D\left(E^1, E^2\right) = 1$.

> Proof: This is also immediate from (4.26).

Theorem: If one adds a scalar multiple of one column to the other, then the value of the determinant does not change, i.e., for $x \in \mathbb{R}$, $D\left(A^1 + xA^2, A^2\right) = D\left(A^1, A^2\right)$. Proof:

> $$\begin{aligned} D\left(A^1 + xA^2, A^2\right) &= D\left(A^1, A^2\right) + xD\left(A^2, A^2\right) && \text{by Property 1 (linearity)} \\ &= D\left(A^1, A^2\right) && \text{by Property 2.} \end{aligned}$$

Theorem: If the two columns are interchanged, then the determinant changes by a sign. That is, $D\left(A^2, A^1\right) = -D\left(A^1, A^2\right)$.

Proof: Writing out the components, the theorem claims

$$\operatorname{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\operatorname{Det}\begin{pmatrix} b & a \\ d & c \end{pmatrix},$$

which is true, from applying (4.26) to both sides. We can also derive it from properties 1 and 2:

$$\begin{aligned} 0 &= D\left(A^1 + A^2, A^1 + A^2\right) \quad \text{(Propery 2)} \\ &= D\left(A^1, A^1 + A^2\right) + D\left(A^2, A^1 + A^2\right) \quad \text{(Property 1)} \\ &= D\left(A^1, A^1\right) + D\left(A^1, A^2\right) + D\left(A^2, A^1\right) + D\left(A^2, A^2\right) \quad \text{(Property 1)} \\ &= D\left(A^1, A^2\right) + D\left(A^2, A^1\right) \quad \text{(Propery 2)}. \end{aligned}$$

Thus we see that $D\left(A^2, A^1\right) = -D\left(A^1, A^2\right)$.

Theorem: The determinant of $A$ is equal to the determinant of its transpose.

Proof: Apply (4.26) to confirm

$$\operatorname{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \operatorname{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix}' = \operatorname{Det}\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Theorem: The vectors $A^1$, $A^2$ are linearly dependent if and only if the determinant is 0.

Proof: First suppose that $A^1, A^2$ are linearly dependent, so there is a linear relation $xA^1 + yA^2 = 0$ with not both $x$ and $y$ equal to 0. Say $x \neq 0$. Then $A^1 = zA^2$, where $z = -y/x$, and, from properties 1 and 2, $D\left(A^1, A^2\right) = D\left(zA^2, A^2\right) = zD\left(A^2, A^2\right) = 0$.

Conversely, suppose that $A^1, A^2$ are linearly independent. Then they must form a basis of $\mathbb{R}^2$, which has dimension 2. Hence we can express the unit vectors $E^1$, $E^2$ as linear combinations of $A^1$, $A^2$, say $E^1 = xA^1 + yA^2$ and $E^2 = zA^1 + wA^2$, where $x, y, z, w$ are scalars. From properties 3, 1, and 2,

$$\begin{aligned} 1 = D\left(E^1, E^2\right) &= D\left(xA^1 + yA^2, zA^1 + wA^2\right) \\ &= xzD\left(A^1, A^1\right) + xwD\left(A^1, A^2\right) + yzD\left(A^2, A^1\right) + ywD\left(A^2, A^2\right) \\ &= (xw - yz)D\left(A^1, A^2\right), \end{aligned}$$

which implies $D\left(A^1, A^2\right) \neq 0$.

Theorem: Let $\varphi$ be a function of two vector variables $A^1, A^2 \in \mathbb{R}^2$ such that:

- $\varphi$ is bilinear; that is $\varphi$ is linear in each variable.
- $\varphi\left(A^1, A^1\right) = 0$ for all $A^1 \in \mathbb{R}^2$.
- $\varphi\left(E^1, E^2\right) = 1$ if $E^1$ and $E^2$ are the unit vectors $\begin{pmatrix}1 & 0\end{pmatrix}'$, $\begin{pmatrix}0 & 1\end{pmatrix}'$, respectively.

Then

$$\varphi\left(A^1, A^2\right) \text{ is the determinant.} \tag{4.27}$$

> Proof: Write $A^1 = aE^1 + cE^2$ and $A^2 = bE^1 + dE^2$. Then
>
> $$\begin{aligned}\varphi\left(A^1, A^2\right) &= \varphi\left(aE^1 + cE^2, bE^1 + dE^2\right) \\ &= ab\varphi\left(E^1, E^1\right) + ad\varphi\left(E^1, E^2\right) + cb\varphi\left(E^2, E^1\right) + cd\varphi\left(E^2, E^2\right) \\ &= ad\varphi\left(E^1, E^2\right) - bc\varphi\left(E^1, E^2\right) \\ &= (ad - bc)\varphi\left(E^1, E^2\right) = ad - bc.\end{aligned}$$

Now consider the $3 \times 3$ matrix

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

We define its determinant according to the formula known as the expansion by a row, say the first row. That is, using the first row, we define

$$\mathrm{Det}(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \tag{4.28}$$

and we will see below that any row can be used. We may describe this sum as follows. Let $A_{ij}$ be the matrix obtained from $A$ by deleting the $i$th row and the $j$th column. Then the sum expressing $\mathrm{Det}(A)$ can be written

$$a_{11}\,\mathrm{Det}\left(A_{11}\right) - a_{12}\,\mathrm{Det}\left(A_{12}\right) + a_{13}\,\mathrm{Det}\left(A_{13}\right).$$

In other words, each term consists of the product of an element of the first row and the determinant of the $2 \times 2$ matrix obtained by deleting the first row and the $j$th column, and putting the appropriate sign to this term as shown. The determinant of a $3 \times 3$ matrix can be written as

$$D(A) = \mathrm{Det}(A) = D\left(A^1, A^2, A^3\right).$$

We use this last expression if we wish to consider the determinant as a function of the columns of $A$. Later, we shall define the determinant of an $n \times n$ matrix, and we use the same notation

$$|A| = D(A) = \mathrm{Det}(A) = D\left(A^1, \ldots, A^n\right).$$

Already in the $3 \times 3$ case we can prove the properties expressed in the next theorem, which we state, however, in the general case.

Theorem: The determinant satisfies the following properties:

1. As a function of each column vector, the determinant is linear, i.e., if the $j$th column $A^j$ is equal to a sum of two column vectors, say $A^j = C + C'$, then (additivity)

$$D\left(A^1, \ldots, C + C', \ldots, A^n\right) = D\left(A^1, \ldots, C, \ldots, A^n\right) + D\left(A^1, \ldots, C', \ldots, A^n\right); \tag{4.29}$$

   and, for $t \in \mathbb{R}$, (homogeneity)

$$D\left(A^1, \ldots, tA^j, \ldots, A^n\right) = tD\left(A^1, \ldots, A^j, \ldots, A^n\right). \tag{4.30}$$

2. If two adjacent columns are equal, i.e., if $A^j = A^{j+1}$ for some $j = 1, \ldots, n-1$, then

$$D(A) = 0. \tag{4.31}$$

3. If $I$ is the identity, or unit, matrix, then

$$D(I) = 1. \tag{4.32}$$

Proof (in the $3 \times 3$ case): The proof is by direct computations. Suppose that the first column is a sum of two columns:

$$A^1 = B + C, \qquad \text{that is,} \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Substituting in each term of (4.28), we see that each term splits into a sum of two terms corresponding to $B$ and $C$. For instance,

$$a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = b_1 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + c_1 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},$$
$$a_{12} \begin{vmatrix} b_2 + c_2 & a_{23} \\ b_3 + c_3 & a_{33} \end{vmatrix} = a_{12} \begin{vmatrix} b_2 & a_{23} \\ b_3 & a_{33} \end{vmatrix} + a_{12} \begin{vmatrix} c_2 & a_{23} \\ c_3 & a_{33} \end{vmatrix},$$

and similarly for the third term. The proof with respect to the other columns are analogous. Next,

$$\begin{aligned} \operatorname{Det}\left(tA^1, A^2, A^3\right) &= ta_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} ta_{21} & a_{23} \\ ta_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} ta_{21} & a_{22} \\ ta_{31} & a_{32} \end{vmatrix} \\ &= t \operatorname{Det}\left(A^1, A^2, A^3\right), \end{aligned}$$

because each $2 \times 2$ determinant is linear in the first column, and we can take $t$ outside each one of the second and third terms. Again the proof is similar with respect to the other columns. A direct substitution shows that, if two adjacent columns are equal, then (4.28), yields 0 for the determinant. Finally, one sees at once that, if $A$ is the unit matrix, then $\operatorname{Det}(A) = 1$. Thus the three properties are verified.

There is no particular reason why we selected the expansion according to the first row, as in (4.28). We can also use the second row, and write a similar sum, namely:

$$\begin{aligned} &- a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ &= -a_{21} \operatorname{Det}\left(A_{21}\right) + a_{22} \operatorname{Det}\left(A_{22}\right) - a_{23} \operatorname{Det}\left(A_{23}\right). \end{aligned}$$

Again, each term is the product of $a_{2j}$ times the determinant of the $2 \times 2$ matrix obtained by deleting the second row and $j$th column, and putting the appropriate sign in front of each term. This sign is determined according to the pattern:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}.$$

One can see directly that the determinant can be expanded according to any row by multiplying out all the terms, and expanding the $2 \times 2$ determinants, thus obtaining the determinant as an alternating sum of six terms:

$$\begin{aligned}\operatorname{Det}(A) = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} \\ + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.\end{aligned} \tag{4.33}$$

In particular, the reader should check this for all three rows, obtaining (4.33) for each. Furthermore, we can also expand according to columns following the same principle. For instance, expanding out according to the first column:

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

yields precisely the same six terms as in (4.33). This implies the following for $3 \times 3$ matrices:

Theorem: The determinant satisfies the rule for expansion according to rows and columns. And the determinant of a matrix is equal to the determinant of its transpose.

Definition: Since the determinant of a $3 \times 3$ matrix is linear as a function of its columns, we may say that it is trilinear; just as a $2 \times 2$ determinant is bilinear. In the $n \times n$ case, we would say $n$-linear, or multilinear.

We now begin with the general $n \times n$ case. Let $F : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ be a function of $n$ variables, where each variable ranges over $\mathbb{R}^n$.

Definition: We say that $F$ is multilinear if $F$ satisfies (4.29) and (4.30); that is,

$$F\left(A^1, \ldots, C + C', \ldots, A^n\right) = F\left(A^1, \ldots, C, \ldots, A^n\right) + F\left(A^1, \ldots, C', \ldots, A^n\right)$$

and $F\left(A^1, \ldots, tC, \ldots, A^n\right) = tF\left(A^1, \ldots, C, \ldots, A^n\right)$. This means that, if we consider some index $j$, and fix the $A^k$ for all $k \neq j$, then the function $X^j \mapsto F\left(A^1, \ldots, X^j, \ldots, A^n\right)$ is linear in the $j$th variable. This is the first property of determinants.

Definition: We say that $F$ is alternating if, whenever $A^j = A^{j+1}$ for some $j$, we have

$$F\left(A^1, \ldots, A^j, A^j, \ldots, A^n\right) = 0.$$

This is the second property of determinants.

Theorem: There exists a multilinear, alternating function $F : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ such that $F(I) = 1$. Such a function is uniquely determined by these three properties.

> Proof: Uniqueness will be shown in §4.3.4. We have already proved existence for $n = 2$ and $n = 3$. The general case of $n \times n$ determinants is done by induction. Suppose that we have been able to define determinants for $(n-1) \times (n-1)$ matrices. Let $i, j$ be a pair of integers between 1 and $n$. If we cross out the $i$th row and $j$th column in the $n \times n$ matrix $A$, we obtain an $(n-1) \times (n-1)$ matrix, which we denote by $A_{ij}$. It looks like
>
> $$i\left(\begin{array}{cc|cc} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & & & \vdots \\ \hline \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{array}\right),$$
>
> with the vertical line indicating the $j$th column. We give an expression for the determinant of an $n \times n$ matrix in terms of determinants of $(n-1) \times (n-1)$ matrices. Let $i$ be an integer, $1 \leq i \leq n$. We define
>
> $$D(A) = (-1)^{i+1} a_{i1} \operatorname{Det}\left(A_{i1}\right) + \cdots + (-1)^{i+n} a_{in} \operatorname{Det}\left(A_{in}\right). \tag{4.34}$$

Each $A_{ij}$ is an $(n-1) \times (n-1)$ matrix. **This sum is called the expansion of the determinant according to the $i$th row.**

This sum can be described in words. For each element of the $i$th row, we have a contribution of one term in the sum. This term is equal to $+$ or $-$ the product of this element, times the determinant of the matrix obtained from $A$ by deleting the $i$th row and the corresponding column. The sign $+$ or $-$ is determined according to the chess-board pattern:

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ & & \cdots & & \end{pmatrix}.$$

We now need to show that this function $D$ satisfies properties (4.29)-(4.32). Note that $D(A)$ is a sum of the terms $\sum(-1)^{i+j} a_{ij} \operatorname{Det}(A_{ij})$ as $j$ ranges from 1 to $n$.

1. Consider $D$ as a function of the $k$th column, and consider any term

   $$(-1)^{i+j} a_{ij} \operatorname{Det}(A_{ij}).$$

   If $j \neq k$, then $a_{ij}$ does not depend on the $k$th column, and $\operatorname{Det}(A_{ij})$ depends linearly on the $k$th column. $\operatorname{Det}(A_{ij})$ depends linearly on the $k$th column because we assume it holds for the $(n-1) \times (n-1)$ case via induction, and note it was explicitly shown for $n = 2$ and $n = 3$.

   If $j = k$, then $a_{ij}$ depends linearly on the $k$th column, and $\operatorname{Det}(A_{ij})$ does not depend on the $k$th column.

   In any case, our term depends linearly on the $k$th column. Since $D(A)$ is a sum of such terms, it depends linearly on the $k$th column, and properties (4.29) and (4.30) follow.

2. Suppose two adjacent columns of $A$ are equal, namely $A^k = A^{k+1}$. Let $j$ be an index such that $j \neq k$ and $j \neq (k+1)$. Then the matrix $A_{ij}$ has two adjacent equal columns, and hence via the induction assumption its determinant is equal to 0. Thus the term corresponding to an index $j$, for $j \neq k$ and $j \neq (k+1)$, gives a zero contribution to $D(A)$. The other two terms can be written

   $$(-1)^{i+k} a_{ik} \operatorname{Det}(A_{ik}) \quad + \quad (-1)^{i+k+1} a_{i,k+1} \operatorname{Det}(A_{i,k+1}).$$

   The two matrices $A_{ik}$ and $A_{i,k+1}$ are equal because of our assumption that the $k$th column of $A$ is equal to the $(k+1)$th column. Similarly, $a_{ik} = a_{i,k+1}$. Hence these two terms cancel (they occur with opposite signs). This proves property (4.31).

3. Let $A$ be the unit matrix. Then $a_{ij} = 0$ unless $i = j$, in which case $a_{ii} = 1$. Each $A_{ii}$ is the unit $(n-1) \times (n-1)$ matrix. The only term in the sum that gives a non-zero contribution is

   $$(-1)^{i+i} a_{ii} \operatorname{Det}(A_{ii}), \tag{4.35}$$

   which is equal to 1. In particular, for fixed row $i$, only $j = i$ had $a_{ij}$ nonzero, so interest centers on $A_{ii}$, which is $A$ after deleting the $i$th row and $j$th equals $i$th column. $A_{ii}$ is the $n-1$ identity matrix. So, the only nonzero term out of the $n$ is (4.35). This proves property (4.32).

Theorem: If $n \times n$ matrix $A = \{a_{ij}\}$ is (upper or lower) triangular, then

$$\operatorname{Det}(A) = a_{11} \cdot a_{22} \cdot \dots \cdot a_{nn}, \tag{4.36}$$

i.e., it is the product of the diagonal terms.

> Proof: Take $i = 1$ in (4.34).

Theorem: Determinants satisfy the rule for expansion according to rows and columns. In particular, for the latter, for any column $A^j$ of the matrix $A = (a_{ij})$, we have

$$D(A) = (-1)^{1+j} a_{1j} D(A_{1j}) + \dots + (-1)^{n+j} a_{nj} D(A_{nj}). \tag{4.37}$$

> Proof: The rule was shown for rows, in (4.34). We have seen above that the result holds for columns for $3 \times 3$ matrices. It will be shown later that the determinant of a general $n \times n$ matrix is equal to the determinant of its transpose. When that is established, then column expansion (4.37) also holds.

Recall properties (4.29)-(4.32). Here is another.

Theorem: Let $i$ and $j$ be integers with $1 \leq i, j \leq n$ and $i \neq j$. If the $i$th and $j$th columns of $n \times n$ matrix $A$ are interchanged, then

$$\textbf{the determinant of } A \textbf{ changes by a sign}. \tag{4.38}$$

> Proof: We first prove this for the case in which we interchange the $j$th and $(j+1)$th columns. Below, in (4.40), the general case is proven.
>
> In the matrix $A$, we replace the $j$th and $(j+1)$th columns by $A^j + A^{j+1}$. We obtain a matrix with two equal adjacent columns, and by (4.31) we have
>
> $$0 = D\left(\dots, A^j + A^{j+1}, A^j + A^{j+1}, \dots\right).$$
>
> Expanding out and using additivity property (4.29) repeatedly yields
>
> $$\begin{aligned} 0 = {} & D\left(\dots, A^j, A^j, \dots\right) + D\left(\dots, A^{j+1}, A^j, \dots\right) \\ & + D\left(\dots, A^j, A^{j+1}, \dots\right) + D\left(\dots, A^{j+1}, A^{j+1}, \dots\right). \end{aligned}$$
>
> Again from (4.31), we see that two of these four terms are equal to 0, and hence that
>
> $$0 = D\left(\dots, A^{j+1}, A^j, \dots\right) + D\left(\dots, A^j, A^{j+1}, \dots\right).$$
>
> In this last sum, one term must be equal to minus the other, as desired.

Theorem: If two columns $A^j$, and $A^i$ of $A$ are equal, $j \neq i$, then

$$\textbf{the determinant of } A \textbf{ is equal to 0}. \tag{4.39}$$

> Proof: Assume that two columns of the matrix $A$ are equal. We can change the matrix by a successive interchange of adjacent columns until we obtain a matrix with equal adjacent columns. (This could be proved formally by induction.) Each time that we make such an adjacent interchange, the determinant changes by a sign, which does not affect its being 0 or not. Hence we conclude by property (4.31) that $D(A) = 0$ if two columns are equal.

We can now return to the proof of (4.38) for any $i \neq j$.

Proof: Exactly the same argument as given in the proof of (4.38) for $j$ and $j+1$ works in the general case if we use (4.39). We just note that

$$0 = D\left(\ldots, A^i + A^j, \ldots, A^i + A^j, \ldots\right) \tag{4.40}$$

and expand as before. This concludes the proof of (4.38).

Theorem: If one adds a scalar multiple of one column to another of $n \times n$ matrix $A$, then

$$\textbf{the value of the determinant does not change}. \tag{4.41}$$

Proof: Consider two distinct columns, say the $k$th and $j$th columns $A^k$ and $A^j$ with $k \neq j$. Let $t$ be a scalar. We add $tA^j$ to $A^k$. By linearity properties (4.29) and (4.30), the determinant becomes (with the sole indicated entries all being in the $k$th column)

$$D\left(\ldots, A^k + tA^j, \ldots\right) = D\left(\ldots, A^k, \ldots\right) + D\left(\ldots, tA^j, \ldots\right),$$

and $D\left(\ldots, A^k, \ldots\right)$ is simply $D(A)$. Furthermore, $D\left(\ldots, tA^j, \ldots\right) = tD\left(\ldots, A^j, \ldots\right)$. Since $k \neq j$, the determinant on the right has two equal columns, because $A^j$ occurs in the $k$th place and also in the $j$th place. Hence, it is equal to 0, and thus $D\left(\ldots, A^k + tA^j, \ldots\right) = D\left(\ldots, A^k, \ldots\right)$, as was to be shown.

Theorem (Cramer's rule): Let $A^1, \ldots, A^n$ be column vectors such that $D\left(A^1, \ldots, A^n\right) \neq 0$. Let $B$ be a column vector. If $x_1, \ldots, x_n$ are numbers such that $x_1 A^1 + \cdots + x_n A^n = B$, then, for each $j = 1, \ldots, n$, we have

$$x_j = \frac{D\left(A^1, \ldots, B, \ldots, A^n\right)}{D\left(A^1, \ldots, A^n\right)}, \tag{4.42}$$

where $B$ occurs in the $j$th column instead of $A^j$. In other words,

$$x_j = \frac{\begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}}.$$

(The numerator is obtained from $A$ by replacing the $j$th column $A^j$ by $B$. The denominator is the determinant of the matrix $A$.)

Cramer's rule gives us an explicit way of finding the coordinates of $B$ with respect to $A^1, \ldots, A^n$. In the language of linear equations, it allows us to solve explicitly in terms of determinants the system of $n$ linear equations in $n$ unknowns:

$$\begin{gathered} x_1 a_{11} + \cdots + x_n a_{1n} = b_1 \\ \cdots \\ x_1 a_{n1} + \cdots + x_n a_{nn} = b_n. \end{gathered}$$

Remark: Cramer's rule is theoretically interesting, but is never used for actual computation when solving a system of linear equations. An application of Cramer's rule for the partial autocorrelation function for autoregressive models in time series analysis is given in Paolella, Linear Models and Time-Series Analysis, §8.2.2.2.

> Proof: Let $B$ be written as in the statement of the theorem, and consider the determinant of the matrix obtained by replacing the $j$th column of $A$ by $B$. Then
>
> $$D\left(A^1, \ldots, B, \ldots, A^n\right) = D\left(A^1, \ldots, x_1 A^1 + \cdots + x_n A^n, \ldots, A^n\right).$$
>
> We use additivity property (4.29) and obtain a sum:
>
> $$\begin{aligned}&D\left(A^1, \ldots, x_1 A^1, \ldots, A^n\right) + \cdots + D\left(A^1, \ldots, x_j A^j, \ldots, A^n\right)\\&+ \cdots + D\left(A^1, \ldots, x_n A^n, \ldots, A^n\right),\end{aligned}$$
>
> which, by homogeneity property (4.30), is equal to
>
> $$\begin{aligned}&x_1 D\left(A^1, \ldots, A^1, \ldots, A^n\right) + \cdots + x_j D\left(A^1, \ldots, A^n\right)\\&+ \cdots + x_n D\left(A^1, \ldots, A^n, \ldots, A^n\right).\end{aligned}$$
>
> In every term of this sum except the $j$th term, two column vectors are equal. Hence every term except the $j$th term is equal to 0, by property (4.39). The $j$th term is equal to
>
> $$x_j D\left(A^1, \ldots, A^n\right),$$
>
> and is therefore equal to the determinant we started with, namely $D\left(A^1, \ldots, B, \ldots, A^n\right)$. We can solve for $x_j$, and obtain precisely the expression given in the statement of the theorem.

Theorem: Let $A^1, \ldots, A^n$ be column vectors (of dimension $n$). If they are linearly dependent, then

$$D\left(A^1, \ldots, A^n\right) = 0. \tag{4.43}$$

If $D\left(A^1, \ldots, A^n\right) \neq 0$, then $A^1, \ldots, A^n$ are linearly independent.

> Proof: The second assertion is merely an equivalent formulation of the first, so it suffices to prove the first. Assuming $A^1, \ldots, A^n$ are linearly dependent, we can find numbers $x_1, \ldots, x_n$, not all 0, such that $x_1 A^1 + \cdots + x_n A^n = O$. Suppose $x_j \neq 0$. Then $x_j A^j = -\sum_{k \neq j} x_k A^k$. Note that there is no $j$th term on the right hand side. Dividing by $x_j$, we obtain $A^j$ as a linear combination of the vectors $A^k$ with $k \neq j$. In other words, there are numbers $y_k$, with $k \neq j$, such that $A^j = \sum_{k \neq j} y_k A^k$, namely $y_k = -x_k / x_j$. By the linearity properties (4.29) and (4.30), we get
>
> $$\begin{aligned}D\left(A^1, \ldots, A^n\right) &= D\left(A^1, \ldots, \sum_{k \neq j} y_k A^k, \ldots, A^n\right)\\&= \sum_{k \neq j} y_k D\left(A^1, \ldots, A^k, \ldots, A^n\right),\end{aligned}$$
>
> with $A^k$ in the $j$th column, and $k \neq j$. In the sum on the right, each determinant has the $k$th column equal to the $j$th column and is therefore equal to 0 by (4.39).

Corollary: If $A^1, \ldots, A^n$ are column vectors of $\mathbb{R}^n$ such that $D\left(A^1, \ldots, A^n\right) \neq 0$, and if $B$ is a column vector of $\mathbb{R}^n$, then there exist numbers $x_1, \ldots, x_n$ such that

$$x_1 A^1 + \cdots + x_n A^n = B. \tag{4.44}$$

> Proof: According to (the contrapositive of) (4.43), $A^1, \ldots, A^n$ are linearly independent, and hence form a basis of $\mathbb{R}^n$. Thus, any vector of $\mathbb{R}^n$ can be written as a linear combination of $A^1, \ldots, A^n$.

In terms of linear equations, this corollary shows: If a system of $n$ linear equations in $n$ unknowns has a matrix of coefficients whose determinant is not 0, then this system has a solution. Later we prove the converse of the corollary, resulting in the following:

Theorem: For vectors $A^1, \ldots, A^n \in \mathbb{R}^n$,

$$D\left(A^1, \ldots, A^n\right) = 0 \quad \Longleftrightarrow \quad \left\{A^1, \ldots, A^n\right\} \text{ is linearly dependent.} \tag{4.45}$$

We have so far used two column operations when working with determinants, namely (i) adding a scalar multiple of one column to another column; and (ii) interchanging two columns.

Definition: Two $n \times n$ matrices $A$ and $B$ are said to be column equivalent if $B$ can be obtained from $A$ by making a succession of column operations (i) and (ii).

Proposition: Let $A$ and $B$ be column equivalent. Then:
(a) $\operatorname{rank} A = \operatorname{rank} B$.
(b) $A$ is invertible if and only if $B$ is invertible.
(c) $\operatorname{Det}(A) = 0$ if and only if $\operatorname{Det}(B) = 0$.

> Proof: Let $A$ be an $n \times n$ matrix. If we interchange two columns of $A$, then the column space is unchanged. Let $A^1, \ldots, A^n$ be the columns of $A$. Let $x$ be a scalar. Then the space generated by $A^1 + xA^2, A^2, \ldots, A^n$ is clearly the same as the space generated by $A^1, \ldots, A^n$. Hence, if $B$ is column equivalent to $A$, it follows that the column space of $B$ is equal to the column space of $A$, so $\operatorname{rank} A = \operatorname{rank} B$.
>
> The determinant changes only by a sign when we make a column operation, so $\operatorname{Det}(A) = 0$ if and only if $\operatorname{Det}(B) = 0$.
>
> Finally, if $A$ is invertible, then $\operatorname{rank} A = n$ by (4.25), so $\operatorname{rank} B = n$, and so, again from (4.25), $B$ is invertible.

Theorem: Let $A$ be an $n \times n$ matrix. Then $A$ is column equivalent to a triangular matrix

$$B = \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}.$$

> Proof: By induction on $n$. Let $A = (a_{ij})$. There is nothing to prove if $n = 1$. Let $n > 1$. If all elements of the first row of $A$ are 0, then we conclude the proof by induction by making column operations on the $(n-1) \times (n-1)$ matrix
>
> $$\begin{pmatrix} a_{22} & \cdots & a_{nn} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Suppose some element of the first row of $A$ is not 0. By column operations, we can suppose that $a_{11} \neq 0$. By adding a scalar multiple of the first column to each of the other columns, we can then get an equivalent matrix $B$ such that $b_{12} = \cdots = b_{1n} = 0$; that is, all elements of the first row are 0 except for $a_{11}$. We can again apply induction to the matrix obtained by deleting the first row and first column. This concludes the proof.

Theorem: Let $A = (A^1, \ldots, A^n)$ be a square matrix. The following are equivalent:

$$\begin{aligned} &\text{(a)} \quad A \text{ is invertible.} \\ &\text{(b)} \quad \text{The columns } A^1, \ldots, A^n \text{ are linearly independent.} \\ &\text{(c)} \quad D(A) \neq 0. \end{aligned} \tag{4.46}$$

Proof: That (a) is equivalent to (b) is from (4.23). From the previous proposition and theorem, $A$ is column equivalent to a (lower) triangular matrix with the same rank. Thus, we may assume that $A$ is a triangular matrix. The determinant is then the product of the diagonal elements from (4.36), and is 0 if and only if some diagonal element is 0. But this condition is equivalent to the column vectors being linearly independent, thus concluding the proof.

Since determinants can be used to test linear independence, they can be used to determine the rank of a matrix.

**Example 4.14** *For the* $3 \times 4$ *matrix*

$$A = \begin{pmatrix} 3 & 1 & 2 & 5 \\ 1 & 2 & -1 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix},$$

*its rank is at most 3, from* (4.24)*. If we can find three linearly independent columns, then we know that its rank is exactly 3. But the determinant*

$$\begin{vmatrix} 1 & 2 & 5 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 4$$

*is not equal to 0. Hence* $\operatorname{rank} A = 3$. ■

**Example 4.15** *Let*

$$C = \begin{pmatrix} 3 & 1 & 2 & 5 \\ 1 & 2 & -1 & 2 \\ 4 & 3 & 1 & 7 \end{pmatrix}.$$

*If we compute every* $3 \times 3$ *sub-determinant, we find 0. Hence the rank of* $C$ *is at most equal to 2. However, the first two rows are linearly independent, for instance because the determinant*

$$\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$$

*is not equal to 0. It is the determinant of the first two columns of the* $2 \times 4$ *matrix*

$$\begin{pmatrix} 3 & 1 & 2 & 5 \\ 1 & 2 & -1 & 2 \end{pmatrix}.$$

*Hence the rank of* $C$ *is equal to 2.* ■

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a $2 \times 2$ matrix, and assume that its determinant $ad - bc \neq 0$. We wish to find an inverse for $A$, that is a $2 \times 2$ matrix

$$X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

such that $AX = XA = I$. Let us look at the first requirement, $AX = I$, which, written out in full, looks like this:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The first column of $AX$ implies that we must solve the equations

$$\begin{aligned} ax + bz &= 1, \\ cx + dz &= 0. \end{aligned}$$

This is a system of two equations in two unknowns, $x$ and $z$, which we know how to solve. Similarly, looking at the second column, we see that we must solve a system of two equations in the unknowns $y$ and $w$, namely

$$\begin{aligned} ay + bw &= 0, \\ cy + dw &= 1. \end{aligned}$$

Similarly, in the $3 \times 3$ case, we would find three systems of linear equations, corresponding to the first column, the second column, and the third column. Each system could be solved to yield the inverse.

Definition: Let $A$ be an $n \times n$ matrix. If $B$ is a matrix such that $AB = I$ and $BA = I$ (where $I$ is the $n \times n$ identity matrix), then we called $B$ an inverse of $A$, and we write $B = A^{-1}$.

Theorem: If there exists an inverse of $A$, then it is unique.

> Proof: Let $B$ and $C$ be inverses of $A$. Then $CA = I$. Multiplying by $B$ on the right, we obtain $CAB = B$. But $CAB = C(AB) = CI = C$. Hence $C = B$. A similar argument works for $AC = I$.

Theorem: Let $A = (a_{ij})$ be an $n \times n$ matrix, and assume that $D(A) \neq 0$. Then $A$ is invertible. Let $E^j$ be the $j$th column unit vector, and let

$$b_{ij} = \frac{D\left(A^1, \ldots, E^j, \ldots, A^n\right)}{D(A)},$$

where $E^j$ occurs in the $i$th place. Then

$$B = (b_{ij}) \text{ is an inverse for } A. \tag{4.47}$$

> Proof: Let $X = (x_{ij})$ be an unknown $n \times n$ matrix. We wish to solve for the components $x_{ij}$, so that they satisfy $AX = I$. From the definition of products of matrices, this means that for each $j$, we must solve
>
> $$E^j = x_{1j}A^1 + \cdots + x_{nj}A^n.$$

This is a system of linear equations, which can be solved uniquely by Cramer's rule (4.42), and we obtain

$$x_{ij} = \frac{D\left(A^1, \ldots, E^j, \ldots, A^n\right)}{D(A)},$$

which is the formula given in the theorem.

We must still prove that $XA = I$. Note that $D\left(A'\right) \neq 0$. Hence by what we have already proved, we can find a matrix $Y$ such that $A'Y = I$. Taking transposes, we obtain $Y'A = I$. Now we have

$$I = Y'(AX)A = Y'A(XA) = XA,$$

thereby proving what we want, namely that $X = B$ is an inverse for $A$.

We can write out the components of the matrix $B$ in Theorem 5.1 as follows:

$$b_{ij} = \frac{\begin{vmatrix} a_{11} & \cdots & 0 & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{j1} & \cdots & 1 & \cdots & a_{jn} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & 0 & \cdots & a_{nn} \end{vmatrix}}{\operatorname{Det}(A)},$$

where the 1 occurs in the $j$th row and $i$th column. If we expand the determinant in the numerator according to the $i$th column, then all terms but one are equal to 0, and hence we obtain the numerator of $b_{ij}$ as a sub-determinant of $\operatorname{Det}(A)$. Let $A_{ij}$ be the matrix obtained from $A$ by deleting the $i$th row and the $j$th column. Then (note the reversal of indices!)

$$b_{ij} = \frac{(-1)^{i+j} \operatorname{Det}\left(A_{ji}\right)}{\operatorname{Det}(A)}$$

and, thus, we have the formula (note $ij$ is used, not $ji$; hence the transpose)

$$A^{-1} = \text{ transpose of } \left(\frac{(-1)^{i+j} \operatorname{Det}\left(A_{ij}\right)}{\operatorname{Det}(A)}\right). \tag{4.48}$$

Remark: Relation (4.48) is arguably fascinating, giving the inverse of a matrix in terms of determinants. It is, however, not computationally efficient. In fact, the method shown above, involving solving $n$ sets of equations, each set with $n$ equations and $n$ unknowns, to arrive at the $n \times n$ matrix inverse, is also slow. In practice, inverses are not computed for solving systems of linear equations, but are rather of only theoretical interest. See just above (4.96) for some discussion of "reality" for large-scale computations.

Definition: A square matrix whose determinant is nonzero, or, equivalently, which admits an inverse, is called non-singular.

### 4.3.2 Determinants as Area and Volume

It is remarkable that the determinant has an interpretation as a volume. We discuss first the 2-dimensional case, and thus speak of area, although we write Vol for the area of a 2-dimensional figure, to keep the terminology which generalizes to higher dimensions. Consider the parallelogram spanned by two vectors $v$ and $w$. By definition, this parallelogram is the set of all linear combinations

$$t_1 v + t_2 w \quad \text{with} \quad 0 \leq t_i \leq 1.$$

We view $v$ and $w$ as column vectors, and can thus form their determinant $D(v, w)$. This determinant may be positive or negative, because $D(v, w) = -D(w, v)$. Thus the determinant itself cannot be the area of this parallelogram, since area is always $\geq 0$. However, we shall prove:

Theorem: The area of the parallelogram spanned by $v$ and $w$ is equal to the absolute value of the determinant, namely

$$|D(v, w)|. \tag{4.49}$$

To prove this, we introduce the notion of oriented area. Let $P(v, w)$ be the parallelogram spanned by $v$ and $w$. We denote by $\mathrm{Vol}_0(v, w)$ the area of $P(v, w)$ if the determinant $D(v, w) \geq 0$, and minus the area of $P(v, w)$ if the determinant $D(v, w) < 0$. Thus at least $\mathrm{Vol}_0(v, w)$ has the same sign as the determinant, and we call $\mathrm{Vol}_0(v, w)$ the oriented area. We denote by $\mathrm{Vol}(v, w)$ the area of the parallelogram spanned by $v$ and $w$. Hence $\mathrm{Vol}_0(v, w) = \pm \mathrm{Vol}(v, w)$.

To prove (4.49), it will suffice to prove: The oriented area is equal to the determinant. In other words,

$$\mathrm{Vol}_0(v, w) = D(v, w). \tag{4.50}$$

Now, to prove (4.50), it will suffice to prove:

Theorem: $\mathrm{Vol}_0$ satisfies the three properties characteristic of a determinant, namely:

1. $\mathrm{Vol}_0$ is linear in each variable $v$ and $w$.

2. $\mathrm{Vol}_0(v, v) = 0$ for all $v$.

3. $\mathrm{Vol}_0 \left(E^1, E^2\right) = 1$ if $E^1$ and $E^2$ are the standard unit vectors.

We already know from (4.27) that these three properties characterize determinants, but repeat the argument briefly here: We assume that we have a function $g$ satisfying these three properties (with $g$ replacing $\mathrm{Vol}_0$). Then, for any vectors

$$v = aE^1 + cE^2 \quad \text{and} \quad w = bE^1 + dE^2,$$

we have

$$\begin{aligned} g\left(aE^1 + cE^2, bE^1 + dE^2\right) &= ab\, g\left(E^1, E^1\right) + ad\, g\left(E^1, E^2\right) \\ &\quad + cb\, g\left(E^2, E^1\right) + cd\, g\left(E^2, E^2\right). \end{aligned}$$

The first and fourth terms equal 0. It is easy to see that $g\left(E^2, E^1\right) = -g\left(E^1, E^2\right)$, and, hence, $g(v, w) = (ad - bc) g\left(E^1, E^2\right) = ad - bc$. This proves what we wanted.

In order to prove that $\mathrm{Vol}_0$ satisfies the three properties, we shall use simple properties of area (or volume) like the following: The area of a line segment is equal to 0. If $A$ is a certain region, then the area of $A$ is the same as the area of a translation of $A$, i.e., the same as the area of the region $A_w$ (consisting of all points $v+w$ with $v \in A$). If $A$ and $B$ are regions that are disjoint or such that their common points have area equal to 0, then

$$\mathrm{Vol}(A \cup B) = \mathrm{Vol}(A) + \mathrm{Vol}(B).$$

Consider now $\mathrm{Vol}_0$. The last two properties are obvious. Indeed, the parallelogram spanned by $v$ and $v$ is simply a line segment, and its 2-dimensional area is therefore equal to 0. Thus property 2 is satisfied. As for the third property, the parallelogram spanned by the unit vectors $E^1$ and $E^2$ is simply the unit square, whose area is 1. Hence, in this case, we have $\mathrm{Vol}_0\left(E^1, E^2\right) = 1$. The harder property is the first. We shall need a lemma.

Lemma: If $v, w$ are linearly dependent, then

$$\mathrm{Vol}_0(v, w) = 0. \tag{4.51}$$

> Proof: Suppose that we can write $av + bw = 0$, with $a$ or $b \neq 0$. Say $a \neq 0$. Then
>
> $$v = -\frac{b}{a} w = cw,$$
>
> so that $v$ and $w$ lie on the same straight line, and the parallelogram spanned by $v$ and $w$ is a line segment. Hence $\mathrm{Vol}_0(v, w) = 0$, thus proving the lemma.

We also know that, when $v$ and $w$ are linearly dependent, then $D(v, w) = 0$, so, in this trivial case, our theorem is proved. In the subsequent lemmas, we assume that $v$ and $w$ are linearly independent.

Lemma: Assume that $v, w$ are linearly independent, and let $n$ be a positive integer. Then

$$\mathrm{Vol}(nv, w) = n\,\mathrm{Vol}(v, w). \tag{4.52}$$

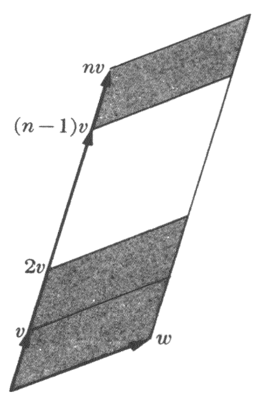



Figure 27: The parallelogram spanned by $nv$ and $w$

> Proof: The parallelogram spanned by $nv$ and $w$ consists of $n$ parallelograms as shown in Figure 27. These $n$ parallelograms are simply the translations of $P(v, w)$ by $v, 2v, \ldots, (n-1)v$, and each translation of $P(v, w)$ has the same area as $P(v, w)$. These translations have only line segments in common, and, hence, $\mathrm{Vol}(nv, w) = n\,\mathrm{Vol}(v, w)$, as desired.

Corollary: Assume that $v$ and $w$ are linearly independent, and let $n \in \mathbb{N}$. Then

$$\mathrm{Vol}\left(\frac{1}{n}v, w\right) = \frac{1}{n}\mathrm{Vol}(v, w).$$

More generally, if $m, n$ are positive integers, then

$$\mathrm{Vol}\left(\frac{m}{n}v, w\right) = \frac{m}{n}\mathrm{Vol}(v, w). \tag{4.53}$$

Proof: Let $v_1 = (1/n)v$. From (4.52), $\mathrm{Vol}\,(nv_1, w) = n\,\mathrm{Vol}\,(v_1, w)$. This is merely a reformulation of our first assertion, since $nv_1 = v$. As for (4.53), we write $m/n = m \cdot 1/n$ and apply the proved statements successively:

$$\mathrm{Vol}\left(m \cdot \frac{1}{n}v, w\right) = m\,\mathrm{Vol}\left(\frac{1}{n}v, w\right) = m \cdot \frac{1}{n}\mathrm{Vol}(v, w) = \frac{m}{n}\mathrm{Vol}(v, w).$$

Lemma:

$$\mathrm{Vol}(-v, w) = \mathrm{Vol}(v, w). \tag{4.54}$$

Proof: The parallelogram spanned by $-v$ and $w$ is a translation by $-v$ of the parallelogram $P(v, w)$. Hence $P(v, w)$ and $P(-v, w)$ have the same area; see the left panel of Figure 28.

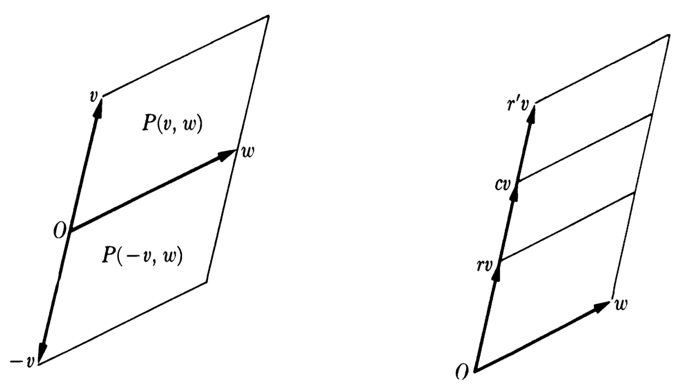



Figure 28: For illustrating (4.54) and (4.55).

Lemma: If $c$ is any real number $> 0$, then

$$\mathrm{Vol}(cv, w) = c\,\mathrm{Vol}(v, w). \tag{4.55}$$

Proof: Let $r$ and $r'$ be rational numbers such that $0 < r < c < r'$. See the right panel of Figure 28. Then $P(rv, w) \subset P(cv, w) \subset P\,(r'v, w)$. Hence, by (4.52),

$$r\,\mathrm{Vol}(v, w) = \mathrm{Vol}(rv, w) \leq \mathrm{Vol}(cv, w) \leq \mathrm{Vol}\,(r'v, w) = r'\,\mathrm{Vol}(v, w).$$

Letting $r$ and $r'$ approach $c$ as a limit, $\mathrm{Vol}(cv, w) = c\,\mathrm{Vol}(v, w)$, as was to be shown.

From (4.54) and (4.55), we can now prove $\mathrm{Vol}_0(cv, w) = c\,\mathrm{Vol}_0(v, w)$ for any real number $c$, and any vectors $v$ and $w$. Indeed, if $v$ and $w$ are linearly dependent, then both sides are equal to 0. If $v$ and $w$ are linearly independent, we use the definition of $\mathrm{Vol}_0$, and (4.54) and (4.55). Say $D(v, w) > 0$ and $c$ is negative, $c = -d$. Then $D(cv, w) \leq 0$ and, consequently,

$$\begin{aligned}\mathrm{Vol}_0(cv, w) = -\mathrm{Vol}(cv, w) &= -\mathrm{Vol}(-dv, w) \\ &= -\mathrm{Vol}(dv, w) \\ &= -d\,\mathrm{Vol}(v, w) \\ &= c\,\mathrm{Vol}(v, w) = c\,\mathrm{Vol}_0(v, w).\end{aligned}$$

A similar argument works when $D(v, w) \leq 0$. We have therefore proved one of the conditions of linearity of the function $\mathrm{Vol}_0$. The analogous property of course works on the other side, namely $\mathrm{Vol}_0(v, cw) = c\,\mathrm{Vol}_0(v, w)$.

For the other condition, we again have a lemma.

Lemma: Assume that $v, w$ are linearly independent. Then

$$\mathrm{Vol}(v + w, w) = \mathrm{Vol}(v, w). \tag{4.56}$$

> Proof: We have to prove that the parallelogram spanned by $v$ and $w$ has the same area as the parallelogram spanned by $v + w$ and $w$. The parallelogram spanned by $v$ and $w$ consists of two triangles $A$ and $B$, as shown in Figure 29. The parallelogram spanned by $v + w$ and $w$ consists of the triangles $B$ and the translation of $A$ by $w$. Since $A$ and $A + w$ have the same area, we get
>
> $$\mathrm{Vol}(v, w) = \mathrm{Vol}(A) + \mathrm{Vol}(B) = \mathrm{Vol}(A + w) + \mathrm{Vol}(B) = \mathrm{Vol}(v + w, w),$$
>
> as was to be shown.

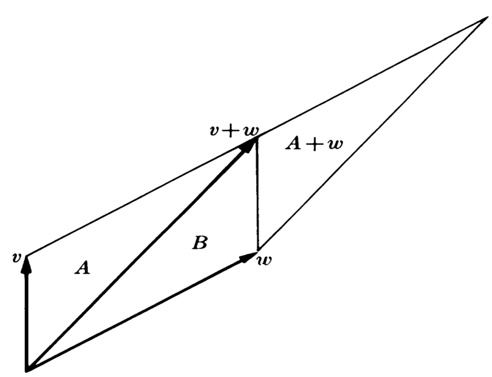



Figure 29: For illustrating (4.56).

We are now in a position to deal with the second property of linearity. Let $w$ be a fixed non-zero vector in the plane, and let $v$ be a vector such that $\{v, w\}$ is a basis of the plane. We shall prove that, for any numbers $c$ and $d$, we have

$$\mathrm{Vol}_0(cv + dw, w) = c\,\mathrm{Vol}_0(v, w). \tag{4.57}$$

Indeed, if $d = 0$, this is nothing but what we have shown previously. If $d \neq 0$, then again by what has been shown previously,

$$d\,\mathrm{Vol}_0(cv + dw, w) = \mathrm{Vol}_0(cv + dw, dw) = c\,\mathrm{Vol}_0(v, dw) = cd\,\mathrm{Vol}_0(v, w).$$

Canceling $d$ yields (4.57). From this last formula, the linearity now follows. Indeed, if

$$v_1 = c_1 v + d_1 w \quad \text{and} \quad v_2 = c_2 v + d_2 w,$$

then

$$\begin{aligned} \mathrm{Vol}_0\,(v_1 + v_2, w) &= \mathrm{Vol}_0\big((c_1 + c_2)\,v + (d_1 + d_2)\,w, w\big) \\ &= (c_1 + c_2)\,\mathrm{Vol}_0(v, w) \\ &= c_1\,\mathrm{Vol}_0(v, w) + c_2\,\mathrm{Vol}_0(v, w) \\ &= \mathrm{Vol}_0\,(v_1, w) + \mathrm{Vol}_0\,(v_2, w)\,. \end{aligned}$$

This concludes the proof of the fact that $\mathrm{Vol}_0(v, w) = D(v, w)$ and, hence, of (4.49).

Remark 1. The proof given above is slightly long, but each step is quite simple. Furthermore, when one wishes to generalize the proof to higher dimensional space (even 3-space), one can give an entirely similar proof. The reason for this is that the conditions characterizing a determinant involve only two coordinates at a time and thus always take place in some 2-dimensional plane. Keeping all but two coordinates fixed, the above proof then can be extended at once. Thus for instance in 3-space, let us denote by $P(u, v, w)$ the box spanned by vectors $u, v, w$, namely all combinations

$$t_1 u + t_2 v + t_3 w \quad \text{with} \quad 0 \leq t_i \leq 1.$$

See Figure 30. Let $\mathrm{Vol}(u, v, w)$ be the volume of this box.

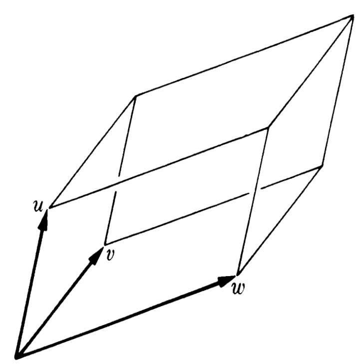



Figure 30: Box spanned by vectors $u, v, w$.

Theorem: The volume of the box spanned by $u, v, w$ is the absolute value of the determinant $|D(u, v, w)|$. That is,

$$\mathrm{Vol}(u, v, w) = |D(u, v, w)|. \tag{4.58}$$

> Proof: The proof follows exactly the same pattern as in the two-dimensional case. Indeed, the volume of the cube spanned by the unit vectors is 1. If two of the vectors $u, v, w$ are equal, then the box is actually a 2-dimensional parallelogram, whose 3-dimensional volume is 0. Finally, the proof of linearity is the same, because all the action took place either in one or in two variables. The other variables can just be carried on in the notation but they did not enter in an essential way in the proof.

Similarly, one can define $n$-dimensional volumes, and the corresponding theorem runs as follows.

Theorem: Let $v_1, \ldots, v_n$ be elements of $\mathbb{R}^n$. Let $\mathrm{Vol}\,(v_1, \ldots, v_n)$ be the $n$-dimensional volume of the $n$-dimensional box spanned by $v_1, \ldots, v_n$. Then

$$\mathrm{Vol}\,(v_1, \ldots, v_n) = |D\,(v_1, \ldots, v_n)|\,. \tag{4.59}$$

Of course, the $n$-dimensional box spanned by $v_1, \ldots, v_n$ is the set of linear combinations

$$\sum_{i=1}^{n} t_i v_i \quad \text{with} \quad 0 \leq t_i \leq 1.$$

Remark 2. We have used geometric properties of area to carry out the above proof. One can lay foundations for all this purely analytically. If the reader is interested, see Lang, Undergraduate Analysis.

Remark 3. In the special case of dimension 2, one could actually have given a simpler proof that the determinant is equal to the area. But we chose to give the slightly more complicated proof because it is the one that generalizes to the 3-dimensional, or $n$-dimensional, case.

We interpret (4.49) in terms of linear maps. Given vectors $v$ and $w$ in the plane, we know that there exists a unique linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L\,(E^1) = v$ and $L\,(E^2) = w$. In fact, if

$$v = aE^1 + cE^2, \quad w = bE^1 + dE^2,$$

then the matrix associated with the linear map is $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Furthermore, if we denote by $C$ the unit square spanned by $E^1$ and $E^2$, and by $P$ the parallelogram spanned by $v$ and $w$, then $P$ is the image under $L$ of $C$, that is $L(C) = P$. Indeed, as we have seen, for $0 \leq t_i \leq 1$ we have

$$L\left(t_1 E^1 + t_2 E^2\right) = t_1 L\left(E^1\right) + t_2 L\left(E^2\right) = t_1 v + t_2 w.$$

If we define the determinant of a linear map to be the determinant of its associated matrix, we conclude that

$$(\text{Area of } P) = |\,\mathrm{Det}(L)|. \tag{4.60}$$

To take a numerical example, the area of the parallelogram spanned by the vectors $(2, 1)$ and $(3, -1)$ is equal to the absolute value of

$$\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -5$$

and hence is equal to 5. See Figure 31.

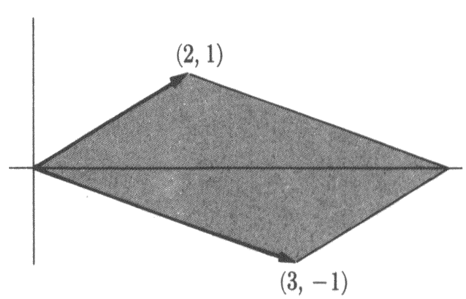



Figure 31: Parallelogram spanned by the vectors $(2, 1)$ and $(3, -1)$.

Theorem: Let $P$ be a parallelogram spanned by two vectors. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map. Then

$$\text{Area of } L(P) = |\text{Det } L| \ (\text{Area of } P) . \tag{4.61}$$

Proof: Suppose that $P$ is spanned by two vectors $v$ and $w$. Then $L(P)$ is spanned by $L(v)$ and $L(w)$. See Figure 32. There is a linear map $L_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L_1 (E^1) = v$ and $L_1 (E^2) = w$. Then $P = L_1(C)$, where $C$ is the unit square, and

$$L(P) = L(L_1(C)) = (L \circ L_1)(C).$$

From (4.60), $\text{Vol } L(P) = |\text{Det } (L \circ L_1)| = |\text{Det}(L) \text{ Det } (L_1)| = |\text{ Det}(L)| \text{ Vol}(P)$.

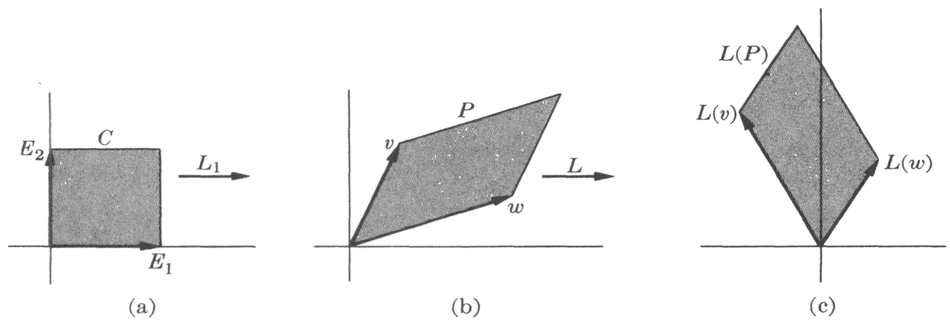



Figure 32: For the proof of (4.61).

Corollary: For any rectangle $R$ with sides parallel to the axes, and any linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$, we have

$$\text{Vol } L(R) = |\text{ Det}(L)| \text{ Vol}(R). \tag{4.62}$$

Proof: Let $c_1$ and $c_2$ be the lengths of the sides of $R$. Let $R_1$ be the rectangle spanned by $c_1E^1$ and $c_2E^2$. Then $R$ is the translation of $R_1$ by some vector, say $R = R_1 + u$. Then

$$L(R) = L(R_1 + u) = L(R_1) + L(u)$$

is the translation of $L(R_1)$ by $L(u)$. See Figure 33. Since area does not change under translation, we need only apply (4.61) to conclude the proof.

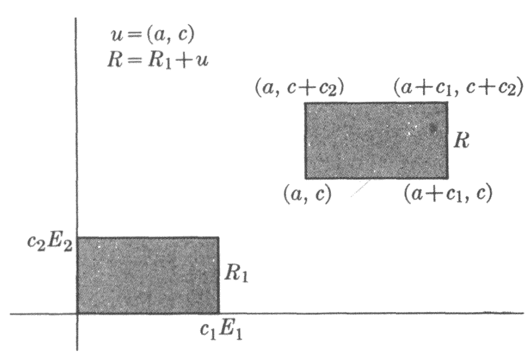



Figure 33: For the proof of (4.62).

### 4.3.3 Permutations

We shall deal only with permutations of the set of integers $\{1, \ldots, n\}$, denoted $J_n$.

Definition: A permutation of $J_n$ is a map $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ of $J_n$ into itself such that, if $i, j \in J_n$ and $i \neq j$, then $\sigma(i) \neq \sigma(j)$.

Thus a permutation is a bijection of $J_n$ with itself. If $\sigma$ is such a permutation, then the set of integers $\{\sigma(1), \ldots, \sigma(n)\}$ has $n$ distinct elements, and hence consists again of the integers $1, \ldots, n$ in a different arrangement. Thus to each integer $j \in J_n$ there exists a unique integer $k$ such that $\sigma(k) = j$.

Definition: The inverse permutation, denoted by $\sigma^{-1}$, is the map $\sigma^{-1} : J_n \to J_n$ such that $\sigma^{-1}(k) =$ unique integer $j \in J_n$ such that $\sigma(j) = k$.

Definition: We denote by id the identity permutation: It is the permutation such that $\mathrm{id}(i) = i$ for all $i = 1, \ldots, n$.

If $\sigma$ and $\tau$ are permutations of $J_n$, then we can form their composite map $\sigma \circ \tau$, and this map will again be a permutation. We shall usually omit the small circle, and write $\sigma\tau$ for the composite map. Thus $(\sigma\tau)(i) = \sigma(\tau(i))$. For any permutation $\sigma$, we have $\sigma\sigma^{-1} = \mathrm{id}$ and $\sigma^{-1}\sigma = \mathrm{id}$.

Proposition: If $\sigma_1, \ldots, \sigma_r$ are permutations of $J_n$, then the inverse of the composite map $\sigma_1 \cdots \sigma_r$ is the permutation $\sigma_r^{-1} \cdots \sigma_1^{-1}$.

> Proof: This is trivially seen by direct multiplication.

Definition: A transposition is a permutation that interchanges two numbers and leaves the others fixed.

The inverse of a transposition $\tau$ is obviously equal to the transposition $\tau$ itself, so that $\tau^2 = \mathrm{id}$.

Proposition: Every permutation of $J_n$ can be expressed as a product of transpositions.

> Proof: We use induction on $n$. For $n = 1$, there is nothing to prove. Let $n > 1$ and assume the assertion proved for $n - 1$. Let $\sigma$ be a permutation of $J_n$. Let $\sigma(n) = k$. If $k \neq n$ let $\tau$ be the transposition of $J_n$ such that $\tau(k) = n$ and $\tau(n) = k$. If $k = n$, let $\tau = \mathrm{id}$. Then $\tau\sigma$ is a permutation such that $\tau\sigma(n) = \tau(k) = n$. In other words, $\tau\sigma$ leaves $n$ fixed. We may therefore view $\tau\sigma$ as a permutation of $J_{n-1}$, and by induction, there exist transpositions $\tau_1, \ldots, \tau_s$ of $J_{n-1}$, leaving $n$ fixed, such that $\tau\sigma = \tau_1 \cdots \tau_s$. We can now write (recalling $\tau^{-1} = \tau$)
>
> $$\sigma = \tau^{-1}\tau_1 \cdots \tau_s = \tau\tau_1 \cdots \tau_s,$$
>
> thereby proving our proposition.

**Example 4.16** *A permutation $\sigma$ of the integers $\{1, \ldots, n\}$ is denoted by*

$$\sigma = \begin{bmatrix} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{bmatrix}.$$

*Thus*

$$\sigma = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$$

*denotes the permutation $\sigma$ such that $\sigma(1) = 2$, $\sigma(2) = 1$, and $\sigma(3) = 3$. This permutation is in fact a transposition. If $\sigma'$ is the permutation*

$$\sigma' = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix},$$

*then $\sigma\sigma' = \sigma \circ \sigma'$ is the permutation such that*

$$\begin{aligned} \sigma\sigma'(1) &= \sigma(\sigma'(1)) = \sigma(3) = 3, \\ \sigma\sigma'(2) &= \sigma(\sigma'(2)) = \sigma(1) = 2, \\ \sigma\sigma'(3) &= \sigma(\sigma'(3)) = \sigma(2) = 1, \end{aligned}$$

*so that we can write*

$$\sigma\sigma' = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}.$$

*Furthermore, the inverse of $\sigma'$ is the permutation*

$$\sigma'^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix},$$

*as is immediately determined from the definitions: Since $\sigma'(1) = 3$, we must have $\sigma'^{-1}(3) = 1$. Since $\sigma'(2) = 1$, we must have $\sigma'^{-1}(1) = 2$. Finally, since $\sigma'(3) = 2$, we must have $\sigma'^{-1}(2) = 3$.* ■

**Example 4.17** *We wish to express the permutation*

$$\sigma = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

*as a product of transpositions. Let $\tau$ be the transposition that interchanges 3 and 1, and leaves 2 fixed. Then*

$$\begin{aligned} \tau\sigma(1) &= \tau(\sigma(1)) = \tau(3) = 1, \\ \tau\sigma(2) &= \tau(\sigma(2)) = \tau(1) = 3, \\ \tau\sigma(3) &= \tau(\sigma(3)) = \tau(2) = 2, \end{aligned}$$

*so*

$$\tau\sigma = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

*so that $\tau\sigma$ is a transposition, which we denote by $\tau'$. We can then write $\tau\sigma = \tau'$, so that*

$$\sigma = \tau^{-1}\tau' = \tau\tau',$$

*because $\tau^{-1} = \tau$. This is the desired product.* ■

**Example 4.18** *Express the permutation*

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

*as a product of transpositions. Let $\tau_1$ be the transposition that interchanges 1 and 2, and leaves 3 and 4 fixed. Then $\tau_1(\sigma(1)) = \tau_1(2) = 1$, $\tau_1(\sigma(2)) = \tau_1(3) = 3$, etc.*

$$\tau_1\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{bmatrix}.$$

*Now let $\tau_2$ be the transposition which interchanges 2 and 3, and leaves 1 and 4 fixed. Then*

$$\tau_2\tau_1\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{bmatrix},$$

*and we see that $\tau_2\tau_1\sigma$ is a transposition, which we may denote by $\tau_3$. Then we get $\tau_2\tau_1\sigma = \tau_3$ so that (again recalling $\tau^{-1} = \tau$) $\sigma = \tau_1\tau_2\tau_3$.* ■

The previous example helps to indicate a general algorithm. The following example, using $J_{11}$, spells this out in yet more clarity. It was inspired from https://math.stackexchange.com/questions/319979.

**Example 4.19** *We wish to decompose permutation*

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 4 & 2 & 9 & 10 & 6 & 5 & 11 & 7 & 8 & 1 & 3 \end{bmatrix}$$

*into a sequence of transpositions $\tau_1\tau_2\cdots\tau_s$. Denote the transposition that switches $j,k \in J_n$ as $\begin{pmatrix} j & k \end{pmatrix}$. Then, with $\tau_1 = \begin{pmatrix} 1 & \sigma(1) \end{pmatrix} = \begin{pmatrix} 1 & 4 \end{pmatrix}$, we have (note the **bold faced**)*

$$\tau_1\sigma = \tau_1 \circ \sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \boldsymbol{1} & 2 & 9 & 10 & 6 & 5 & 11 & 7 & 8 & \boldsymbol{4} & 3 \end{bmatrix}.$$

*As $\tau_1\sigma(2) = 2$, we can skip it. Next let $\tau_2 = \begin{pmatrix} 3 & (\tau_1\sigma)(3) \end{pmatrix} = \begin{pmatrix} 3 & 9 \end{pmatrix}$, so that*

$$\tau_2\tau_1\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 2 & \boldsymbol{3} & 10 & 6 & 5 & 11 & 7 & 8 & 4 & \boldsymbol{9} \end{bmatrix}.$$

*Next let $\tau_3 = \begin{pmatrix} 4 & (\tau_2\tau_1\sigma)(4) \end{pmatrix} = \begin{pmatrix} 4 & 10 \end{pmatrix}$, so that*

$$\tau_3\tau_2\tau_1\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 2 & 3 & \boldsymbol{4} & 6 & 5 & 11 & 7 & 8 & \boldsymbol{10} & 9 \end{bmatrix}.$$

*Continuing, we arrive at $\sigma = \tau_1\tau_2\cdots\tau_7 = (1\ \ 4)(3\ \ 9)(4\ \ 10)(5\ \ 6)(7\ \ 11)(8\ \ 11)(9\ \ 11)$.* ■

Proposition: To each permutation $\sigma$ of $J_n$ it is possible to assign a sign 1 or $-1$, denoted by $\epsilon(\sigma)$, satisfying the following conditions:

(a) If $\tau$ is a transposition, then $\epsilon(\tau) = -1$.

(b) If $\sigma, \sigma'$ are permutations of $J_n$, then $\epsilon(\sigma\sigma') = \epsilon(\sigma)\epsilon(\sigma')$.

In fact, if $A = (A^1, \ldots, A^n)$ is an $n \times n$ matrix, then $\epsilon(\sigma)$ can be defined by the condition

$$D\left(A^{\sigma(1)}, \ldots, A^{\sigma(n)}\right) = \epsilon(\sigma) D\left(A^1, \ldots, A^n\right). \tag{4.63}$$

> Proof: Observe that $\left(A^{\sigma(1)}, \ldots, A^{\sigma(n)}\right)$ is simply a different ordering from $(A^1, \ldots, A^n)$. Let $\sigma$ be a permutation of $J_n$. Then from (4.38), $D\left(A^{\sigma(1)}, \ldots, A^{\sigma(n)}\right) = \pm D\left(A^1, \ldots, A^n\right)$, and the sign $+$ or $-$ is determined by $\sigma$, and does not depend on $A^1, \ldots, A^n$. Indeed, by making a succession of transpositions, we can return $\left(A^{\sigma(1)}, \ldots, A^{\sigma(n)}\right)$ to the standard ordering $(A^1, \ldots, A^n)$, and each transposition changes the determinant by a sign. Thus we may *define*
>
> $$\epsilon(\sigma) = \frac{D\left(A^{\sigma(1)}, \ldots, A^{\sigma(n)}\right)}{D\left(A^1, \ldots, A^n\right)}$$

for any choice of $A^1, \ldots, A^n$ whose determinant is not 0, say the unit vectors $E^1, \ldots, E^n$. There are of course many ways of applying a succession of transpositions to return $\left(A^{\sigma(1)}, \ldots, A^{\sigma(n)}\right)$ to the standard ordering, but since the determinant is a well defined function, it follows that the sign $\epsilon(\sigma)$ is also well defined, and is the same, no matter which way we select. Thus we have $D\left(A^{\sigma(1)}, \ldots, A^{\sigma(n)}\right) = \epsilon(\sigma) D\left(A^1, \ldots, A^n\right)$, and of course this holds if $D\left(A^1, \ldots, A^n\right) = 0$ because both sides are equal to 0.

If $\tau$ is a transposition, then assertion (a) follows from (4.38).

Finally, let $\sigma$ and $\sigma'$ be permutations of $J_n$. Let $C^j = A^{\sigma'(j)}$ for $j = 1, \ldots, n$. Then, on the one hand, we have

$$D\left(A^{\sigma'\sigma(1)}, \ldots, A^{\sigma'\sigma(n)}\right) = \epsilon\left(\sigma'\sigma\right) D\left(A^1, \ldots, A^n\right); \tag{4.64}$$

and, on the other hand, we have Note $C^{\sigma(1)} = A^{\sigma'(\sigma(1))}$, etc.

$$\begin{aligned} D\left(A^{\sigma'\sigma(1)}, \ldots, A^{\sigma'\sigma(n)}\right) &= D\left(C^{\sigma(1)}, \ldots, C^{\sigma(n)}\right) \\ &= \epsilon(\sigma) D\left(C^1, \ldots, C^n\right) \\ &= \epsilon(\sigma) D\left(A^{\sigma'(1)}, \ldots, A^{\sigma'(n)}\right) \\ &= \epsilon(\sigma)\epsilon\left(\sigma'\right) D\left(A^1, \ldots, A^n\right). \end{aligned} \tag{4.65}$$

Let $A^1, \ldots, A^n$ be the unit vectors $E^1, \ldots, E^n$. From the equality between (4.64) and (4.65), we conclude that $\epsilon\left(\sigma'\sigma\right) = \epsilon\left(\sigma'\right)\epsilon(\sigma)$, thus proving our proposition.

Corollary: If a permutation $\sigma$ of $J_n$ is expressed as a product of transpositions, $\sigma = \tau_1 \cdots \tau_s$, where each $\tau_i$ is a transposition, then $s$ is even or odd according as $\epsilon(\sigma) = 1$ or $-1$.

Proof: We have $\epsilon(\sigma) = \epsilon\left(\tau_1\right) \cdots \epsilon\left(\tau_s\right) = (-1)^s$, from which the assertion follows.

Corollary: If $\sigma$ is a permutation of $J_n$, then

$$\epsilon(\sigma) = \epsilon\left(\sigma^{-1}\right). \tag{4.66}$$

Proof: We have $1 = \epsilon(\mathrm{id}) = \epsilon\left(\sigma\sigma^{-1}\right) = \epsilon(\sigma)\epsilon\left(\sigma^{-1}\right)$. Hence either $\epsilon(\sigma)$ and $\epsilon\left(\sigma^{-1}\right)$ are both equal to 1, or both equal to $-1$, as desired.

Definition: A permutation is called even, if its sign is 1, and odd, if its sign is $-1$. Thus, every transposition is odd.

**Example 4.20** *The sign of the permutation $\sigma$ in Example 4.17 is equal to 1 because $\sigma = \tau\tau'$. The sign of the permutation $\sigma$ in Example 4.18 is equal to $-1$ because $\sigma = \tau_1\tau_2\tau_3$.* ■

### 4.3.4 Expansion Formula and Uniqueness of Determinants

Let $X^1, X^2, X^3 \in \mathbb{R}^3$, and let $(b_{ij})$, $i,j = 1,2,3$, be a $3 \times 3$ matrix. Let

$$A^1 = b_{11}X^1 + b_{21}X^2 + b_{31}X^3 = \sum_{k=1}^{3} b_{k1}X^k,$$

$$A^2 = b_{12}X^1 + b_{22}X^2 + b_{32}X^3 = \sum_{l=1}^{3} b_{l2}X^l,$$

$$A^3 = b_{13}X^1 + b_{23}X^2 + b_{33}X^3 = \sum_{m=1}^{3} b_{m3}X^m.$$

Then we can expand using linearity,

$$\begin{aligned} D\left(A^1, A^2, A^3\right) &= D\left(\sum_{k=1}^{3} b_{k1}X^k, \sum_{l=1}^{3} b_{l2}X^l, \sum_{m=1}^{3} b_{m3}X^m\right) \\ &= \sum_{k=1}^{3} b_{k1} D\left(X^k, \sum_{l=1}^{3} b_{l2}X^l, \sum_{m=1}^{3} b_{m3}X^m\right) \\ &= \sum_{k=1}^{3}\sum_{l=1}^{3} b_{k1}b_{l2} D\left(X^k, X^l, \sum_{m=1}^{3} b_{m3}X^m\right) \\ &= \sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} b_{k1}b_{l2}b_{m3} D\left(X^k, X^l, X^m\right). \end{aligned}$$

If we wish to get a similar expansion for the $n \times n$ case, we must obviously adjust the notation, otherwise we run out of letters $k$, $l$, and $m$. Thus instead of using $k, l, m$, we observe that these values $k, l, m$ correspond to an arbitrary choice of an integer 1, 2, or 3, for each one of the numbers $1, 2, 3$ occurring as the second index in $b_{ij}$. Thus, if we let $\sigma$ denote such a choice, we can write $k = \sigma(1)$, $l = \sigma(2)$, $m = \sigma(3)$; and $b_{k1}b_{l2}b_{m3} = b_{\sigma(1),1}b_{\sigma(2),2}b_{\sigma(3),3}$.

Thus $\sigma : \{1,2,3\} \to \{1,2,3\}$ is nothing but an association, i.e., a function, from $J_3$ to $J_3$, and we can write

$$D\left(A^1, A^2, A^3\right) = \sum_{\sigma} b_{\sigma(1),1}b_{\sigma(2),2}b_{\sigma(3),3} D\left(X^{\sigma(1)}, X^{\sigma(2)}, X^{\sigma(3)}\right),$$

the sum being taken for all such possible $\sigma$. We shall find an expression for the determinant that corresponds to the six-term expansion (4.33) for the $3 \times 3$ case, and the general $n$ case.

In the following, observe that the properties used in the proof are only properties (4.29)-(4.32), and their consequences, (4.38), (4.39), and (4.41), so that our proof applies to any function $D$ satisfying these properties.

We first give the argument in the $2 \times 2$ case. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $A^1 = \begin{pmatrix} a \\ c \end{pmatrix}$ and $A^2 = \begin{pmatrix} b \\ d \end{pmatrix}$ its column vectors. We can write

$$A^1 = aE^1 + cE^2 \quad \text{and} \quad A^2 = bE^1 + dE^2,$$

where $E^1$ and $E^2$ are the unit column vectors. Then

$$\begin{aligned}
D(A) &= D\left(A^1, A^2\right) = D\left(aE^1 + cE^2, bE^1 + dE^2\right) \\
&= abD\left(E^1, E^1\right) + cbD\left(E^2, E^1\right) + adD\left(E^1, E^2\right) + cdD\left(E^2, E^2\right) \\
&= -bcD\left(E^1, E^2\right) + adD\left(E^1, E^2\right) = ad - bc.
\end{aligned}$$

This proves that any function $D$ satisfying the basic properties of a determinant is given by (4.26), i.e., $ad - bc$. The proof in general is entirely similar, taking into account the $n$ components. It is based on an expansion similar to the one we have just used in the $2 \times 2$ case.

Lemma: Let $X^1, \ldots, X^n$ be $n$ vectors in $n$-space. Let $B = (b_{ij})$ be an $n \times n$ matrix, and let

$$\begin{aligned}
A^1 &= b_{11}X^1 + \cdots + b_{n1}X^n \\
&\vdots \\
A^n &= b_{1n}X^1 + \cdots + b_{nn}X^n.
\end{aligned}$$

Then

$$D\left(A^1, \ldots, A^n\right) = \sum_\sigma \epsilon(\sigma) b_{\sigma(1),1} \cdots b_{\sigma(n),n} D\left(X^1, \ldots, X^n\right), \tag{4.67}$$

where the sum is taken over all permutations $\sigma$ of $\{1, \ldots, n\}$.

> Proof: We must compute
>
> $$D\left(b_{11}X^1 + \cdots + b_{n1}X^n, \ldots, b_{1n}X^1 + \cdots + b_{nn}X^n\right).$$
>
> Using the linearity property with respect to each column, we can express this as a sum
>
> $$\sum_\sigma b_{\sigma(1),1} \cdots b_{\sigma(n),n} D\left(X^{\sigma(1)}, \ldots, X^{\sigma(n)}\right), \tag{4.68}$$
>
> where $\sigma(1), \ldots, \sigma(n)$ denote a choice of an integer between 1 and $n$ for each value of $1, \ldots, n$. Thus each $\sigma$ is a mapping of the set of integers $\{1, \ldots, n\}$ into itself, and the sum is taken over all such maps. If some $\sigma$ assigns the same integer to distinct values $i$ and $j$ between 1 and $n$, then the determinant on the right has two equal columns, and hence is equal to 0. Consequently, we can take our sum only for those $\sigma$ such that $\sigma(i) \neq \sigma(j)$ whenever $i \neq j$, namely permutations. From (4.63) we have
>
> $$D\left(X^{\sigma(1)}, \ldots, \mathrm{X}^{\sigma(n)}\right) = \epsilon(\sigma) D\left(X^1, \ldots, X^n\right).$$
>
> Substituting this into the expression (4.68) of $D\left(A^1, \ldots, A^n\right)$ yields (4.67).

Theorem: Determinants are uniquely determined by properties (4.29)-(4.32). Let $A = (a_{ij})$. The determinant satisfies the expression

$$D\left(A^1, \ldots, A^n\right) = \sum_\sigma \epsilon(\sigma) a_{\sigma(1),1} \cdots a_{\sigma(n),n}, \tag{4.69}$$

where the sum is taken over all permutations of the integers $\{1, \ldots, n\}$.

Proof: We let $X^j = E^j$ be the unit vector having 1 in the $j$th component, and we let $b_{ij} = a_{ij}$ in (4.67). Since by hypothesis we have $D\left(E^1, \ldots, E^n\right) = 1$, we see that (4.69) drops out at once.

We obtain further applications of (4.67).

Theorem: Let $A, B$ be two $n \times n$ matrices. Then

$$\operatorname{Det}(AB) = \operatorname{Det}(A)\operatorname{Det}(B), \tag{4.70}$$

i.e., the determinant of a product is equal to the product of the determinants.

Proof: Let $A = (a_{ij})$ and $B = (b_{jk})$, with product

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1k} & \cdots & b_{1n} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nk} & \cdots & b_{nn} \end{pmatrix}.$$

Let $AB = C$, and let $C^k$ be the $k$th column of $C$, $C^k = b_{1k}A^1 + \cdots + b_{nk}A^n$. Thus

$$\begin{aligned} D(AB) &= D\left(C^1, \ldots, C^n\right) \\ &= D\left(b_{11}A^1 + \cdots + b_{n1}A^n, \ldots, b_{1n}A^1 + \cdots + b_{nn}A^n\right) \\ &= \sum_\sigma b_{\sigma(1),1} \cdots b_{\sigma(n),n} D\left(A^{\sigma(1)}, \ldots, A^{\sigma(n)}\right) \\ &= \sum_\sigma \epsilon(\sigma) b_{\sigma(1),1} \cdots b_{\sigma(n),n} D\left(A^1, \ldots, A^n\right) \qquad \text{from (4.67)} \\ &= D(B)D(A) \qquad \text{from (4.69).} \end{aligned}$$

Corollary: Let $A$ be an invertible $n \times n$ matrix. Then

$$\operatorname{Det}\left(A^{-1}\right) = \operatorname{Det}(A)^{-1}. \tag{4.71}$$

Proof: We have $1 = D(I) = D\left(AA^{-1}\right) = D(A)D\left(A^{-1}\right)$.

Theorem: Let $A$ be a square matrix. Then

$$\operatorname{Det}(A) = \operatorname{Det}\left(A'\right). \tag{4.72}$$

Proof: From (4.69),

$$\operatorname{Det}(A) = \sum_\sigma \epsilon(\sigma) a_{\sigma(1),1} \cdots a_{\sigma(n),n}. \tag{4.73}$$

Let $\sigma$ be a permutation of $\{1, \ldots, n\}$. If $\sigma(j) = k$, then $\sigma^{-1}(k) = j$. We can therefore write $a_{\sigma(j),j} = a_{k,\sigma^{-1}(k)}$. In a product $a_{\sigma(1),1} \cdots a_{\sigma(n),n}$, each integer $k$ from 1 to $n$ occurs precisely once among the integers $\sigma(1), \ldots, \sigma(n)$. Hence this product can be written

$$a_{1,\sigma^{-1}(1)} \cdots a_{n,\sigma^{-1}(n)},$$

and sum (4.73) is equal to

$$\sum_{\sigma} \epsilon\left(\sigma^{-1}\right) a_{1,\sigma^{-1}(1)} \cdots a_{n,\sigma^{-1}(n)},$$

because $\epsilon(\sigma) = \epsilon\left(\sigma^{-1}\right)$. In this sum, each term corresponds to a permutation $\sigma$. However, as $\sigma$ ranges over all permutations, so does $\sigma^{-1}$ because a permutation determines its inverse uniquely. Hence our sum is equal to

$$\sum_{\sigma} \epsilon(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}. \tag{4.74}$$

The sum (4.74) is precisely the sum giving the expanded form of the determinant of the transpose of $A$. Hence we have proved what we wanted.

## 4.4 The Cross Product

We base our presentation on (an augmented version of that in) Shifrin, Multivariable Mathematics, 2005; and (mostly verbatim) Lang, Calculus of Several Variables, 3rd ed., 1987. We begin with the former.

Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^2$ and consider the parallelogram $\mathcal{P}$ they span. The area of $\mathcal{P}$ is nonzero as long as $\mathbf{x}$ and $\mathbf{y}$ are not collinear. We want to express the area of $\mathcal{P}$ in terms of the coordinates of $\mathbf{x}$ and $\mathbf{y}$. First notice that the area of the parallelogram pictured in the left panel of Figure 34 is the same as the area of the rectangle obtained by moving the shaded triangle from the right side to the left. This rectangle has area $bh$, where $b = \|\mathbf{x}\|$ is the base and $h = \|\mathbf{y}\| \sin\theta$ is the height.

We could calculate $\sin\theta$ from (4.19), namely $\cos\theta = (\mathbf{x}\cdot\mathbf{y})/\|\mathbf{x}\|\|\mathbf{y}\|$, but instead we note from the middle panel of Figure 34, and recalling (4.5), or (4.14), or (4.19),

$$\operatorname{area}(\mathcal{P}) = bh = \|\mathbf{x}\|\|\mathbf{y}\|\sin\theta = \|\mathbf{x}\|\|\mathbf{y}\|\cos\left(\frac{\pi}{2}-\theta\right) = \rho(\mathbf{x})\cdot\mathbf{y}, \tag{4.75}$$

where $\rho(\mathbf{x})$ is the vector obtained by rotating $\mathbf{x}$ an angle $\pi/2$ counterclockwise.

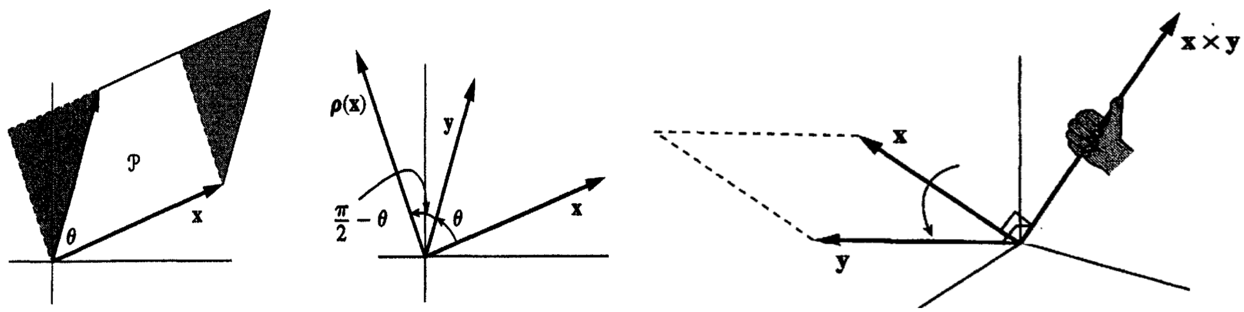



Figure 34: From Shifrin, p. 44 and p. 48. In the middle panel, imagine a perpendicular line, dropped from the end of vector $\mathbf{y}$ onto $\rho(\mathbf{x})$, which is the projection of $\mathbf{y}$ onto $\rho(\mathbf{x})$. The length of this projection is, from the usual determination of cosine from an angle drawn in a cirlce, $\|\mathbf{y}\|\cos\left(\frac{\pi}{2}-\theta\right)$. In the third panel: If you curl the fingers of your right hand from $\mathbf{x}$ toward $\mathbf{y}$, your thumb points in the direction of $\mathbf{x}\times\mathbf{y}$.

Note the known identities (e.g., Wikipedia) also from (2.81) and (2.83):

$$\cos\theta = \sin\left(\frac{\pi}{2}-\theta\right),\quad \sin\theta = \cos\left(\frac{\pi}{2}-\theta\right),\quad \sin\left(\theta\pm\frac{\pi}{2}\right) = \pm\cos\theta,\quad \cos\left(\theta\pm\frac{\pi}{2}\right) = \mp\sin\theta. \tag{4.76}$$

If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, then we have $\rho(\mathbf{x})\cdot\mathbf{y} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}\cdot\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, i.e.,

$$\|\mathbf{x}\|\|\mathbf{y}\|\sin\theta = \operatorname{area}(\mathcal{P}) = \rho(\mathbf{x})\cdot\mathbf{y} = x_1y_2 - x_2y_1 = \operatorname{Det}(\mathbf{x},\mathbf{y}). \tag{4.77}$$

If $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, then the area of the parallelogram spanned by $\mathbf{x}$ and $\mathbf{y}$ is $x_1y_2 - x_2y_1 = 3\cdot3 - 1\cdot4 = 5$. On the other hand, if we interchange the two, letting $\mathbf{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, then we get $x_1y_2 - x_2y_1 = 4\cdot1 - 3\cdot3 = -5$. Certainly the parallelogram hasn't changed; nor does it make sense to have negative area. What is the explanation?

In deriving our formula for the area above, we assumed $0 < \theta < \pi$; but if we must turn clockwise to get from $\mathbf{x}$ to $\mathbf{y}$, this means that $\theta$ is negative, resulting in a sign discrepancy in the area calculation. So we should amend our earlier result. We define the signed area of the parallelogram $\mathcal{P}$ to be the area of $\mathcal{P}$ when one turns counterclockwise from $\mathbf{x}$ to $\mathbf{y}$ and to be negative the area of $\mathcal{P}$ when one turns clockwise from $\mathbf{x}$ to $\mathbf{y}$. Then we have

$$\text{signed area } (\mathcal{P}) = x_1y_2 - x_2y_1.$$

Definition: Given two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, define a vector, called their *cross product*, by

$$(x_2y_3 - x_3y_2,\ \ x_3y_1 - x_1y_3,\ \ x_1y_2 - x_2y_1) \tag{4.78}$$

or, with $\mathbf{e}_i$ the $i$th unit vector in $\mathbb{R}^3$ and a "formal"[26] use of the determinant,

$$\mathbf{x} \times \mathbf{y} = (x_2y_3 - x_3y_2)\,\mathbf{e}_1 + (x_3y_1 - x_1y_3)\,\mathbf{e}_2 + (x_1y_2 - x_2y_1)\,\mathbf{e}_3 \tag{4.79}$$

$$= \begin{vmatrix} \mathbf{e}_1 & x_1 & y_1 \\ \mathbf{e}_2 & x_2 & y_2 \\ \mathbf{e}_3 & x_3 & y_3 \end{vmatrix}. \tag{4.80}$$

Let $\mathbf{z} = (z_1, z_2, z_3)' = \sum z_i\mathbf{e}_i$, and note from (4.79) that

$$\mathbf{z} \cdot (\mathbf{x} \times \mathbf{y}) = \mathrm{Det}(\mathbf{z}, \mathbf{x}, \mathbf{y}). \tag{4.81}$$

The geometric interpretation of the cross product is indicated in right panel of Figure 34, and given as follows. The cross product $\mathbf{x} \times \mathbf{y}$ of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ is orthogonal to both $\mathbf{x}$ and $\mathbf{y}$, and

$$\mathrm{area}(\mathcal{P}) = \|\mathbf{x} \times \mathbf{y}\|. \tag{4.82}$$

Moreover, when $\mathbf{x}$ and $\mathbf{y}$ are nonparallel, the vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{x} \times \mathbf{y}$ determine a parallelepiped of positive signed volume. To see this, note that the orthogonality is a consequence of (4.81) and properties of the determinant. In particular, $\mathbf{x} \cdot (\mathbf{x} \times \mathbf{y}) = \mathrm{Det}(\mathbf{x}, \mathbf{x}, \mathbf{y}) = \mathbf{0}$. From the interpretation of determinant as the volume of a parallelepiped, $\mathrm{Det}(\mathbf{x}, \mathbf{y}, \mathbf{x} \times \mathbf{y})$ is the signed volume of the parallelepiped spanned by $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x} \times \mathbf{y}$. As $\mathbf{x} \times \mathbf{y}$ is orthogonal to the plane spanned by $\mathbf{x}$ and $\mathbf{y}$, that volume is the product of the area of $\mathcal{P}$ and $\|\mathbf{x} \times \mathbf{y}\|$. On the other hand, again from (4.81), simply substituting $\mathbf{x} \times \mathbf{y}$ in place of $\mathbf{z}$,

$$\mathrm{Det}(\mathbf{x}, \mathbf{y}, \mathbf{x} \times \mathbf{y}) = \mathrm{Det}(\mathbf{x} \times \mathbf{y}, \mathbf{x}, \mathbf{y}) = (\mathbf{x} \times \mathbf{y}) \cdot (\mathbf{x} \times \mathbf{y}) = \|\mathbf{x} \times \mathbf{y}\|^2. \tag{4.83}$$

Setting the two expressions equal yields $\|\mathbf{x} \times \mathbf{y}\| = \mathrm{area}(\mathcal{P})$. When $\mathbf{x}$ and $\mathbf{y}$ are nonparallel, $\mathrm{area}(\mathcal{P}) > 0$, or $\|\mathbf{x} \times \mathbf{y}\|^2 > 0$, so the three vectors span a parallelepiped of positive signed volume, as desired.

**Example 4.21** *We can use the cross product to find the equation of the subspace $\mathcal{P}$ spanned by the vectors $u = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. For the normal vector to $P$ is*

$$\mathbf{A} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & 1 & 1 \\ \mathbf{e}_2 & 1 & 2 \\ \mathbf{e}_3 & -1 & 1 \end{vmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

[26] According to Wikipedia, a formal calculation, or formal operation, is a calculation that is systematic but without a rigorous justification. It involves manipulating symbols in an expression using a generic substitution without proving that the necessary conditions hold.

*and so*

$$\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{A} \cdot \mathbf{x} = 0 \right\} = \left\{ \mathbf{x} \in \mathbb{R}^3 : 3x_1 - 2x_2 + x_3 = 0 \right\}.$$

*Moreover, the affine plane* $\mathcal{P}_1$ *parallel to* $\mathcal{P}$ *and passing through the point* $x_0 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ *is given by*

$$\begin{aligned} \mathcal{P}_1 &= \left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{A} \cdot (\mathbf{x} - \mathbf{x}_0) = 0 \right\} = \left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{A} \cdot \mathbf{x} = \mathbf{A} \cdot \mathbf{x}_0 \right\} \\ &= \left\{ \mathbf{x} \in \mathbb{R}^3 : 3x_1 - 2x_2 + x_3 = 7 \right\}. \qquad \blacksquare \end{aligned}$$

We now turn to the presentation in Lang.

Definition: Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ be two vectors in 3-space. We define their *cross product* to be the vector

$$A \times B = (a_2 b_3 - a_3 b_2,\ \ a_3 b_1 - a_1 b_3,\ \ a_1 b_2 - a_2 b_1), \tag{4.84}$$

which is symbolically (or "formally") the determinant

$$A \times B = \begin{vmatrix} E_1 & E_2 & E_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \tag{4.85}$$

(Note (4.79) is the same as (4.84); and (4.80) agrees with (4.85).) Indeed, the rhs of (4.85) is, upon expansion, $E_1 (a_2 b_3 - a_3 b_2) - E_2 (a_1 b_3 - a_3 b_1) + E_3 (a_1 b_2 - a_2 b_1)$, which agrees with the definition of $A \times B$ in (4.84).

Theorem: The following results on the cross product can be verified directly from (4.84):

CP 1. $A \times B = -(B \times A)$.
CP 2. $A \times (B + C) = (A \times B) + (A \times C)$, and $(B + C) \times A = B \times A + C \times A$.
CP 3. For any number $a$, we have

$$(aA) \times B = a(A \times B) = A \times (aB).$$

CP 4. $(A \times B) \times C = (A \cdot C)B - (B \cdot C)A$.
CP 5. $A \times B$ is perpendicular to both $A$ and $B$.
CP 6. $(A \times B)^2 = (A \times B) \cdot (A \times B) = (A \cdot A)(B \cdot B) - (A \cdot B)^2$.

The first three also follow immediately from results on determinants. For CP5,

$$A \cdot (A \times B) = a_1 (a_2 b_3 - a_3 b_2) + a_2 (a_3 b_1 - a_1 b_3) + a_3 (a_1 b_2 - a_2 b_1) = 0,$$

because all terms cancel. Similarly for $B \cdot (A \times B)$. The vector $A \times B$ is perpendicular to the plane spanned by $A$ and $B$. So is $B \times A$, but $B \times A$ points in the opposite direction.

For CP6, the first equality is just Lang's notation mentioned just before (4.16). For the second equality, simple algebra based on definitions yields

$$\begin{aligned} (A \times B) \cdot (A \times B) &= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2, \\ (A \cdot A)(B \cdot B) - (A \cdot B)^2 &= \left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2. \end{aligned}$$

Expanding everything out and comparing confirms relation CP6.

From (4.5), or (4.14), or (4.19), CP6 is $\|A \times B\|^2 = \|A\|^2\|B\|^2 - \|A\|^2\|B\|^2 \cos^2\theta$, where $\theta$ is the angle between $A$ and $B$. Hence, we obtain $\|A \times B\|^2 = \|A\|^2\|B\|^2 \sin^2\theta$, or

$$\|A \times B\| = \|A\|\|B\||\sin\theta|, \tag{4.86}$$

(which, of course, agrees with the juxtaposition of (4.75) and (4.82) in our first presentation above).

Lang concludes by showing what we saw above, namely (4.82): Identity (4.86) can be used to make another interpretation of the cross product. Indeed, we see that $\|A \times B\|$ is the area of the parallelogram spanned by $A$ and $B$. If we consider the plane containing the located vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$, then the picture looks like that in Figure 35, and our assertion amounts simply to the statement that the area of a parallelogram is equal to the base times the altitude.

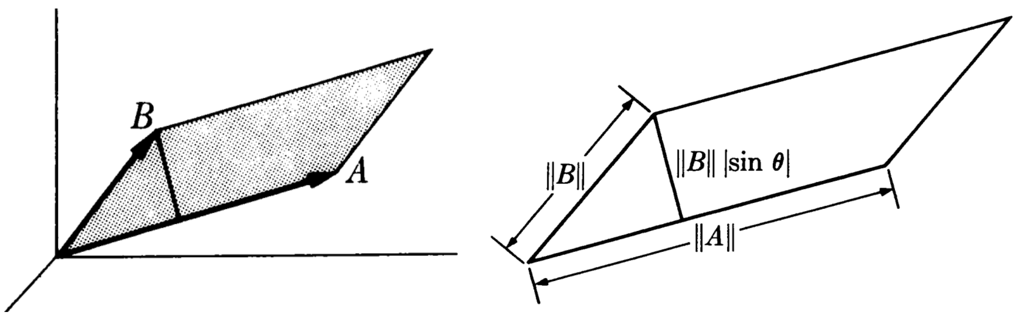



Figure 35: From Lang, p. 46 and p. 47

## 4.5 More Advanced Linear Algebra: Projection and Least Squares

This subsection is taken from the advanced undergraduate linear algebra book of Olver and Shakiban (Applied Linear Algebra, 2nd ed., 2018), and will presuppose an acquaintance with the numerous *basic* concepts from an introductory course in linear algebra, e.g., definitions of a vector space, subspaces, an independent set of vectors, the span of a set of vectors, a basis of a subspace, and the kernel and image of a matrix. We review the kernel and image below. We do not require previous knowledge of concepts such as change of basis, LDU, QR, and spectral (i.e., eigenvalue, eigenvector) factorizations or decompositions, Gram-Schmidt, or complex vector spaces. The Gram-Schmidt method and the resulting QR decomposition will be developed herein.

The image of an $m \times n$ matrix $A$ is the subspace $\operatorname{img} A \subset \mathbb{R}^m$ spanned by its columns. The kernel of $A$ is the subspace $\ker A \subset \mathbb{R}^n$ consisting of all vectors that are annihilated by $A$, so

$$\ker A = \{\mathbf{z} \in \mathbb{R}^n \mid A\mathbf{z} = \mathbf{0}\} \subset \mathbb{R}^n.$$

The image is also known as the column space or the range of the matrix. By definition, a vector $\mathbf{b} \in \mathbb{R}^m$ belongs to img $A$ if can be written as a linear combination,

$$\mathbf{b} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$$

of the columns of $A = (\mathbf{v}_1\mathbf{v}_2 \ldots \mathbf{v}_n)$. The right-hand side of this equation equals the product $A\mathbf{x}$ of the matrix $A$ with the column vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, and, hence, $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. Thus,

$$\operatorname{img} A = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\} \subset \mathbb{R}^m,$$

and so a vector $\mathbf{b}$ lies in the image of $A$ if and only if the linear system $A\mathbf{x} = \mathbf{b}$ has a solution.

A common alternative name for the kernel is the null space. The kernel or null space of $A$ is the set of solutions $\mathbf{z}$ to the homogeneous system $A\mathbf{z} = \mathbf{0}$. The proof that $\ker A$ is a subspace requires us to verify the usual closure conditions: Suppose that $\mathbf{z}, \mathbf{w} \in \ker A$, so that $A\mathbf{z} = \mathbf{0} = A\mathbf{w}$. Then, by the compatibility of scalar and matrix multiplication, for any scalars $c, d$,

$$A(c\mathbf{z} + d\mathbf{w}) = cA\mathbf{z} + dA\mathbf{w} = \mathbf{0},$$

which implies that $c\mathbf{z} + d\mathbf{w} \in \ker A$.

### 4.5.1 Inner Product Spaces and Gram Matrices

A vector space equipped with an inner product and its associated norm, e.g., (4.1) and (4.2), is known as an *inner product space*. Other inner products are possible, and used. The general definition is:

Definition: An inner product on the real vector space $V$ is a pairing that takes two vectors $\mathbf{v}, \mathbf{w} \in V$ and produces a real number $\langle \mathbf{v}, \mathbf{w} \rangle \in \mathbb{R}$. The inner product is required to satisfy the following three axioms for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and scalars $c, d \in \mathbb{R}$, termed, respectively, bilinearity, symmetry, and positivity:

(i) $\langle c\mathbf{u} + d\mathbf{v}, \mathbf{w} \rangle = c\langle \mathbf{u}, \mathbf{w} \rangle + d\langle \mathbf{v}, \mathbf{w} \rangle, \quad \langle \mathbf{u}, c\mathbf{v} + d\mathbf{w} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle + d\langle \mathbf{u}, \mathbf{w} \rangle.$
(ii) $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle.$
(iii) $\langle \mathbf{v}, \mathbf{v} \rangle > 0 \quad \text{whenever} \quad \mathbf{v} \neq \mathbf{0}$, while $\langle \mathbf{0}, \mathbf{0} \rangle = 0$.

Every inner product gives rise to a norm that can be used to measure the magnitude or length of the elements of the underlying vector space. However, not every norm that is used in analysis and applications arises from an inner product. To define a general norm on a vector space, we will extract those properties that do not directly rely on the inner product structure.

Definition: A norm on a vector space $V$ assigns a non-negative real number $\|\mathbf{v}\|$ to each vector $\mathbf{v} \in V$, subject to the following axioms, valid for every $\mathbf{v}, \mathbf{w} \in V$ and $c \in \mathbb{R}$:

(i) Positivity: $\|\mathbf{v}\| \geq 0$, with $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
(ii) Homogeneity: $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$.
(iii) Triangle inequality: $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.

Two important examples of norms that do not come from inner products (and that we will not use subsequently) are as follows. First, let $V = \mathbb{R}^n$. The 1-norm of a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$ is defined as the sum of the absolute values of its entries:

$$\|\mathbf{v}\|_1 = |v_1| + |v_2| + \cdots + |v_n|.$$

The max- or $\infty$-norm is equal to its maximal entry (in absolute value):

$$\|\mathbf{v}\|_\infty = \max \{|v_1|, |v_2|, \ldots, |v_n|\}.$$

Every norm defines a distance between vector space elements, namely

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|. \tag{4.87}$$

For the standard dot product norm, we recover the usual notion of distance between points in Euclidean space. Other types of norms produce alternative (and sometimes quite useful) notions of distance that are, nevertheless, subject to all the familiar properties:

(a) Symmetry: $d(\mathbf{v}, \mathbf{w}) = d(\mathbf{w}, \mathbf{v})$;
(b) Positivity: $d(\mathbf{v}, \mathbf{w}) = 0$ if and only if $\mathbf{v} = \mathbf{w}$;
(c) Triangle inequality: $d(\mathbf{v}, \mathbf{w}) \leq d(\mathbf{v}, \mathbf{z}) + d(\mathbf{z}, \mathbf{w})$.

For the Euclidean norm, we wish to verify that the distance (4.87) obeys the triangle inequality, i.e., for $\mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathbb{R}^n$, $\|\mathbf{v} - \mathbf{w}\| \leq \|\mathbf{v} - \mathbf{z}\| + \|\mathbf{z} - \mathbf{w}\|$. This is confirmed by recalling the triangle inequality (1.23) or (4.21), which states that, for $\mathbf{A}, \mathbf{B} \in \mathbb{R}^n$, $\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$. Now apply this, taking $\mathbf{A} = \mathbf{v} - \mathbf{z}$ and $\mathbf{B} = \mathbf{z} - \mathbf{w}$.

Just as the distance between vectors measures how close they are to each other keeping in mind that this measure of proximity depends on the underlying choice of norm, so the distance between functions in a normed function space tells something about how close they are to each other, which is related, albeit subtly, to how close their graphs are. Thus, the norm serves to define the topology of the underlying vector space, which determines notions of open and closed sets, convergence, and so on.[27]

Suppose we are given an inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ between vectors $\mathbf{x} = (x_1 x_2 \ldots x_n)^T$ and $\mathbf{y} = (y_1 y_2 \ldots y_n)^T$ in $\mathbb{R}^n$. Our goal is to determine its explicit formula. We begin by writing the vectors in terms of the standard basis vectors:

$$\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n = \sum_{i=1}^{n} x_i\mathbf{e}_i, \quad \mathbf{y} = y_1\mathbf{e}_1 + \cdots + y_n\mathbf{e}_n = \sum_{j=1}^{n} y_j\mathbf{e}_j.$$

[27] For interested readers, there are several excellent books on (and with titles that include) Metric Space Theory. I highly recommend Heil's Metrics, Norms, Inner Products, and Operator Theory; Sasane's A Friendly Approach to Functional Analysis; and Lindstrøm's Spaces: An Introduction to Real Analysis.

To evaluate their inner product, we will appeal to the three basic axioms. We first employ bilinearity to expand

$$\langle \mathbf{x}, \mathbf{y} \rangle = \left\langle \sum_{i=1}^{n} x_i \mathbf{e}_i, \sum_{j=1}^{n} y_j \mathbf{e}_j \right\rangle = \sum_{i,j=1}^{n} x_i y_j \left\langle \mathbf{e}_i, \mathbf{e}_j \right\rangle .$$

Therefore,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i,j=1}^{n} k_{ij} x_i y_j = \mathbf{x}^T K \mathbf{y}, \tag{4.88}$$

where $K$ denotes the $n \times n$ matrix of inner products of the basis vectors, with entries

$$k_{ij} = \left\langle \mathbf{e}_i, \mathbf{e}_j \right\rangle, \quad i, j = 1, \ldots, n.$$

We conclude that any inner product must be expressed in the general bilinear form (4.88).

The two remaining inner product axioms will impose certain constraints on the inner product matrix $K$. Symmetry implies that

$$k_{ij} = \left\langle \mathbf{e}_i, \mathbf{e}_j \right\rangle = \left\langle \mathbf{e}_j, \mathbf{e}_i \right\rangle = k_{ji}, \quad i, j = 1, \ldots, n.$$

Consequently, the inner product matrix $K$ must be symmetric, i.e., $K = K^T$. Conversely, symmetry of $K$ ensures symmetry of the bilinear form:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T K \mathbf{y} = \left( \mathbf{x}^T K \mathbf{y} \right)^T = \mathbf{y}^T K^T \mathbf{x} = \mathbf{y}^T K \mathbf{x} = \langle \mathbf{y}, \mathbf{x} \rangle,$$

where the second equality follows from the fact that the quantity $\mathbf{x}^T K \mathbf{y}$ is a scalar, and hence equals its transpose. The final condition for an inner product is positivity:

$$\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^T K \mathbf{x} = \sum_{i,j=1}^{n} k_{ij} x_i x_j \geq 0 \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n,$$

with equality if and only if $\mathbf{x} = \mathbf{0}$. The precise meaning of this positivity condition on the matrix $K$ is not so immediately evident, and so will be encapsulated in a definition.

Definition: An $n \times n$ matrix $K$ is called *positive definite* if it is symmetric, and satisfies the positivity condition $\mathbf{x}^T K \mathbf{x} > 0$ for all $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n$. We will sometimes write $K > 0$ to mean that $K$ is a positive definite matrix.

Our preliminary analysis has resulted in the following general characterization of inner products on a finite-dimensional vector space:

Theorem: Every inner product on $\mathbb{R}^n$ is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T K \mathbf{y} \quad \text{for} \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \tag{4.89}$$

where $K$ is a symmetric, positive definite $n \times n$ matrix.

Definition: Given a symmetric matrix $K$, the homogeneous quadratic polynomial

$$q(\mathbf{x}) = \mathbf{x}^T K \mathbf{x} = \sum_{i,j=1}^{n} k_{ij} x_i x_j \tag{4.90}$$

is known as a quadratic form on $\mathbb{R}^n$.

Definition: The quadratic form is called *positive definite* if $q(\mathbf{x}) > 0$ for all $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n$.

So the quadratic form (4.90) is positive definite if and only if its coefficient matrix $K$ is. It is easy to show that the coefficient matrix $K$ in any quadratic form can be taken to be symmetric without any loss of generality. If a matrix is positive definite, then it is nonsingular.

A quadratic form and its associated symmetric coefficient matrix are called positive semi-definite if $q(\mathbf{x}) = \mathbf{x}^T K \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, in which case we write $K \geq 0$.

Definition: Let $V$ be an inner product space, and let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$. The associated Gram matrix

$$K = \begin{pmatrix} \langle \mathbf{v}_1, \mathbf{v}_1 \rangle & \langle \mathbf{v}_1, \mathbf{v}_2 \rangle & \ldots & \langle \mathbf{v}_1, \mathbf{v}_n \rangle \\ \langle \mathbf{v}_2, \mathbf{v}_1 \rangle & \langle \mathbf{v}_2, \mathbf{v}_2 \rangle & \ldots & \langle \mathbf{v}_2, \mathbf{v}_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{v}_n, \mathbf{v}_1 \rangle & \langle \mathbf{v}_n, \mathbf{v}_2 \rangle & \ldots & \langle \mathbf{v}_n, \mathbf{v}_n \rangle \end{pmatrix} \tag{4.91}$$

is the $n \times n$ matrix whose entries are the inner products between the selected vector space elements.

Symmetry of the inner product implies symmetry of the Gram matrix:

$$k_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \langle \mathbf{v}_j, \mathbf{v}_i \rangle = k_{ji}, \qquad \text{and hence} \qquad K^T = K. \tag{4.92}$$

In fact, the most direct method for producing positive definite and semi-definite matrices is through the Gram matrix construction.

Theorem: All Gram matrices are positive semi-definite. The Gram matrix above is positive definite if and only if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

> Proof: To prove positive (semi-)definiteness of $K$, we need to examine the associated quadratic form $q(\mathbf{x}) = \mathbf{x}^T K \mathbf{x} = \sum_{i,j=1}^n k_{ij} x_i x_j$. Substituting the values (4.92) for the matrix entries, we obtain
>
> $$q(\mathbf{x}) = \sum_{i,j=1}^n \langle \mathbf{v}_i, \mathbf{v}_j \rangle \, x_i x_j.$$
>
> Bilinearity of the inner product on $V$ implies that we can assemble this summation into a single inner product
>
> $$q(\mathbf{x}) = \left\langle \sum_{i=1}^n x_i \mathbf{v}_i, \sum_{j=1}^n x_j \mathbf{v}_j \right\rangle = \langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{v}\|^2 \geq 0, \qquad \text{where} \qquad \mathbf{v} = \sum_{i=1}^n x_i \mathbf{v}_i$$
>
> lies in the subspace of $V$ spanned by the given vectors. This immediately proves that $K$ is positive semi-definite. Moreover, $q(\mathbf{x}) = \|\mathbf{v}\|^2 > 0$ as long as $\mathbf{v} \neq \mathbf{0}$. If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, then
>
> $$\mathbf{v} = x_1 \mathbf{v}_1 + \cdots + x_n \mathbf{v}_n = \mathbf{0} \qquad \text{if and only if} \qquad x_1 = \cdots = x_n = 0,$$
>
> and hence $q(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$. This implies that $q(\mathbf{x})$ and hence $K$ are positive definite.

In the case of the Euclidean dot product, the construction of the Gram matrix $K$ can be directly implemented as follows. Given column vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$, let us form the

$m \times n$ matrix $A = (\mathbf{v}_1 \mathbf{v}_2 \ldots \mathbf{v}_n)$. In view of the identification between the dot product and multiplication of row and column vectors, the $(i, j)$ entry of $K$ is given as the product

$$k_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j = \mathbf{v}_i^T \mathbf{v}_j$$

of the $i$th row of the transpose $A^T$ and the $j$th column of $A$. In other words, the Gram matrix can be evaluated as a matrix product:

$$K = A^T A. \tag{4.93}$$

Changing the underlying inner product will, of course, change the Gram matrix. As noted in (4.89), every inner product on $\mathbb{R}^m$ has the form

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T C \mathbf{w} \quad \text{for} \quad \mathbf{v}, \mathbf{w} \in \mathbb{R}^m, \tag{4.94}$$

where $C > 0$ is a symmetric, positive definite $m \times m$ matrix. Therefore, given $n$ vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$, the entries of the Gram matrix with respect to this inner product are

$$k_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \mathbf{v}_i^T C \mathbf{v}_j.$$

If, as above, we assemble the column vectors into an $m \times n$ matrix $A = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{pmatrix}$, then the Gram matrix entry $k_{ij}$ is obtained by multiplying the $i$th row of $A^T$ by the $j$th column of the product matrix $CA$. Therefore, the Gram matrix based on the alternative inner product (4.94) is given by

$$K = A^T C A. \tag{4.95}$$

Recall the above theorem stating that all Gram matrices are positive semi-definite, and that a Gram matrix is positive definite if and only if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent. Thus, provided that the matrix $A$ has full rank $n$, $K$ is positive definite.

Theorem: Suppose $A$ is an $m \times n$ matrix with linearly independent columns. Suppose $C$ is any positive definite $m \times m$ matrix. Then the Gram matrix $K = A^T C A$ is a positive definite $n \times n$ matrix.

### 4.5.2 Orthogonal and Orthonormal Bases

Let $V$ be a real inner product space. Recall that two elements $\mathbf{v}, \mathbf{w} \in V$ are called orthogonal if their inner product vanishes: $\langle \mathbf{v}, \mathbf{w} \rangle = 0$. In the case of vectors in Euclidean space, orthogonality under the dot product means that they meet at a right angle.

A particularly important configuration arises when $V$ admits a basis consisting of mutually orthogonal elements.

Definition: A basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of an $n$-dimensional inner product space $V$ is called orthogonal if $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ for all $i \neq j$. The basis is called orthonormal if, in addition, each vector has unit length: $\|\mathbf{u}_i\| = 1$, for all $i = 1, \ldots, n$.

For the Euclidean space $\mathbb{R}^n$ equipped with the standard dot product, the simplest example of an orthonormal basis is the standard basis

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Orthogonality follows because $\mathbf{e}_i \cdot \mathbf{e}_j = 0$, for $i \neq j$, while $\|\mathbf{e}_i\| = 1$ implies normality. Since a basis cannot contain the zero vector, there is an easy way to convert an orthogonal basis to an orthonormal basis. Namely, we replace each basis vector with a unit vector pointing in the same direction.

Lemma: If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an orthogonal basis of a vector space $V$, then the normalized vectors $\mathbf{u}_i = \mathbf{v}_i / \|\mathbf{v}_i\|, i = 1, \ldots, n$, form an orthonormal basis.

A useful observation is that every orthogonal collection of nonzero vectors is automatically linearly independent.

Proposition: Let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$ be nonzero, mutually orthogonal elements, so $\mathbf{v}_i \neq \mathbf{0}$ and $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for all $i \neq j$. Then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

> Proof: Suppose
> $$c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k = \mathbf{0}.$$
> Let us take the inner product of this equation with any $\mathbf{v}_i$. Using linearity of the inner product and orthogonality, we compute
> $$0 = \langle c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k, \mathbf{v}_i \rangle = c_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + \cdots + c_k \langle \mathbf{v}_k, \mathbf{v}_i \rangle = c_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle = c_i \|\mathbf{v}_i\|^2 .$$
> Therefore, given that $\mathbf{v}_i \neq \mathbf{0}$, we conclude that $c_i = 0$. Since this holds for all $i = 1, \ldots, k$, the linear independence of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ follows.

As a direct corollary, we infer that every collection of nonzero orthogonal vectors forms a basis for its span:

Theorem: Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ are nonzero, mutually orthogonal elements of an inner product space $V$. Then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form an orthogonal basis for their span $W = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$, which is therefore a subspace of dimension $n = \dim W$. In particular, if $\dim V = n$, then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a orthogonal basis for $V$.

What are the advantages of orthogonal and orthonormal bases? Once one has a basis of a vector space, a key issue is how to express other elements as linear combinations of the basis elements. That is, to find their coordinates in the prescribed basis. In general, this is not so easy, since it requires solving a system of linear equations. In high-dimensional situations arising in applications, computing the solution may require a considerable, if not infeasible, amount of time and effort.

However, if the basis is orthogonal, or, even better, orthonormal, then the change of basis computation requires almost no work. This is the crucial insight underlying the efficacy of both discrete and continuous Fourier analysis in signal, image, and video processing, least squares approximations, the statistical analysis of large data sets, and a multitude of other applications, both classical and modern.

Theorem: Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be an orthonormal basis for an inner product space $V$. Then one can write any element $\mathbf{v} \in V$ as a linear combination

$$\mathbf{v} = c_1 \mathbf{u}_1 + \cdots + c_n \mathbf{u}_n \tag{4.96}$$

in which its coordinates

$$c_i = \langle \mathbf{v}, \mathbf{u}_i \rangle, \quad i = 1, \ldots, n, \tag{4.97}$$

are explicitly given as inner products. Moreover, its norm is given by the Pythagorean formula

$$\|\mathbf{v}\| = \sqrt{c_1^2 + \cdots + c_n^2} = \sqrt{\sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{u}_i \rangle^2}, \tag{4.98}$$

namely, the square root of the sum of the squares of its orthonormal basis coordinates.

> Proof: Let us compute the inner product of the element (4.96) with one of the basis vectors. Using the orthonormality conditions
>
> $$\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \begin{cases} 0 & i \neq j, \\ 1 & i = j, \end{cases}$$
>
> and bilinearity of the inner product, we obtain
>
> $$\langle \mathbf{v}, \mathbf{u}_i \rangle = \left\langle \sum_{j=1}^{n} c_j \mathbf{u}_j, \mathbf{u}_i \right\rangle = \sum_{j=1}^{n} c_j \langle \mathbf{u}_j, \mathbf{u}_i \rangle = c_i \|\mathbf{u}_i\|^2 = c_i.$$
>
> To prove formula (4.98), we similarly expand
>
> $$\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle = \left\langle \sum_{j=1}^{n} c_i \mathbf{u}_i, \sum_{j=1}^{n} c_j \mathbf{u}_j \right\rangle = \sum_{i,j=1}^{n} c_i c_j \langle \mathbf{u}_i, \mathbf{u}_j \rangle = \sum_{i=1}^{n} c_i^2,$$
>
> again making use of the orthonormality of the basis elements.

While passage from an orthogonal basis to its orthonormal version is elementary—one simply divides each basis element by its norm—we shall often find it more convenient to work directly with the unnormalized version. The next result provides the corresponding formula expressing a vector in terms of an orthogonal, but not necessarily orthonormal basis. The proof proceeds exactly as in the orthonormal case, and details are left to the reader.

Theorem: If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form an orthogonal basis, then the corresponding coordinates of a vector

$$\mathbf{v} = a_1 \mathbf{v}_1 + \cdots + a_n \mathbf{v}_n \qquad \text{are given by} \qquad a_i = \frac{\langle \mathbf{v}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}. \tag{4.99}$$

In this case, its norm can be computed using the formula

$$\|\mathbf{v}\|^2 = \sum_{i=1}^{n} a_i^2 \|\mathbf{v}_i\|^2 = \sum_{i=1}^{n} \left( \frac{\langle \mathbf{v}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|} \right)^2.$$

### 4.5.3 Gram–Schmidt, Orthogonal Matrices, QR Factorization

Once we become convinced of the utility of orthogonal and orthonormal bases, a natural question arises: How can we construct them? A practical algorithm was first discovered by the French mathematician Pierre-Simon Laplace in the eighteenth century. Today the algorithm is known as the Gram-Schmidt process, after its rediscovery by Gram and the twentieth-century German mathematician Erhard Schmidt. The Gram-Schmidt process is one of the premier algorithms of applied and computational linear algebra.

Let $W$ denote a finite-dimensional inner product space. (To begin with, you might wish to think of $W$ as a subspace of $\mathbb{R}^m$, equipped with the standard Euclidean dot product, although the algorithm will be formulated in complete generality.) We assume that we already know some basis $\mathbf{w}_1, \ldots, \mathbf{w}_n$ of $W$, where $n = \dim W$. Our goal is to use this information to construct an orthogonal basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

We will construct the orthogonal basis elements one by one. Since initially we are not worrying about normality, there are no conditions on the first orthogonal basis element $\mathbf{v}_1$, and so there is no harm in choosing $\mathbf{v}_1 = \mathbf{w}_1$. Note that $\mathbf{v}_1 \neq \mathbf{0}$, since $\mathbf{w}_1$ appears in the original basis. Starting with $\mathbf{w}_2$, the second basis vector $\mathbf{v}_2$ must be orthogonal to the first: $\langle \mathbf{v}_2, \mathbf{v}_1 \rangle = 0$. Let us try to arrange this by subtracting a suitable multiple of $\mathbf{v}_1$, and set

$$\mathbf{v}_2 = \mathbf{w}_2 - c\,\mathbf{v}_1,$$

where $c$ is a scalar to be determined. The orthogonality condition

$$0 = \langle \mathbf{v}_2, \mathbf{v}_1 \rangle = \langle \mathbf{w}_2, \mathbf{v}_1 \rangle - c\,\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \langle \mathbf{w}_2, \mathbf{v}_1 \rangle - c\,\|\mathbf{v}_1\|^2$$

requires that $c = \langle \mathbf{w}_2, \mathbf{v}_1 \rangle \,/\, \|\mathbf{v}_1\|^2$, and therefore

$$\mathbf{v}_2 = \mathbf{w}_2 - \frac{\langle \mathbf{w}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\,\mathbf{v}_1.$$

Linear independence of $\mathbf{v}_1 = \mathbf{w}_1$ and $\mathbf{w}_2$ ensures that $\mathbf{v}_2 \neq \mathbf{0}$. Next, we construct

$$\mathbf{v}_3 = \mathbf{w}_3 - c_1\mathbf{v}_1 - c_2\mathbf{v}_2$$

by subtracting suitable multiples of the first two orthogonal basis elements from $\mathbf{w}_3$. We want $\mathbf{v}_3$ to be orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. Since we already arranged that $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$, this requires

$$0 = \langle \mathbf{v}_3, \mathbf{v}_1 \rangle = \langle \mathbf{w}_3, \mathbf{v}_1 \rangle - c_1 \langle \mathbf{v}_1, \mathbf{v}_1 \rangle, \qquad 0 = \langle \mathbf{v}_3, \mathbf{v}_2 \rangle = \langle \mathbf{w}_3, \mathbf{v}_2 \rangle - c_2 \langle \mathbf{v}_2, \mathbf{v}_2 \rangle,$$

and hence

$$c_1 = \frac{\langle \mathbf{w}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}, \qquad c_2 = \frac{\langle \mathbf{w}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}.$$

Therefore, the next orthogonal basis vector is given by the formula

$$\mathbf{v}_3 = \mathbf{w}_3 - \frac{\langle \mathbf{w}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\,\mathbf{v}_1 - \frac{\langle \mathbf{w}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\,\mathbf{v}_2.$$

Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linear combinations of $\mathbf{w}_1$ and $\mathbf{w}_2$, we must have $\mathbf{v}_3 \neq \mathbf{0}$, since otherwise, this would imply that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are linearly dependent, and hence could not come from a basis.

Continuing in the same manner, suppose we have already constructed the mutually orthogonal vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$ as linear combinations of $\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}$. The next orthogonal basis element $\mathbf{v}_k$ will be obtained from $\mathbf{w}_k$ by subtracting off a suitable linear combination of the previous orthogonal basis elements:

$$\mathbf{v}_k = \mathbf{w}_k - c_1\mathbf{v}_1 - \cdots - c_{k-1}\mathbf{v}_{k-1}.$$

Since $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$ are already orthogonal, the orthogonality constraint

$$0 = \langle \mathbf{v}_k, \mathbf{v}_j \rangle = \langle \mathbf{w}_k, \mathbf{v}_j \rangle - c_j \langle \mathbf{v}_j, \mathbf{v}_j \rangle$$

requires

$$c_j = \frac{\langle \mathbf{w}_k, \mathbf{v}_j \rangle}{\|\mathbf{v}_j\|^2} \quad \text{for} \quad j = 1, \ldots, k-1.$$

In this fashion, we establish the general Gram-Schmidt formula

$$\mathbf{v}_k = \mathbf{w}_k - \sum_{j=1}^{k-1} \frac{\langle \mathbf{w}_k, \mathbf{v}_j \rangle}{\|\mathbf{v}_j\|^2} \mathbf{v}_j, \quad k = 1, \ldots, n. \tag{4.100}$$

The iterative Gram-Schmidt process (4.100), where we start with $\mathbf{v}_1 = \mathbf{w}_1$ and successively construct $\mathbf{v}_2, \ldots, \mathbf{v}_n$, defines an explicit, recursive procedure for constructing the desired orthogonal basis vectors. If we are actually after an orthonormal basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$, we merely normalize the resulting orthogonal basis vectors, setting $\mathbf{u}_k = \mathbf{v}_k / \|\mathbf{v}_k\|$ for each $k = 1, \ldots, n$.

**Example 4.22** *The vectors*

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix},$$

*are readily seen to form a basis of* $\mathbb{R}^3$. *To construct an orthogonal basis (with respect to the standard dot product) using the Gram-Schmidt process, we begin by setting*

$$\mathbf{v}_1 = \mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

*The next basis vector is*

$$\mathbf{v}_2 = \mathbf{w}_2 - \frac{\mathbf{w}_2 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \frac{-1}{3} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{1}{3} \\ \frac{5}{3} \end{pmatrix}.$$

*The last orthogonal basis vector is*

$$\mathbf{v}_3 = \mathbf{w}_3 - \frac{\mathbf{w}_3 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\mathbf{w}_3 \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} - \frac{-3}{3} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \frac{7}{\frac{14}{3}} \begin{pmatrix} \frac{4}{3} \\ \frac{1}{3} \\ \frac{5}{3} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

*The reader can easily validate the orthogonality of* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. *An orthonormal basis is obtained by dividing each vector by its length. Since*

$$\|\mathbf{v}_1\| = \sqrt{3}, \quad \|\mathbf{v}_2\| = \sqrt{\frac{14}{3}}, \quad \|\mathbf{v}_3\| = \sqrt{\frac{7}{2}},$$

*we produce the corresponding orthonormal basis vectors*

$$\mathbf{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{42}} \\ \frac{5}{\sqrt{42}} \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} \frac{2}{\sqrt{14}} \\ -\frac{3}{\sqrt{14}} \\ -\frac{1}{\sqrt{14}} \end{pmatrix}. \qquad \blacksquare$$

**Example 4.23** *Here is a typical problem: find an orthonormal basis, with respect to the dot product, for the subspace $W \subset \mathbb{R}^4$ consisting of all vectors that are orthogonal to the given vector $\mathbf{a} = (1,2,-1,-3)^T$. The first task is to find a basis for the subspace. Now, a vector $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$ is orthogonal to $\mathbf{a}$ if and only if*

$$\mathbf{x} \cdot \mathbf{a} = x_1 + 2x_2 - x_3 - 3x_4 = 0.$$

*Solving this homogeneous linear system by the usual method, we observe that the free variables are $x_2, x_3, x_4$, and so a (non-orthogonal) basis for the subspace is*

$$\mathbf{w}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

*To obtain an orthogonal basis, we apply the Gram-Schmidt process. First,*

$$\mathbf{v}_1 = \mathbf{w}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

*The next element is*

$$\mathbf{v}_2 = \mathbf{w}_2 - \frac{\mathbf{w}_2 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{-2}{5}\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ 1 \\ 0 \end{pmatrix}.$$

*The last element of our orthogonal basis is*

$$\mathbf{v}_3 = \mathbf{w}_3 - \frac{\mathbf{w}_3 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\mathbf{w}_3 \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{-6}{5}\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{\frac{3}{5}}{\frac{6}{5}}\begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \end{pmatrix}.$$

*An orthonormal basis can then be obtained by dividing each $\mathbf{v}_i$ by its length:*

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ \frac{5}{\sqrt{30}} \\ 0 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \end{pmatrix}. \qquad \blacksquare$$

With the basic Gram-Schmidt algorithm now in hand, it is worth looking at a couple of reformulations that have both practical and theoretical advantages. The first can be used to construct the orthonormal basis vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ directly from the basis $\mathbf{w}_1, \ldots, \mathbf{w}_n$.

We begin by replacing each orthogonal basis vector in the basic Gram-Schmidt formula (4.100) by its normalized version $\mathbf{u}_j = \mathbf{v}_j / \|\mathbf{v}_j\|$. The original basis vectors can be expressed in terms of the orthonormal basis via a "triangular" system

$$\begin{aligned}
\mathbf{w}_1 &= r_{11}\mathbf{u}_1 \\
\mathbf{w}_2 &= r_{12}\mathbf{u}_1 + r_{22}\mathbf{u}_2, \\
\mathbf{w}_3 &= r_{13}\mathbf{u}_1 + r_{23}\mathbf{u}_2 + r_{33}\mathbf{u}_3, \\
\vdots \;\; & \;\;\; \vdots \qquad \vdots \qquad \ddots \\
\mathbf{w}_n &= r_{1n}\mathbf{u}_1 + r_{2n}\mathbf{u}_2 + \cdots + r_{nn}\mathbf{u}_n.
\end{aligned} \tag{4.101}$$

Before proceeding, it is useful to cast this in matrix terms. It will be used below by the authors in (4.107). For illustration, take $n = 2$, so that

$$\begin{bmatrix} w_{1,1} \\ w_{2,1} \end{bmatrix} = \mathbf{w}_1 = r_{11}\mathbf{u}_1 = \begin{bmatrix} r_{11}u_{1,1} \\ r_{11}u_{2,1} \end{bmatrix}, \quad \begin{bmatrix} w_{1,2} \\ w_{2,2} \end{bmatrix} = \mathbf{w}_2 = r_{12}\mathbf{u}_1 + r_{22}\mathbf{u}_2 = \begin{bmatrix} r_{12}u_{1,1} + r_{22}u_{1,2} \\ r_{12}u_{2,1} + r_{22}u_{2,2} \end{bmatrix}$$

or, with $\mathbf{W} = [\mathbf{w}_1\ \mathbf{w}_2]$ and $\mathbf{U} = [\mathbf{u}_1\ \mathbf{u}_2]$,

$$\mathbf{W} = \begin{bmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{bmatrix} = \begin{bmatrix} r_{11}u_{1,1} & r_{12}u_{1,1} + r_{22}u_{1,2} \\ r_{11}u_{2,1} & r_{12}u_{2,1} + r_{22}u_{2,2} \end{bmatrix} = \begin{bmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = \mathbf{UR}, \tag{4.102}$$

where $\mathbf{R}$ is the $2 \times 2$ indicated matrix, and the case of general $n$ is clear.

The coefficients $r_{ij}$ can, in fact, be computed directly from these formulas. Indeed, taking the inner product of the equation for $\mathbf{w}_j$ with the orthonormal basis vector $\mathbf{u}_i$ for $i \le j$, we obtain, in view of the orthonormality constraints,

$$\langle \mathbf{w}_j, \mathbf{u}_i \rangle = \langle r_{1j}\mathbf{u}_1 + \cdots + r_{jj}\mathbf{u}_j, \mathbf{u}_i \rangle = r_{1j}\langle \mathbf{u}_1, \mathbf{u}_i \rangle + \cdots + r_{jj}\langle \mathbf{u}_j, \mathbf{u}_i \rangle = r_{ij},$$

and hence

$$r_{ij} = \langle \mathbf{w}_j, \mathbf{u}_i \rangle\,. \tag{4.103}$$

On the other hand,

$$\|\mathbf{w}_j\|^2 = \|r_{1j}\mathbf{u}_1 + \cdots + r_{jj}\mathbf{u}_j\|^2 = r_{1j}^2 + \cdots + r_{j-1,j}^2 + r_{jj}^2. \tag{4.104}$$

The pair of equations (4.103) and (4.104) can be rearranged to devise a recursive procedure to compute the orthonormal basis. We begin by setting $r_{11} = \|\mathbf{w}_1\|$ and so $\mathbf{u}_1 = \mathbf{w}_1/r_{11}$. At each subsequent stage $j \ge 2$, we assume that we have already constructed $\mathbf{u}_1, \ldots, \mathbf{u}_{j-1}$. We then compute

$$r_{ij} = \langle \mathbf{w}_j, \mathbf{u}_i \rangle\,, \qquad \text{for each} \qquad i = 1, \ldots, j-1. \tag{4.105}$$

We obtain the next orthonormal basis vector $\mathbf{u}_j$ by computing

$$r_{jj} = \sqrt{\|\mathbf{w}_j\|^2 - r_{1j}^2 - \cdots - r_{j-1,j}^2}, \qquad \mathbf{u}_j = \frac{\mathbf{w}_j - r_{1j}\mathbf{u}_1 - \cdots - r_{j-1,j}\mathbf{u}_{j-1}}{r_{jj}}. \tag{4.106}$$

Running through the formulas (4.105) and (4.106) for $j = 1, \ldots, n$ leads to the same orthonormal basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$ produced by the previous version of the Gram-Schmidt procedure.

In practical, large-scale computations, both versions of the Gram-Schmidt process suffer from a serious flaw. They are subject to numerical instabilities, and so accumulating round-off errors may seriously corrupt the computations, leading to inaccurate, non-orthogonal vectors. Fortunately, there is a simple rearrangement of the calculation that ameliorates this difficulty and leads to the numerically robust algorithm that is most often used in practice. The idea is to treat the vectors simultaneously rather than sequentially, making full use of the orthonormal basis vectors as they arise. More specifically, the algorithm begins as before—we take $\mathbf{u}_1 = \mathbf{w}_1/\|\mathbf{w}_1\|$. We then subtract off the appropriate multiples of $\mathbf{u}_1$ from all of the remaining basis vectors so as to arrange their orthogonality to $\mathbf{u}_1$. This is accomplished by setting

$$\mathbf{w}_k^{(2)} = \mathbf{w}_k - \langle \mathbf{w}_k, \mathbf{u}_1 \rangle\, \mathbf{u}_1 \qquad \text{for} \qquad k = 2, \ldots, n.$$

The second orthonormal basis vector $\mathbf{u}_2 = \mathbf{w}_2^{(2)} / \left\| \mathbf{w}_2^{(2)} \right\|$ is then obtained by normalizing. We next modify the remaining $\mathbf{w}_3^{(2)}, \ldots, \mathbf{w}_n^{(2)}$ to produce vectors

$$\mathbf{w}_k^{(3)} = \mathbf{w}_k^{(2)} - \left\langle \mathbf{w}_k^{(2)}, \mathbf{u}_2 \right\rangle \mathbf{u}_2, \quad k = 3, \ldots, n,$$

that are orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. Then $\mathbf{u}_3 = \mathbf{w}_3^{(3)} / \left\| \mathbf{w}_3^{(3)} \right\|$ is the next orthonormal basis element, and the process continues. The full algorithm starts with the initial basis vectors $\mathbf{w}_j = \mathbf{w}_j^{(1)}, j = 1, \ldots, n$, and then recursively computes, for $j = 1, \ldots, n$ and $k = j+1, \ldots, n$,

$$\mathbf{u}_j = \frac{\mathbf{w}_j^{(j)}}{\left\| \mathbf{w}_j^{(j)} \right\|}, \quad \mathbf{w}_k^{(j+1)} = \mathbf{w}_k^{(j)} - \left\langle \mathbf{w}_k^{(j)}, \mathbf{u}_j \right\rangle \mathbf{u}_j.$$

(In the final phase, when $j = n$, the second formula is no longer needed.) The result is a numerically stable computation of the same orthonormal basis vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$.

Matrices whose columns form an orthonormal basis of $\mathbb{R}^n$ relative to the standard Euclidean dot product play a distinguished role. Such “orthogonal matrices” appear in a wide range of applications in geometry, physics, quantum mechanics, crystallography, partial differential equations, symmetry theory, and special functions. Rotational motions of bodies in three-dimensional space are described by orthogonal matrices, and hence they lie at the foundations of rigid body mechanics, including satellites, airplanes, drones, and underwater vehicles, as well as three-dimensional computer graphics and animation for video games and movies. Furthermore, orthogonal matrices are an essential ingredient in one of the most important methods of numerical linear algebra: the QR algorithm for computing eigenvalues of matrices.

Definition: A square matrix $Q$ is called orthogonal if it satisfies $Q^T Q = Q Q^T = \mathrm{I}$.

This is referred to as the orthogonality condition, or requirement. It implies that one can easily invert an orthogonal matrix: $Q^{-1} = Q^T$. In fact, the two conditions are equivalent, and hence a matrix is orthogonal if and only if its inverse is equal to its transpose. In particular, the identity matrix $I$ is orthogonal. Also note that, if $Q$ is orthogonal, then so is $Q^T$. The second important characterization of orthogonal matrices relates them directly to orthonormal bases.

Proposition: A matrix $Q$ is orthogonal if and only if its columns form an orthonormal basis with respect to the Euclidean dot product on $\mathbb{R}^n$.

> Proof: Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be the columns of $Q$. Then $\mathbf{u}_1^T, \ldots, \mathbf{u}_n^T$ are the rows of the transposed matrix $Q^T$. The $(i, j)$ entry of the product $Q^T Q$ is given as the product of the $i$th row of $Q^T$ and the $j$th column of $Q$. Thus, the orthogonality requirement implies
>
> $$\mathbf{u}_i \cdot \mathbf{u}_j = \mathbf{u}_i^T \mathbf{u}_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$
>
> which are precisely the conditions for $\mathbf{u}_1, \ldots, \mathbf{u}_n$ to form an orthonormal basis.

In particular, the columns of the identity matrix produce the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of $\mathbb{R}^n$. Also, the rows of an orthogonal matrix $Q$ also produce an (in general different) orthonormal basis.

Warning: Technically, we should be referring to an "orthonormal" matrix, not an "orthogonal" matrix. But the terminology is so standard throughout mathematics and physics that we have no choice but to adopt it here. There is no commonly accepted name for a matrix whose columns form an orthogonal but not orthonormal basis.

**Example 4.24** *A* $2 \times 2$ *matrix* $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is orthogonal if and only if its columns* $\mathbf{u}_1 = \begin{pmatrix} a \\ c \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} b \\ d \end{pmatrix}$, *form an orthonormal basis of* $\mathbb{R}^2$. *Equivalently, the requirement*

$$Q^T Q = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

*implies that its entries must satisfy the algebraic equations* $a^2+c^2 = 1$, $ab+cd = 0$, $b^2+d^2 = 1$. *The first and last equations say that the points* $(a,c)^T$ *and* $(b,d)^T$ *lie on the unit circle in* $\mathbb{R}^2$, *and so*

$$a = \cos\theta, \quad c = \sin\theta, \quad b = \cos\psi, \quad d = \sin\psi$$

*for some choice of angles* $\theta, \psi$. *The remaining orthogonality condition is*

$$0 = ab + cd = \cos\theta\cos\psi + \sin\theta\sin\psi = \cos(\theta - \psi),$$

*which implies that* $\theta$ *and* $\psi$ *differ by a right angle:* $\psi = \theta \pm \frac{1}{2}\pi$. *The* $\pm$ *sign leads to two cases: It is useful to recall the relations in (4.76).*

$$b = -\sin\theta, \quad d = \cos\theta, \qquad \text{or} \qquad b = \sin\theta, \quad d = -\cos\theta.$$

*As a result, every* $2 \times 2$ *orthogonal matrix has one of two possible forms*

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \qquad \text{where } 0 \le \theta < 2\pi.$$

*The corresponding orthonormal bases are illustrated in Figure 36.* ■

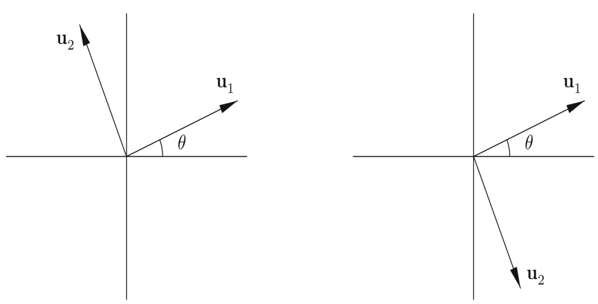



Figure 36: Orthonormal bases in $\mathbb{R}^2$

Lemma: An orthogonal matrix $Q$ has determinant $\det Q = \pm 1$.

> Proof: Taking the determinant of $Q^T Q = Q Q^T = I$ shows that
>
> $$1 = \det I = \det\left(Q^T Q\right) = \det Q^T \det Q = (\det Q)^2,$$
>
> which immediately proves the lemma.

An orthogonal matrix is called proper or special if it has determinant $+1$. An improper orthogonal matrix, with determinant $-1$, corresponds to a left handed basis that lives in a mirror-image world.

Proposition: The product of two orthogonal matrices is also orthogonal.

> Proof: If
>
> $$Q_1^T Q_1 = I = Q_2^T Q_2, \quad \text{then} \quad (Q_1 Q_2)^T (Q_1 Q_2) = Q_2^T Q_1^T Q_1 Q_2 = Q_2^T Q_2 = I,$$
>
> and so the product matrix $Q_1 Q_2$ is also orthogonal.

This multiplicative property, combined with the fact that the inverse of an orthogonal matrix is also orthogonal, says that the set of all orthogonal matrices forms a group. The orthogonal group lies at the foundation of everyday Euclidean geometry, as well as rigid body mechanics, atomic structure and chemistry, computer graphics and animation, and many other areas.

The Gram-Schmidt procedure for orthonormalizing bases of $\mathbb{R}^n$ can be reinterpreted as a matrix factorization. This is more subtle than the LU factorization that resulted from Gaussian Elimination, but is of comparable significance, and is used in a broad range of applications in mathematics, statistics, physics, engineering, and numerical analysis.

Let $\mathbf{w}_1, \ldots, \mathbf{w}_n$ be a basis of $\mathbb{R}^n$, and let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be the corresponding orthonormal basis that results from any one of the three implementations of the Gram-Schmidt process. We assemble both sets of column vectors to form nonsingular $n \times n$ matrices

$$A = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_n \end{pmatrix}, \quad Q = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_n \end{pmatrix}.$$

Since the $\mathbf{u}_i$ form an orthonormal basis, $Q$ is an orthogonal matrix. The Gram-Schmidt equations (4.101) can be recast into an equivalent matrix form: Recall (4.102)

$$A = QR, \qquad \text{where} \qquad R = \begin{pmatrix} r_{11} & r_{12} & \ldots & r_{1n} \\ 0 & r_{22} & \ldots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & r_{nn} \end{pmatrix} \tag{4.107}$$

is an upper triangular matrix whose entries are the coefficients in (4.105) and (4.106). Since the Gram-Schmidt process works on any basis, the only requirement on the matrix $A$ is that its columns form a basis of $\mathbb{R}^n$, and hence $A$ can be any nonsingular matrix. We have therefore established the celebrated QR factorization of nonsingular matrices.

Theorem: Every nonsingular matrix can be factored, $A = QR$, into the product of an orthogonal matrix $Q$ and an upper triangular matrix $R$. The factorization is unique if $R$ is positive upper triangular, meaning that all its diagonal entries of are positive.

### 4.5.4 Orthogonal Projections and Orthogonal Subspaces

Definition: A vector $\mathbf{z} \in V$ is said to be orthogonal to the subspace $W \subset V$ if it is orthogonal to every vector in $W$, so $\langle \mathbf{z}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in W$.

Given a basis $\mathbf{w}_1, \ldots, \mathbf{w}_n$ of the subspace $W$, we note that $\mathbf{z}$ is orthogonal to $W$ if and only if it is orthogonal to every basis vector: $\langle \mathbf{z}, \mathbf{w}_i \rangle = 0$ for $i = 1, \ldots, n$. Indeed, any other vector in $W$ has the form $\mathbf{w} = c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n$, and hence, by linearity, $\langle \mathbf{z}, \mathbf{w} \rangle = c_1 \langle \mathbf{z}, \mathbf{w}_1 \rangle + \cdots + c_n \langle \mathbf{z}, \mathbf{w}_n \rangle = 0$, as required.

Definition: The orthogonal projection of $\mathbf{v}$ onto the subspace $W$ is the element $\mathbf{w} \in W$ that makes the difference $\mathbf{z} = \mathbf{v} - \mathbf{w}$ orthogonal to $W$.

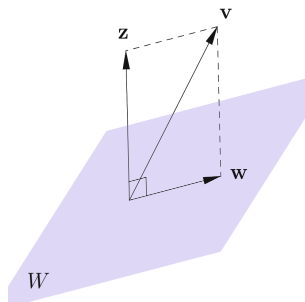



Figure 37: The orthogonal projection of a vector onto a subspace

The geometric configuration underlying orthogonal projection is sketched in Figure 37. As we shall see, the orthogonal projection is unique. Note that $\mathbf{v} = \mathbf{w} + \mathbf{z}$ is the sum of its orthogonal projection $\mathbf{w} \in V$ and the perpendicular vector $\mathbf{z} \perp W$. The explicit construction is greatly simplified by taking an orthonormal basis of the subspace, which, if necessary, can be arranged by applying the Gram-Schmidt process to a known basis.

Theorem: Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be an orthonormal basis for the subspace $W \subset V$. Then the orthogonal projection of $\mathbf{v} \in V$ onto $\mathbf{w} \in W$ is given by

$$\mathbf{w} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n \quad \text{where} \quad c_i = \langle \mathbf{v}, \mathbf{u}_i \rangle, \quad i = 1, \ldots, n. \tag{4.108}$$

Proof: First, since $\mathbf{u}_1, \ldots, \mathbf{u}_n$ form a basis of the subspace, the orthogonal projection element must be some linear combination thereof: $\mathbf{w} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$. The definition of the orthogonal projection of $\mathbf{v}$ onto the subspace $W$ requires that the difference $\mathbf{z} = \mathbf{v} - \mathbf{w}$ be orthogonal to $W$, and, as noted above, it suffices to check orthogonality to the basis vectors. By our orthonormality assumption,

$$\begin{aligned} 0 = \langle \mathbf{z}, \mathbf{u}_i \rangle = \langle \mathbf{v} - \mathbf{w}, \mathbf{u}_i \rangle &= \langle \mathbf{v} - c_1\mathbf{u}_1 - \cdots - c_n\mathbf{u}_n, \mathbf{u}_i \rangle \\ &= \langle \mathbf{v}, \mathbf{u}_i \rangle - c_1 \langle \mathbf{u}_1, \mathbf{u}_i \rangle - \cdots - c_n \langle \mathbf{u}_n, \mathbf{u}_i \rangle = \langle \mathbf{v}, \mathbf{u}_i \rangle - c_i. \end{aligned}$$

The coefficients $c_i = \langle \mathbf{v}, \mathbf{u}_i \rangle$ of the orthogonal projection $\mathbf{w}$ are thus uniquely prescribed by the orthogonality requirement, which thereby proves its uniqueness.

More generally, if we employ an orthogonal basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ for the subspace $W$, then the same argument demonstrates that the orthogonal projection of $\mathbf{v}$ onto $W$ is given by

$$\mathbf{w} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n, \quad \text{where} \quad a_i = \frac{\langle \mathbf{v}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}, \quad i = 1, \ldots, n. \tag{4.109}$$

We could equally well replace the orthogonal basis by the orthonormal basis obtained by dividing each vector by its length: $\mathbf{u}_i = \mathbf{v}_i / \|\mathbf{v}_i\|$. The reader should be able to prove that the two formulas (4.108) and (4.109) for the orthogonal projection yield the same vector $\mathbf{w}$.

An intriguing observation is that the coefficients in the orthogonal projection formulas (4.108) and (4.109) coincide with the formulas (4.97) and (4.99) for writing a vector in terms of an orthonormal or orthogonal basis. Indeed, if $\mathbf{v}$ were an element of $W$, then it would coincide with its orthogonal projection, $\mathbf{w} = \mathbf{v}$. As a result, the orthogonal projection formula include the orthogonal basis formula as a special case.

It is also worth noting that the same formulae occur in the Gram-Schmidt algorithm, cf. (4.100). This observation leads to a useful geometric interpretation of the Gram-Schmidt construction. For each $k = 1, \ldots, n$, let

$$W_k = \operatorname{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_k\} = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$$

denote the $k$-dimensional subspace spanned by the first $k$ basis elements, which is the same as that spanned by their orthogonalized and orthonormalized counterparts. In view of (4.41), the basic Gram-Schmidt formula (4.100) can be re-expressed in the form $\mathbf{v}_k = \mathbf{w}_k - \mathbf{p}_k$, where $\mathbf{p}_k$ is the orthogonal projection of $\mathbf{w}_k$ onto the subspace $W_{k-1}$. The resulting vector $\mathbf{v}_k$ is, by construction, orthogonal to the subspace, and hence orthogonal to all of the previous basis elements, which serves to rejustify the Gram-Schmidt construction.

We now extend the notion of orthogonality from individual elements to entire subspaces of an inner product space $V$.

Definition: Two subspaces $W, Z \subset V$ are called orthogonal if every vector in $W$ is orthogonal to every vector in $Z$.

In other words, $W$ and $Z$ are orthogonal subspaces if and only if $\langle \mathbf{w}, \mathbf{z} \rangle = 0$ for every $\mathbf{w} \in W$ and $\mathbf{z} \in Z$. In practice, one only needs to check orthogonality of basis elements, or, more generally, spanning sets.

Lemma: If $\mathbf{w}_1, \ldots, \mathbf{w}_k$ span $W$ and $\mathbf{z}_1, \ldots, \mathbf{z}_l$ span $Z$, then $W$ and $Z$ are orthogonal subspaces if and only if $\langle \mathbf{w}_i, \mathbf{z}_j \rangle = 0$ for all $i = 1, \ldots, k$ and $j = 1, \ldots, l$. The proof of this lemma is left to the reader.

**Example 4.25** *Let $V = \mathbb{R}^3$ have the ordinary dot product. Then the plane $W \subset \mathbb{R}^3$ defined by the equation $2x - y + 3z = 0$ is orthogonal to the line $Z$ spanned by its normal vector $\mathbf{n} = (2, -1, 3)^T$. Indeed, every $\mathbf{w} = (x, y, z)^T \in W$ satisfies the orthogonality condition $\mathbf{w} \cdot \mathbf{n} = 2x - y + 3z = 0$, which is simply the equation for the plane.* ■

**Example 4.26** *Let $W$ be the span of $\mathbf{w}_1 = (1, -2, 0, 1)^T, \mathbf{w}_2 = (3, -5, 2, 1)^T$, and let $Z$ be the span of the vectors $\mathbf{z}_1 = (3, 2, 0, 1)^T, \mathbf{z}_2 = (1, 0, -1, -1)^T$. We find that $\mathbf{w}_1 \cdot \mathbf{z}_1 = \mathbf{w}_1 \cdot \mathbf{z}_2 = \mathbf{w}_2 \cdot \mathbf{z}_1 = \mathbf{w}_2 \cdot \mathbf{z}_2 = 0$, and so $W$ and $Z$ are orthogonal two-dimensional subspaces of $\mathbb{R}^4$ under the Euclidean dot product.* ■

Definition: The orthogonal complement of a subspace $W \subset V$, denoted $W^\perp$, is defined as the set of all vectors that are orthogonal to $W$:

$$W^\perp = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\}.$$

If $W$ is the one-dimensional subspace (line) spanned by a single vector $\mathbf{w} \neq \mathbf{0}$, then we also denote $W^\perp$ by $\mathbf{w}^\perp$. One easily checks that the orthogonal complement $W^\perp$ is also a subspace. Moreover, $W \cap W^\perp = \{\mathbf{0}\}$. Keep in mind that the orthogonal complement will depend upon which inner product is being used.

**Example 4.27** *Let* $W = \left\{ (t, 2t, 3t)^T \mid t \in \mathbb{R} \right\}$ *be the line (one-dimensional subspace) in the direction of the vector* $\mathbf{w}_1 = (1,2,3)^T \in \mathbb{R}^3$. *Under the dot product, its orthogonal complement* $W^\perp = \mathbf{w}_1^\perp$ *is the plane passing through the origin having normal vector* $\mathbf{w}_1$. *In other words,* $\mathbf{z} = (x, y, z)^T \in W^\perp$ *if and only if*

$$\mathbf{z} \cdot \mathbf{w}_1 = x + 2y + 3z = 0.$$

*Thus,* $W^\perp$ *is characterized as the solution space of this previous homogeneous linear equation, or, equivalently, the kernel of the* $1 \times 3$ *matrix* $A = \mathbf{w}_1^T = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$. *We can write the general solution in the form*

$$\mathbf{z} = \begin{pmatrix} -2y - 3z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} = y\mathbf{z}_1 + z\mathbf{z}_2,$$

*where* $y, z$ *are the free variables. The indicated vectors* $\mathbf{z}_1 = (-2, 1, 0)^T, \mathbf{z}_2 = (-3, 0, 1)^T$, *form a (non-orthogonal) basis for the orthogonal complement* $W^\perp$. ■

Proposition: Suppose that $W \subset V$ is a finite-dimensional subspace of an inner product space. Then every vector $\mathbf{v} \in V$ can be uniquely decomposed into $\mathbf{v} = \mathbf{w} + \mathbf{z}$, where $\mathbf{w} \in W$ and $\mathbf{z} \in W^\perp$.

> Proof: We let $\mathbf{w} \in W$ be the orthogonal projection of $\mathbf{v}$ onto $W$. Then $\mathbf{z} = \mathbf{v} - \mathbf{w}$ is, by definition, orthogonal to $W$ and hence belongs to $W^\perp$. Note that $\mathbf{z}$ can be viewed as the orthogonal projection of $\mathbf{v}$ onto the complementary subspace $W^\perp$ (provided it is finite-dimensional). If we are given two such decompositions, $\mathbf{v} = \mathbf{w} + \mathbf{z} = \widetilde{\mathbf{w}} + \widetilde{\mathbf{z}}$, then $\mathbf{w} - \widetilde{\mathbf{w}} = \widetilde{\mathbf{z}} - \mathbf{z}$. The left-hand side of this equation lies in $W$, while the right-hand side belongs to $W^\perp$. But, as we already noted, the only vector that belongs to both $W$ and $W^\perp$ is the zero vector. Thus, $\mathbf{w} - \widetilde{\mathbf{w}} = \mathbf{0} = \widetilde{\mathbf{z}} - \mathbf{z}$, so $\mathbf{w} = \widetilde{\mathbf{w}}$ and $\mathbf{z} = \widetilde{\mathbf{z}}$, which proves uniqueness.

As a consequence of this previous proposition, we have:

Proposition: If $W$ is a finite-dimensional subspace of an inner product space, then $\left(W^\perp\right)^\perp = W$.

In a finite-dimensional inner product space, a subspace and its orthogonal complement have complementary dimensions:

Proposition: If $W \subset V$ is a subspace with $\dim W = n$ and $\dim V = m$, then $\dim W^\perp = m - n$.

**Example 4.28** *Let* $W \subset \mathbb{R}^4$ *be the two-dimensional subspace spanned by the orthogonal vectors* $\mathbf{w}_1 = (1, 1, 0, 1)^T$ *and* $\mathbf{w}_2 = (1, 1, 1, -2)^T$. *Its orthogonal complement* $W^\perp$ *(with respect to the Euclidean dot product) is the set of all vectors* $\mathbf{v} = (x, y, z, w)^T$ *that satisfy the linear system*

$$\mathbf{v} \cdot \mathbf{w}_1 = x + y + w = 0, \quad \mathbf{v} \cdot \mathbf{w}_2 = x + y + z - 2w = 0.$$

*Applying the usual algorithm (the free variables are* $y$ *and* $w$*), we find that the solution space is spanned by*

$$\mathbf{z}_1 = (-1, 1, 0, 0)^T, \quad \mathbf{z}_2 = (-1, 0, 3, 1)^T,$$

*which form a non-orthogonal basis for $W^\perp$. An orthogonal basis*

$$\mathbf{y}_1 = \mathbf{z}_1 = (-1, 1, 0, 0)^T, \quad \mathbf{y}_2 = \mathbf{z}_2 - \frac{1}{2}\mathbf{z}_1 = \left(-\frac{1}{2}, -\frac{1}{2}, 3, 1\right)^T,$$

*for $W^\perp$ is obtained by a single Gram-Schmidt step. To decompose the vector $\mathbf{v} = (1, 0, 0, 0)^T = \mathbf{w} + \mathbf{z}$, say, we compute the two orthogonal projections: Recall (4.109)*

$$\mathbf{w} = \frac{1}{3}\mathbf{w}_1 + \frac{1}{7}\mathbf{w}_2 = \left(\frac{10}{21}, \frac{10}{21}, \frac{1}{7}, \frac{1}{21}\right)^T \in W$$

$$\mathbf{z} = \mathbf{v} - \mathbf{w} = -\frac{1}{2}\mathbf{y}_1 - \frac{1}{21}\mathbf{y}_2 = \left(\frac{11}{21}, -\frac{10}{21}, -\frac{1}{7}, -\frac{1}{21}\right)^T \in W^\perp. \qquad \blacksquare$$

### 4.5.5 Least Squares Minimization

For us, the most important case is that of a linear system

$$A\mathbf{x} = \mathbf{b} \tag{4.110}$$

consisting of $m$ equations in $n$ unknowns. In this case, the solutions may be obtained by minimizing the function

$$p(\mathbf{x}) = \|A\mathbf{x} - \mathbf{b}\|^2, \tag{4.111}$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^m$. Clearly $p(\mathbf{x})$ has a minimum value of 0, which is achieved if and only if $\mathbf{x}$ is a solution to the linear system (4.110). Of course, it is not clear that we have gained much, since we already know how to solve $A\mathbf{x} = \mathbf{b}$ by Gaussian Elimination. However, this artifice turns out to have profound consequences.

Suppose that the linear system (4.110) does not have a solution, i.e., $\mathbf{b}$ does not lie in the image of the matrix $A$. This situation is very typical when there are more equations than unknowns. Such problems arise in data fitting, when the measured data points are all supposed to lie on a straight line, say, but rarely do so exactly, due to experimental error. Although we know there is no exact solution to the system, we might still try to find an approximate solution — a vector $\mathbf{x}^\star$ that comes as close to solving the system as possible.

One way to measure closeness is by looking at the magnitude of the error as measured by the residual vector $\mathbf{r} = \mathbf{b} - A\mathbf{x}$, i.e., the difference between the right- and left-hand sides of the system. The smaller its norm $\|\mathbf{r}\| = \|A\mathbf{x} - \mathbf{b}\|$, the better the attempted solution. For the Euclidean norm, the vector $\mathbf{x}^\star$ that minimizes the squared residual norm function (4.111) is known as the least squares solution to the linear system, because $\|\mathbf{r}\|^2 = r_1^2 + \cdots + r_n^2$ is the sum of the squares of the individual error components. As before, if the linear system (4.110) happens to have an actual solution, with $A\mathbf{x}^\star = \mathbf{b}$, then $\mathbf{x}^\star$ qualifies as the least squares solution too, since in this case, $\|A\mathbf{x}^\star - \mathbf{b}\| = 0$ achieves its absolute minimum. So least squares solutions include traditional solutions as special cases.

Unlike an exact solution, the least squares minimizer depends on the choice of inner product governing the norm; thus a suitable weighted norm can be introduced to emphasize or de-emphasize the various errors. While not the only possible approach, least squares is certainly the easiest to analyze and solve, and, hence, is often the method of choice for fitting functions to experimental data and performing statistical analysis.

The following minimization problem arises in elementary geometry, although its practical implications cut a much wider swath. Given a point $\mathbf{b} \in \mathbb{R}^m$ and a subset $V \subset \mathbb{R}^m$, find the point $\mathbf{v}^\star \in V$ that is closest to $\mathbf{b}$. In other words, we seek to minimize the Euclidean distance $d(\mathbf{v}, \mathbf{b}) = \|\mathbf{v} - \mathbf{b}\|$ over all possible $\mathbf{v} \in V$.

The simplest situation occurs when $V$ is a subspace of $\mathbb{R}^m$. In this case, the closest point problem can, in fact, be reformulated as a least squares minimization problem. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis for $V$. The general element $\mathbf{v} \in V$ is a linear combination of the basis vectors. We can write the subspace elements in the form

$$\mathbf{v} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = A\mathbf{x},$$

where $A = (\mathbf{v}_1\mathbf{v}_2 \ldots \mathbf{v}_n)$ is the $m \times n$ matrix formed by the (column) basis vectors and $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ are the coordinates of $\mathbf{v}$ relative to the chosen basis. In this manner, we can identify $V$ with the image of $A$, i.e., the subspace spanned by its columns. Consequently, the closest point in $V$ to $\mathbf{b}$ is found by minimizing $\|\mathbf{v} - \mathbf{b}\|^2 = \|A\mathbf{x} - \mathbf{b}\|^2$ over all possible $\mathbf{x} \in \mathbb{R}^n$. But this is exactly the same as the least squares function (4.111)! Thus, if $\mathbf{x}^\star$ is the least squares solution to the system $A\mathbf{x} = \mathbf{b}$, then $\mathbf{v}^\star = A\mathbf{x}^\star$ is the closest point to $\mathbf{b}$ belonging to $V = \operatorname{img} A$. In this way, we have established a profound and fertile connection between least squares solutions to linear systems and the geometrical problem of minimizing distances to subspaces. And, as we shall see, the closest point $\mathbf{v} \in V$ turns out to be the orthogonal projection of $\mathbf{b}$ onto the subspace.

The simplest algebraic equations are linear systems. As such, one must thoroughly understand them before venturing into the far more complicated nonlinear realm. For minimization problems, the starting point is the quadratic function. We shall see how the problem of minimizing a general quadratic function of $n$ variables can be solved by linear algebra techniques.

Let us begin by reviewing the very simplest example: minimizing a scalar quadratic polynomial $p(x) = ax^2 + 2bx + c$ over all possible values of $x \in \mathbb{R}$. If $a > 0$, then the graph of $p$ is a parabola opening upwards, and so there exists a unique minimum value. If $a < 0$, the parabola points downwards, and there is no minimum (although there is a maximum). If $a = 0$, the graph is a straight line, and there is neither minimum nor maximum over all $x \in \mathbb{R}$ - except in the trivial case $b = 0$ also, and the function $p(x) = c$ is constant, with every $x$ qualifying as a minimum (and a maximum).

In the case $a > 0$, the minimum can be found by calculus. The critical points of a function, which are candidates for minima (and maxima), are found by setting its derivative to zero. In this case, differentiating, and solving $p'(x) = 2ax + 2b = 0$, we conclude that the only possible minimum value occurs at

$$x^\star = -\frac{b}{a}, \qquad \text{where} \qquad p\left(x^\star\right) = c - \frac{b^2}{a}. \tag{4.112}$$

Of course, one must check that this critical point is indeed a minimum, and not a maximum or inflection point. The second derivative test will show that $p''\left(x^\star\right) = 2a > 0$, and so $x^\star$ is at least a local minimum.

A more instructive approach to this problem — and one that requires only elementary algebra — is to "complete the square". We rewrite

$$p(x) = a\left(x + \frac{b}{a}\right)^2 + \frac{ac - b^2}{a}.$$

If $a > 0$, then the first term is always $\geq 0$, and, moreover, attains its minimum value 0 only at $x^\star = -b/a$. The second term is constant, and so is unaffected by the value of $x$. Thus, the global minimum of $p(x)$ is at $x^\star = -b/a$. Moreover, its minimal value equals the constant term, $p(x^\star) = c - b^2/a$, thereby reconfirming and strengthening the calculus result in (4.112).

Now that we have the one-variable case firmly in hand, let us turn our attention to the more substantial problem of minimizing quadratic functions of several variables. Thus, we seek to minimize a (real) quadratic polynomial

$$p(\mathbf{x}) = p(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} k_{ij} x_i x_j - 2 \sum_{i=1}^{n} f_i x_i + c, \tag{4.113}$$

depending on $n$ variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. The coefficients $k_{ij}, f_i$ and $c$ are all assumed to be real. Moreover, we can assume, without loss of generality, that the coefficients of the quadratic terms are symmetric: $k_{ij} = k_{ji}$. Note that $p(\mathbf{x})$ is more general than a quadratic form in that it also contains linear and constant terms. We seek a global minimum, and so the variables $\mathbf{x}$ are allowed to vary over all of $\mathbb{R}^n$.

Let us begin by rewriting the quadratic function (4.113) in a more compact matrix notation:

$$p(\mathbf{x}) = \mathbf{x}^T K \mathbf{x} - 2\mathbf{x}^T \mathbf{f} + c, \quad \mathbf{x} \in \mathbb{R}^n, \tag{4.114}$$

in which $K = (k_{ij})$ is a symmetric $n \times n$ matrix, $\mathbf{f} \in \mathbb{R}^n$ is a constant vector, and $c$ is a constant scalar.

Theorem: If $K$ is a positive definite (and hence symmetric) matrix, then the quadratic function (4.114) has a unique minimizer, which is the solution to the linear system

$$K\mathbf{x} = \mathbf{f}, \qquad \text{namely} \qquad \mathbf{x}^\star = K^{-1}\mathbf{f}. \tag{4.115}$$

The minimum value of $p(\mathbf{x})$ is equal to any of the following expressions:

$$p(\mathbf{x}^\star) = p\left(K^{-1}\mathbf{f}\right) = c - \mathbf{f}^T K^{-1} \mathbf{f} = c - \mathbf{f}^T \mathbf{x}^\star = c - (\mathbf{x}^\star)^T K \mathbf{x}^\star. \tag{4.116}$$

> Proof: First recall that positive definiteness implies that $K$ is a nonsingular matrix, and hence the linear system (4.115) has a unique solution $\mathbf{x}^\star = K^{-1}\mathbf{f}$. Then, for all $\mathbf{x} \in \mathbb{R}^n$, since $\mathbf{f} = K\mathbf{x}^\star$, it follows that
>
> $$\begin{aligned} p(\mathbf{x}) &= \mathbf{x}^T K \mathbf{x} - 2\mathbf{x}^T \mathbf{f} + c = \mathbf{x}^T K \mathbf{x} - 2\mathbf{x}^T K \mathbf{x}^\star + c \\ &= (\mathbf{x} - \mathbf{x}^\star)^T K (\mathbf{x} - \mathbf{x}^\star) + \left[c - (\mathbf{x}^\star)^T K \mathbf{x}^\star\right], \end{aligned} \tag{4.117}$$
>
> where we used the symmetry of $K = K^T$ to identify the scalar terms
>
> $$\mathbf{x}^T K \mathbf{x}^\star = \left(\mathbf{x}^T K \mathbf{x}^\star\right)^T = (\mathbf{x}^\star)^T K^T \mathbf{x} = (\mathbf{x}^\star)^T K \mathbf{x}.$$
>
> The first term in the final expression in (4.117) has the form $\mathbf{y}^T K \mathbf{y}$, where $\mathbf{y} = \mathbf{x} - \mathbf{x}^\star$. Since we assumed that $K$ is positive definite, we know that $\mathbf{y}^T K \mathbf{y} > 0$ for all $\mathbf{y} \neq \mathbf{0}$. Thus, the first term achieves its minimum value, namely 0, if and only if $\mathbf{0} = \mathbf{y} = \mathbf{x} - \mathbf{x}^\star$. Since $\mathbf{x}^\star$ is fixed, the second, bracketed, term does not depend on $\mathbf{x}$, and hence the minimizer of $p(\mathbf{x})$ coincides with the minimizer of the first term, namely $\mathbf{x} = \mathbf{x}^\star$. Moreover, the minimum value of $p(\mathbf{x})$ is equal to the constant term: $p(\mathbf{x}^\star) = c - (\mathbf{x}^\star)^T K \mathbf{x}^\star$. The alternative expressions in (4.116) follow from simple substitutions.

We are now ready to solve the geometric problem of finding the element in a prescribed subspace that lies closest to a given point. For simplicity, we work mostly with subspaces of

$\mathbb{R}^m$, equipped with the Euclidean norm and inner product, but the method extends straightforwardly to arbitrary finite-dimensional subspaces of any inner product space. However, it does not apply to more general norms not associated with inner products, such as the 1 norm, the $\infty$ norm and, in fact, the $p$ norms whenever $p \neq 2$. In such cases, finding the closest point problem is a nonlinear minimization problem whose solution requires more sophisticated analytical techniques.

Let $\mathbb{R}^m$ be equipped with an inner product $\langle \mathbf{v}, \mathbf{w} \rangle$ and associated norm $\|\mathbf{v}\|$, and let $W \subset \mathbb{R}^m$ be a subspace. Given $\mathbf{b} \in \mathbb{R}^m$, the goal is to find the point $\mathbf{w}^\star \in W$ that minimizes $\|\mathbf{w} - \mathbf{b}\|$ over all possible $\mathbf{w} \in W$. The minimal distance $d^\star = \|\mathbf{w}^\star - \mathbf{b}\|$ to the closest point is designated as the distance from the point $\mathbf{b}$ to the subspace $W$.

Of course, if $\mathbf{b} \in W$ lies in the subspace, then the answer is easy: the closest point in $W$ is $\mathbf{w}^\star = \mathbf{b}$ itself, and the distance from $\mathbf{b}$ to the subspace is zero. Thus, the problem becomes interesting only when $\mathbf{b} \notin W$.

In solving the closest point problem, the goal is to minimize the squared distance

$$\|\mathbf{w} - \mathbf{b}\|^2 = \langle \mathbf{w} - \mathbf{b}, \mathbf{w} - \mathbf{b} \rangle = \|\mathbf{w}\|^2 - 2\langle \mathbf{w}, \mathbf{b} \rangle + \|\mathbf{b}\|^2 \tag{4.118}$$

over all possible $\mathbf{w}$ belonging to the subspace $W \subset \mathbb{R}^m$. Let us assume that we know a basis $\mathbf{w}_1, \ldots, \mathbf{w}_n$ of $W$, with $n = \dim W$. Then the most general vector in $W$ is a linear combination

$$\mathbf{w} = x_1 \mathbf{w}_1 + \cdots + x_n \mathbf{w}_n \tag{4.119}$$

of the basis vectors. We substitute the formula (4.119) for $\mathbf{w}$ into the squared distance function (4.118). As we shall see, the resulting expression is a quadratic function of the coefficients $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, and so the minimum is provided by (4.115).

First, the quadratic terms come from expanding

$$\|\mathbf{w}\|^2 = \langle x_1 \mathbf{w}_1 + \cdots + x_n \mathbf{w}_n, x_1 \mathbf{w}_1 + \cdots + x_n \mathbf{w}_n \rangle = \sum_{i,j=1}^{n} x_i x_j \langle \mathbf{w}_i, \mathbf{w}_j \rangle .$$

Therefore,

$$\|\mathbf{w}\|^2 = \sum_{i,j=1}^{n} k_{ij} x_i x_j = \mathbf{x}^T K \mathbf{x}$$

where $K$ is the symmetric $n \times n$ Gram matrix (4.91) whose $(i, j)$ entry is the inner product

$$k_{ij} = \langle \mathbf{w}_i, \mathbf{w}_j \rangle \tag{4.120}$$

between the basis vectors of our subspace. Similarly,

$$\langle \mathbf{w}, \mathbf{b} \rangle = \langle x_1 \mathbf{w}_1 + \cdots + x_n \mathbf{w}_n, \mathbf{b} \rangle = \sum_{i=1}^{n} x_i \langle \mathbf{w}_i, \mathbf{b} \rangle ,$$

and so

$$\langle \mathbf{w}, \mathbf{b} \rangle = \sum_{i=1}^{n} x_i f_i = \mathbf{x}^T \mathbf{f},$$

where $\mathbf{f} \in \mathbb{R}^n$ is the vector whose $i$th entry is the inner product

$$f_i = \langle \mathbf{w}_i, \mathbf{b} \rangle \tag{4.121}$$

between the point and the subspace's basis elements. Substituting back, we conclude that the squared distance function (4.118) reduces to the quadratic function

$$p(\mathbf{x}) = \mathbf{x}^T K \mathbf{x} - 2\mathbf{x}^T \mathbf{f} + c = \sum_{i,j=1}^{n} k_{ij} x_i x_j - 2 \sum_{i=1}^{n} f_i x_i + c,$$

in which $K$ and $\mathbf{f}$ are given in (4.120) and (4.121), while $c = \|\mathbf{b}\|^2$. Since we assumed that the basis vectors $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are linearly independent, their associated Gram matrix is positive definite. Therefore, we may directly apply (4.115) to solve the closest point problem.

Theorem: Let $\mathbf{w}_1, \ldots, \mathbf{w}_n$ form a basis for the subspace $W \subset \mathbb{R}^m$. Given $\mathbf{b} \in \mathbb{R}^m$, the closest point $\mathbf{w}^\star = x_1^\star \mathbf{w}_1 + \cdots + x_n^\star \mathbf{w}_n \in W$ is unique and prescribed by the solution $\mathbf{x}^\star = K^{-1}\mathbf{f}$ to the linear system

$$K\mathbf{x} = \mathbf{f}, \tag{4.122}$$

where the entries of $K$ and $\mathbf{f}$ are given in (4.120) and (4.121). The (minimum) distance between the point and the subspace is

$$d^\star = \|\mathbf{w}^\star - \mathbf{b}\| = \sqrt{\|\mathbf{b}\|^2 - \mathbf{f}^T \mathbf{x}^\star}. \tag{4.123}$$

When the standard dot product and Euclidean norm on $\mathbb{R}^m$ are used to measure distance, the entries of the Gram matrix $K$ and the vector $\mathbf{f}$ are given by

$$k_{ij} = \mathbf{w}_i \cdot \mathbf{w}_j = \mathbf{w}_i^T \mathbf{w}_j, \qquad f_i = \mathbf{w}_i \cdot \mathbf{b} = \mathbf{w}_i^T \mathbf{b}.$$

As in (4.93), each set of equations can be combined into a single matrix equation. If $A = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_n \end{pmatrix}$ denotes the $m \times n$ matrix formed by the basis vectors, then

$$K = A^T A, \qquad \mathbf{f} = A^T \mathbf{b}, \qquad c = \mathbf{b}^T \mathbf{b} = \|\mathbf{b}\|^2. \tag{4.124}$$

A direct derivation of these equations is instructive. As

$$\mathbf{w} = x_1 \mathbf{w}_1 + \cdots + x_n \mathbf{w}_n = A\mathbf{x}$$

we have

$$\begin{aligned} \|\mathbf{w} - \mathbf{b}\|^2 = \|A\mathbf{x} - \mathbf{b}\|^2 &= (A\mathbf{x} - \mathbf{b})^T (A\mathbf{x} - \mathbf{b}) = \left(\mathbf{x}^T A^T - \mathbf{b}^T\right)(A\mathbf{x} - \mathbf{b}) \\ &= \mathbf{x}^T A^T A \mathbf{x} - 2\mathbf{x}^T A^T \mathbf{b} + \mathbf{b}^T \mathbf{b} = \mathbf{x}^T K \mathbf{x} - 2\mathbf{x}^T \mathbf{f} + c, \end{aligned}$$

thereby justifying (4.124). Thus, (4.122) and (4.123) imply that the closest point $\mathbf{w}^\star = A\mathbf{x}^\star \in W$ to $\mathbf{b}$ in the Euclidean norm is obtained by solving what are known as the *normal equations*

$$\left(A^T A\right)\mathbf{x} = A^T \mathbf{b} \tag{4.125}$$

for

$$\mathbf{x}^\star = \left(A^T A\right)^{-1} A^T \mathbf{b}, \qquad \text{giving} \qquad \mathbf{w}^\star = A\mathbf{x}^\star = A\left(A^T A\right)^{-1} A^T \mathbf{b}.$$

If, instead of the Euclidean inner product, we adopt a weighted inner product $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T C \mathbf{w}$ on $\mathbb{R}^m$ prescribed by a positive definite $m \times m$ matrix $C > 0$, then the same computations produce

$$K = A^T C A, \qquad \mathbf{f} = A^T C \mathbf{b}, \qquad c = \mathbf{b}^T C \mathbf{b} = \|\mathbf{b}\|^2.$$

The resulting formula for the weighted Gram matrix $K$ was previously derived in (4.95). In this case, the closest point $\mathbf{w}^\star \in W$ in the weighted norm is obtained by solving the *weighted normal equations*

$$A^T C A \mathbf{x} = A^T C \mathbf{b} \tag{4.126}$$

so that

$$\mathbf{x}^\star = \left(A^T C A\right)^{-1} A^T C \mathbf{b}, \quad \mathbf{w}^\star = A\mathbf{x}^\star = A\left(A^T C A\right)^{-1} A^T C \mathbf{b}. \tag{4.127}$$

Remark: The solution to the closest point problem given in (4.122) and (4.123) applies, as stated, to the more general case in which $W \subset V$ is a finite-dimensional subspace of a general inner product space $V$. The underlying inner product space $V$ can even be infinite-dimensional, as, for example, in least squares approximations in function space.

Now, consider what happens if we know an orthonormal basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of the subspace $W$. Since, by definition, $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ for $i \neq j$, while $\langle \mathbf{u}_i, \mathbf{u}_i \rangle = \|\mathbf{u}_i\|^2 = 1$, the associated Gram matrix is the identity matrix: $K = I$. Thus, in this situation, the system (4.122) reduces to simply $\mathbf{x} = \mathbf{f}$, with solution $x_i^\star = f_i = \langle \mathbf{u}_i, \mathbf{b} \rangle$, and the closest point is given by

$$\mathbf{w}^\star = x_1^\star \mathbf{u}_1 + \cdots + x_n^\star \mathbf{u}_n \quad \text{where} \quad x_i^\star = \langle \mathbf{b}, \mathbf{u}_i \rangle, \quad i = 1, \ldots, n. \tag{4.128}$$

We have already seen this formula! According to (4.108), $\mathbf{w}^\star$ is the orthogonal projection of $\mathbf{b}$ onto the subspace $W$. Thus, if we are supplied with an orthonormal basis of our subspace, we can easily compute the closest point using the orthogonal projection formula (4.128). If the basis is orthogonal, one can either normalize it or directly apply the equivalent orthogonal projection formula (4.109).

In this manner, we have established the key connection identifying the closest point in the subspace to a given vector with the orthogonal projection of that vector onto the subspace:

Theorem: Let $W \subset V$ be a finite-dimensional subspace of an inner product space. Given a point $\mathbf{b} \in V$, the closest point $\mathbf{w}^\star \in W$ coincides with the orthogonal projection of $\mathbf{b}$ onto $W$.

As we already observed, the solution to the closest point problem also solves the basic least squares minimization problem. Let us first officially define the notion of a (classical) least squares solution to a linear system.

Definition: A least squares solution to a linear system of equations

$$A\mathbf{x} = \mathbf{b} \tag{4.129}$$

is a vector $\mathbf{x}^\star \in \mathbb{R}^n$ that minimizes the squared Euclidean norm $\|A\mathbf{x} - \mathbf{b}\|^2$.

If the system (4.129) actually has a solution, then it is automatically the least squares solution. The concept of least squares solution is new only when the system does not have a solution, i.e., $\mathbf{b}$ does not lie in the image of $A$. We also want the least squares solution to be unique. As with an ordinary solution, this happens if and only if $\ker A = \{0\}$, or, equivalently, the columns of $A$ are linearly independent, or, equivalently, $\operatorname{rank} A = n$. Indeed, if $\mathbf{z} \in \ker A$, then $\widetilde{\mathbf{x}} = \mathbf{x} + \mathbf{z}$ also satisfies

$$\|A\widetilde{\mathbf{x}} - \mathbf{b}\|^2 = \|A(\mathbf{x} + \mathbf{z}) - \mathbf{b}\|^2 = \|A\mathbf{x} - \mathbf{b}\|^2,$$

and hence is also a minimum. Thus, uniqueness requires $\mathbf{z} = \mathbf{0}$.

As before, to make the connection with the closest point problem, we identify the subspace $W = \operatorname{img} A \subset \mathbb{R}^m$ as the image or column space of the matrix $A$. If the columns of $A$ are linearly independent, then they form a basis for the image $W$. Since every element of the image can be written as $\mathbf{w} = A\mathbf{x}$, minimizing $\|A\mathbf{x} - \mathbf{b}\|^2$ is the same as minimizing the distance $\|\mathbf{w} - \mathbf{b}\|$ between the point and the subspace. The solution $\mathbf{x}^\star$ to the quadratic minimization problem produces the closest point $\mathbf{w}^\star = A\mathbf{x}^\star$ in $W = \operatorname{img} A$, which is thus found using (4.122) and (4.123). In the Euclidean case, we therefore find the least squares solution by solving the normal equations given in (4.125).

Theorem: Assume that $\ker A = \{0\}$. Then the least squares solution to the linear system $A\mathbf{x} = \mathbf{b}$ under the Euclidean norm is the unique solution $\mathbf{x}^\star$ to the normal equations

$$\left(A^T A\right)\mathbf{x} = A^T\mathbf{b}, \quad \text{namely} \quad \mathbf{x}^\star = \left(A^T A\right)^{-1} A^T\mathbf{b}. \tag{4.130}$$

The least squares error is

$$\|A\mathbf{x}^\star - \mathbf{b}\|^2 = \|\mathbf{b}\|^2 - \mathbf{f}^T\mathbf{x}^\star = \|\mathbf{b}\|^2 - \mathbf{b}^T A\left(A^T A\right)^{-1} A^T\mathbf{b}. \tag{4.131}$$

Note that the normal equations (4.125) can be simply obtained by multiplying the original system $A\mathbf{x} = \mathbf{b}$ on both sides by $A^T$. In particular, if $A$ is square and invertible, then $\left(A^T A\right)^{-1} = A^{-1}\left(A^T\right)^{-1}$, and so the least squares solution formula (4.130) reduces to $\mathbf{x} = A^{-1}\mathbf{b}$, while the error formula (4.131) becomes zero. In the rectangular case — when inversion of $A$ itself is not allowed — (4.130) gives a new formula for the solution to the linear system $A\mathbf{x} = \mathbf{b}$ whenever $\mathbf{b} \in \operatorname{img} A$. An alternative approach is to use a *pseudoinverse* of a matrix.

One can extend the basic least squares method by introducing a suitable weighted norm in the measurement of the error. Let $C > 0$ be a positive definite matrix that governs the weighted norm $\|\mathbf{v}\|^2 = \mathbf{v}^T C\mathbf{v}$. In most applications, $C = \operatorname{diag}(c_1, \ldots, c_m)$ is a diagonal matrix whose entries are the assigned weights of the individual coordinates, but the method works equally well for general norms defined by positive definite matrices. The off-diagonal entries of $C$ can be used to weight cross-correlations between data values, although this extra freedom is rarely used in practice.[28] The weighted least squares solution is thus obtained by solving the corresponding weighted normal equations (4.126), as follows.

Theorem: Suppose $A$ is an $m \times n$ matrix such that $\ker A = \{\mathbf{0}\}$, and suppose $C > 0$ is any positive definite $m \times m$ matrix specifying the weighted norm $\|\mathbf{v}\|^2 = \mathbf{v}^T C\mathbf{v}$. Then the least squares solution to the linear system $A\mathbf{x} = \mathbf{b}$ that minimizes the weighted squared error $\|A\mathbf{x} - \mathbf{b}\|^2$ is the unique solution $\mathbf{x}^\star$ to the weighted normal equations

$$A^T CA\mathbf{x}^\star = A^T C\mathbf{b}, \quad \text{so that} \quad \mathbf{x}^\star = \left(A^T CA\right)^{-1} A^T C\mathbf{b}.$$

The weighted least squares error is

$$\|A\mathbf{x}^\star - \mathbf{b}\|^2 = \|\mathbf{b}\|^2 - \mathbf{f}^T\mathbf{x}^\star = \|\mathbf{b}\|^2 - \mathbf{b}^T CA\left(A^T A\right)^{-1} A^T C\mathbf{b}.$$

[28] I could not disagree with the authors more on this point. Regression with autocorrelated errors is one of the most fundamental core topics in time-series econometrics. See, e.g., Paolella, Linear Models and Time-Series Analysis, chapters 4 to 9.

# 5 Multivariate Calculus: Differentiation, Tangent Maps, and Taylor Series

The discovery in 1846 of the planet Neptune was a dramatic and spectacular achievement of mathematical astronomy. The very existence of this new member of the solar system, and its exact location, were demonstrated with pencil and paper; there was left to observers only the routine task of pointing their telescopes at the spot the mathematicians had marked. (James R. Newman)

I was brought up to believe The universe has a plan We are only human. It's not ours to understand (Rush, BU2B)

## 5.1 Sequences, Limits, Functions, and Continuity

This section reviews some key definitions we will require in the multivariate setting. We have seen a few of these definitions already in §3.1. The emphasis is on multivariate sequences and continuity of multivariate functions.

- For any $\mathbf{x} \in \mathbb{R}^n$ and $r \in \mathbb{R}_{>0}$, the *open ball of radius r around* $\mathbf{x}$ is the subset $B_r(\mathbf{x}) \subset \mathbb{R}^n$ with $B_r(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{y}\| < r\}$ (note the strict inequality), where $\|\mathbf{x}\|$ is the desired *norm* of $\mathbf{x}$ (see §3.2). When not specified, the norm is the Euclidean norm from (1.24). We will also use the calligraphic $B$, i.e., $\mathcal{B}_r(\mathbf{x})$.

- A *neighborhood* of a point $\mathbf{x} \in \mathbb{R}^n$ is a subset $A \subset \mathbb{R}^n$ such that there exists an $\epsilon > 0$ with $B_\epsilon(\mathbf{x}) \subset A$.

- If, for some $r \in \mathbb{R}_{>0}$, the set $A \subset \mathbb{R}^n$ is contained in the ball $B_r(\mathbf{0})$, then $A$ is said to be *bounded*.

- The subset $U \subset \mathbb{R}^n$ is *open in* $\mathbb{R}^n$ if, for every point $\mathbf{x} \in U$, $\exists r > 0$ such that $B_r(\mathbf{x}) \subset U$.

  We prove below in (5.1) that, for every $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$, the open ball $B_r(\mathbf{x})$ is open in $\mathbb{R}^n$. For example, the open interval $\{x \in \mathbb{R} : |x - c| < r\} = (c - r, c + r)$, $c \in \mathbb{R}$, $r \in \mathbb{R}_{>0}$, is an open set in $\mathbb{R}$, but it is not open in the plane $\mathbb{R}^2$. Likewise, the square region $S_1 = \{\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2 : |y_1| < 1, |y_2| < 1\}$ is open in $\mathbb{R}^2$, but not in $\mathbb{R}^3$.

- A set $C \subset \mathbb{R}^n$ is *closed* if its complement, $\mathbb{R}^n \setminus C$ is open. By convention, the empty set $\emptyset$ is open (indeed, every point in $\emptyset$ satisfies the requirement), so that its complement, $\mathbb{R}^n$, is closed. But, from the definition, $\mathbb{R}^n$ is open, so that $\emptyset$ is closed. This is not incorrect: sets can be open and closed (or neither). The closed interval $[a, b]$ is a closed set, as is the square region $S_2 = \{\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2 : |y_1| \leq 1,\ |y_2| \leq 1\}$.

  Below, in (5.8), after vector sequences are introduced, we state a definition of a closed set that is equivalent to its above definition in terms of open sets, but adds considerably more intuition into what a closed set represents.

- If $S$ is a subset of a set $X$, then the *complement* of $S$ in $X$ is the set $X \setminus S = \{x \in X : x \notin S\}$. We will abbreviate $X \setminus S$ as $S^{\mathrm{C}}$ if the set $X$ is understood, such as $X = \mathbb{R}^n$.

- The point $\mathbf{x} \in A \subset \mathbb{R}^n$ is an *interior point* of $A$ if $\exists r > 0$ such that $B_r(\mathbf{x}) \subset A$.

- The *interior* of $A$ is the set of all interior points of $A$, denoted $A^o$, or $\text{Int}(A)$, or $\text{int}(A)$. Observe that the biggest open set contained in any set $A \subset \mathbb{R}^n$ is $A^o$.

- The *exterior* of a set $A \subseteq \mathbb{R}^n$ is $\left(A^{\mathrm{C}}\right)^{\circ}$, and can be denoted by $\text{ext}(A)$. An exterior point of $A$ is any point $x \in \left(A^{\mathrm{C}}\right)^{\circ}$.

- The smallest closed set that contains $A$ is the *closure* of $A$, denoted $\overline{A}$; it is the set of $\mathbf{x} \in \mathbb{R}^n$ such that, $\forall r > 0$, $B_r(\mathbf{x}) \cap A \neq \emptyset$.

  The closure of a set is closed. For example, the closure of $(a, b)$, $a < b$, is $[a, b]$ (note that its complement, $(-\infty, a) \cup (b, \infty)$ is open).

  As an example in $\mathbb{R}^2$, using the sets $S_1$ and $S_2$ given above, $\overline{S}_1 = S_2$. In words, the closure of a set includes its "boundary"; see the subsequent discussion.

Definition (Boundary Point): Let $E$ be a subset of $\mathbb{R}^n$. We say that $x \in \mathbb{R}^n$ is a boundary point of $E$ if for every $r > 0$ we have both

$$B_r(x) \cap E \neq \varnothing \quad \text{and} \quad B_r(x) \cap E^{\mathrm{C}} \neq \varnothing.$$

That is, $x$ is a boundary point if every ball centered at $x$ contains both a point in $E$ and a point not in $E$.

We now give two equivalent definitions of the boundary of a set $E$, denoted $\partial E$. (The notation $\partial$ is used because it signifies a line around a region, and has nothing to do with the symbol for the partial derivative.)

Definition (Boundary, first definition): The set of all boundary points of $E$ is called the boundary of $E$, and it is denoted by $\partial E = \{x \in \mathbb{R}^n : x \text{ is a boundary point of } E\}$.

Definition (Boundary, second definition): The *boundary* of a set $E \subset \mathbb{R}^n$ is defined to be the difference between its closure and interior, i.e., $\partial E = \overline{E} - E^o$.

According to the next result, if $E$ is a subset of $\mathbb{R}^n$, then the interior, exterior, and boundary of $E$ form a partition of $\mathbb{R}^n$. That is, every point in $\mathbb{R}^n$ belongs to exactly one of these sets.

Theorem: If $E$ is a subset of $\mathbb{R}^n$, then the interior of $E$, the exterior of $E$, and the boundary of $E$ form a partition of $\mathbb{R}^n$. This means that:

(a) $\mathbb{R}^n = E^\circ \cup \left(E^{\mathrm{C}}\right)^{\circ} \cup \partial E$, and

(b) $E^\circ$, $\left(E^{\mathrm{C}}\right)^{\circ}$, and $\partial E$ are disjoint.

> Proof: (a) Choose any point $x \in \mathbb{R}^n$, We will show that $x$ belongs to at least one of $E^\circ$, $\left(E^{\mathrm{C}}\right)^{\circ}$, or $\partial E$.
>
> If $x$ does belong to either $E^\circ$ or $\left(E^{\mathrm{C}}\right)^{\circ}$, then we are done. So, suppose that $x$ does not belong to either the interior or the exterior of $E$. We must show that this implies that $x$ belongs to the boundary of $E$.
>
> Since $x$ is not an interior point, no matter what radius $r > 0$ that we consider, it is not true that $B_r(x)$ is a subset of $E$. Hence $B_r(x)$ must contain some point $z$ from $E^{\mathrm{C}}$. Similarly, since $x$ is not an exterior point, no matter what $r > 0$ that we choose, $B_r(x)$ is not entirely contained in $E^{\mathrm{C}}$. Hence $B_r(x)$ must contain some point $y$ from $E$. Thus every open ball centered at $x$ contains both a point from $E$ and a point from $E^{\mathrm{C}}$. By definition, this tells us that $x$ is a boundary point of $E$, and hence belongs to $\partial E$.

The above work shows that $\mathbb{R}^n \subseteq E^\circ \cup \left(E^{\mathrm{C}}\right)^\circ \cup \partial E$. The opposite inclusion holds simply by definition, so we conclude that $\mathbb{R}^n = E^\circ \cup \left(E^{\mathrm{C}}\right)^\circ \cup \partial E$.

(b) We have to prove that the intersection of any two of $E^\circ$, $\left(E^{\mathrm{C}}\right)^\circ$, and $\partial E$ is empty. Since $E^\circ \subseteq E$ and $\left(E^{\mathrm{C}}\right)^\circ \subseteq E^{\mathrm{C}}$, we see immediately that these two sets are disjoint. Suppose that $x$ belonged to both $E^\circ$ and $\partial E$. Then $x$ is an interior point, so there is some $r > 0$ such that $B_r(x) \subseteq E$. But $x$ is also a boundary point, so by definition $B_r(x)$ must contain some point not in $E$. This is a contradiction, so we conclude that $E^\circ \cap \partial E = \varnothing$. Similarly, the exterior and the boundary of $E$ are disjoint.

To summarize, if $E$ is a subset of $\mathbb{R}^n$ then every point $x \in \mathbb{R}^n$ belongs to exactly one of:

- the interior of $E$ (the set $E^\circ$),
- the exterior or $E$ (the set $\left(E^{\mathrm{C}}\right)^\circ$), or
- the boundary of $E$ (the set $\partial E$).

In particular, this implies that the boundary of $E$ is the complement of the union of the interior and exterior of $E$:

$$\partial E = \left(E^\circ \cup \left(E^{\mathrm{C}}\right)^\circ\right)^{\mathrm{C}}.$$

Since the interior and the exterior of $E$ are both open, their union is open, and therefore its complement, which is $\partial E$, is closed.

Remark: Recall that we defined Lebesgue measure zero of a subset of $\mathbb{R}$ in (1.5). (See §6.2.3 for the multivariate setting.) It is not true that the boundary of every set $E \subseteq \mathbb{R}$ has measure zero. For example, the set of rationals $\mathbb{Q}$ has measure zero, but its boundary is $\partial\mathbb{Q} = \mathbb{R}$, whose measure is infinite. First note that $\operatorname{int}\mathbb{Q} = \emptyset$, because, from §3.1, $\mathbb{Q}$ is dense in $\mathbb{R}$. That is, if we take any $x \in \mathbb{Q}$ and $r \in \mathbb{R}$, $r > 0$, then any open ball (i.e., open interval) centered at $x$ will contain irrationals, and hence it will not lie entirely in $\mathbb{Q}$. Next, $\partial\mathbb{Q} = \mathbb{R}$ from the definition that the boundary is the closure minus the interior, and the closure is $\mathbb{R}$.

**Example 5.1** *For points $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$, the* line segment *from $\mathbf{x}_1$ to $\mathbf{x}_2$ is the set of points*

$$\mathbf{x}_1 + t(\mathbf{x}_2 - \mathbf{x}_1) = t\mathbf{x}_2 + (1-t)\mathbf{x}_1, \quad 0 \le t \le 1.$$

*For point $\mathbf{c} \in \mathbb{R}^n$ and $r > 0$, let $B_r(\mathbf{c})$ be the open ball of radius $r$ around $\mathbf{c}$. It should be geometrically obvious that, if $\mathbf{x}_1, \mathbf{x}_2 \in B_r(\mathbf{c})$, then so are all the points on the line segment from $\mathbf{x}_1$ to $\mathbf{x}_2$. To see this algebraically, let $\mathbf{x} = \mathbf{x}(t) = t\mathbf{x}_2 + (1-t)\mathbf{x}_1$ for $0 \le t \le 1$, and use the triangle inequality (1.23) to get*

$$\begin{aligned}
\|\mathbf{x} - \mathbf{c}\| &= \|t\mathbf{x}_2 + (1-t)\mathbf{x}_1 - t\mathbf{c} - (1-t)\mathbf{c}\| \\
&= \|t(\mathbf{x}_2 - \mathbf{c}) + (1-t)(\mathbf{x}_1 - \mathbf{c})\| \\
&\le t\,\|\mathbf{x}_2 - \mathbf{c}\| + (1-t)\,\|\mathbf{x}_1 - \mathbf{c}\| \\
&< t \cdot r + (1-t) \cdot r = r.
\end{aligned}$$

*As $B_r(\mathbf{c})$ is open, $\|\mathbf{x} - \mathbf{c}\|$ is strictly less than $r$, though* $\sup\|\mathbf{x} - \mathbf{c}\| = r$. ■

The following presentation of the above mentioned result is from Fitzpatrick, p. 284.

Theorem:

$$\textbf{Every open ball in } \mathbb{R}^n \textbf{ is open in } \mathbb{R}^n. \tag{5.1}$$

Proof: Let $\mathbf{u}$ be a point in $\mathbb{R}^n$ and let $r$ be a positive real number. Consider the open ball $\mathcal{B}_r(\mathbf{u})$. We must show that every point in $\mathcal{B}_r(\mathbf{u})$ is an interior point of $\mathcal{B}_r(\mathbf{u})$. See Figure 38. Let $\mathbf{v}$ be a point in $\mathcal{B}_r(\mathbf{u})$. Define $R = r - \operatorname{dist}(\mathbf{u}, \mathbf{v})$ and observe that $R$ is positive. We claim that

$$\mathcal{B}_R(\mathbf{v}) \subseteq \mathcal{B}_r(\mathbf{u}). \tag{5.2}$$

Indeed, if $\mathbf{w}$ is in $\mathcal{B}_R(\mathbf{v})$, then $\operatorname{dist}(\mathbf{w}, \mathbf{v}) < R = r - \operatorname{dist}(\mathbf{u}, \mathbf{v})$, so, using the triangle inequality, we have

$$\begin{aligned} \operatorname{dist}(\mathbf{w}, \mathbf{u}) &\leq \operatorname{dist}(\mathbf{w}, \mathbf{v}) + \operatorname{dist}(\mathbf{v}, \mathbf{u}) \\ &< [r - \operatorname{dist}(\mathbf{u}, \mathbf{v})] + \operatorname{dist}(\mathbf{v}, \mathbf{u}) = r. \end{aligned}$$

Thus, the inclusion (5.2) holds; so $\mathbf{v}$ is an interior point of $\mathcal{B}_r(\mathbf{u})$.

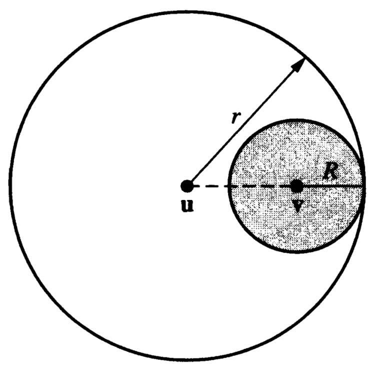



Figure 38: From Fitzpatrick, p. 284. $R = r - \operatorname{dist}(\mathbf{u}, \mathbf{v})$. $\mathcal{B}_R(\mathbf{v}) \subseteq \mathcal{B}_r(\mathbf{u})$ if $R = r - \operatorname{dist}(\mathbf{u}, \mathbf{v})$.

The next result (5.3) offers further practice with the above definitions, and we will need it directly after to prove (5.4), which in turn is used to prove result (5.143). The following two theorems are standard results; I took them from Sasane, A Friendly Approach to Functional Analysis, 2017, pp. 268, 269. The reference to "normed space", for our purposes, can be taken to be the usual vector space in $\mathbb{R}^n$, with the usual vector norm; see the definition in §4.5.1 and §3.2.

Theorem: Let $(X, \|\cdot\|)$ be a normed space, $x \in X$ and $r > 0$. Prove that

$$\overline{B(x,r)} := \{y \in X : \|y - x\| \leqslant r\} \textbf{ is a closed set.} \tag{5.3}$$

Proof: Consider the closed ball $\overline{B(x,r)} = \{y \in X : \|x - y\| \leqslant r\}$ in $X$. To show that $\overline{B(x,r)}$ is closed, we'll show its complement, $U := \{y \in X : \|x - y\| > r\}$, open. If $y \in U$, then $\|x - y\| > r$. Define $r' = \|x - y\| - r > 0$. We claim that $B(y, r') \subset U$. Let $z \in B(y, r')$. Then $\|z - y\| < r' = \|x - y\| - r$ and so, from the reverse triangle (1.21) with $a = x - y$ and $b = y - z$,

$$\|x - z\| \geqslant \|x - y\| - \|y - z\| > \|x - y\| - (\|x - y\| - r) = r.$$

Hence $z \in U$.

Theorem: The unit sphere is closed, i.e.,

$$\mathbb{S} := \{\mathbf{x} \in X : \|\mathbf{x}\| = 1\} \textbf{ is closed.} \tag{5.4}$$

Proof: We know from (5.1) that the interior of $\mathbb{S}$, namely the open ball $B(\mathbf{0}, 1) = \{\mathbf{x} \in X : \|\mathbf{x}\| < 1\}$ is open. Also, it follows from (5.3) that the exterior of the closed ball $\overline{B(\mathbf{0}, 1)}$, namely the set $U = \{\mathbf{x} \in X : \|\mathbf{x}\| > 1\}$ is open as well. Thus, the complement of $\mathbb{S}$, being the union of the two open sets $B(\mathbf{0}, 1)$ and $U$, is open. Consequently, $\mathbb{S}$ is closed.

We continue now with essential concepts and results.

Definition: A (multivariate, or vector) sequence is a *mapping* $\mathbf{f} : \mathbb{N} \to \mathbb{R}^n$ with $k$th term $\mathbf{f}(k)$, $k \in \mathbb{N}$. Many authors reserve the word *function* for the case when $n = 1$. As in the univariate case, the more common notation for sequence $\mathbf{a}_1 = \mathbf{f}(1)$, $\mathbf{a}_2 = \mathbf{f}(2), \ldots$ is $\{\mathbf{a}_k\}$. The $i$th component of $\mathbf{a}_k$ is denoted by $(\mathbf{a}_k)_i$, $i = 1, \ldots, n$. For sequence $\{\mathbf{a}_k\}$ and set $S \subset \mathbb{R}^n$, the notation $\{\mathbf{a}_k\} \subset S$ indicates that, $\forall k \in \mathbb{N}$, $\mathbf{a}_k \in S$.

Definition: The sequence $\{\mathbf{a}_k\} \subset \mathbb{R}^n$ converges to $\mathbf{a} \in \mathbb{R}^n$ if, $\forall \epsilon > 0$, $\exists K \in \mathbb{N}$ such that, $\forall k > K$, $\|\mathbf{a}_k - \mathbf{a}\| < \epsilon$. Point $\mathbf{a}$ is the *limit* of $\{\mathbf{a}_k\}$ if $\{\mathbf{a}_k\}$ converges to $\mathbf{a}$, in which case one writes $\lim_{k\to\infty} \mathbf{a}_k = \mathbf{a}$. As in the univariate case, if the limit exists, then it is unique.

Theorem: In order for $\lim_{k\to\infty} \mathbf{a}_k = \mathbf{a} = (a_1, \ldots, a_n)$ to hold, it is necessary and sufficient that $\lim_{k\to\infty} (\mathbf{a}_k)_i = a_i$, $i = 1, \ldots, n$.

Definition: For each index $i$ with $1 \leq i \leq n$, we define the $i$th component projection function $p_i : \mathbb{R}^n \to \mathbb{R}$ by

$$p_i(\mathbf{u}) \equiv u_i, \qquad \text{for } \mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n. \tag{5.5}$$

It follows directly from this definition that, for $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{u} = (p_1(\mathbf{u}), \ldots, p_n(\mathbf{u}))$, so a point in $\mathbb{R}^n$ is completely determined by the values of the component projection functions at that point.

Definition: A sequence of points $\{\mathbf{u}_k\}$ in $\mathbb{R}^n$ is said to converge componentwise to the point $\mathbf{u}$ in $\mathbb{R}^n$ provided that, for each index $i$ with $1 \leq i \leq n$,

$$\lim_{k\to\infty} p_i(\mathbf{u}_k) = p_i(\mathbf{u}).$$

Theorem (The Componentwise Convergence Criterion): Let $\{\mathbf{u}_k\}$ be a sequence in $\mathbb{R}^n$ and let $\mathbf{u}$ be a point in $\mathbb{R}^n$. Then

$$\mathbf{u}_k \to \mathbf{u} \iff \{\mathbf{u}_k\} \textbf{ converges componentwise to } \mathbf{u}. \tag{5.6}$$

Proof: First we suppose that the sequence $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$. Fix an index $i$ with $1 \leq i \leq n$. Then

$$0 \leq |p_i(\mathbf{u}_k) - p_i(\mathbf{u})| = |p_i(\mathbf{u}_k - \mathbf{u})| \leq \|\mathbf{u}_k - \mathbf{u}\| \qquad \text{for every index } k. \tag{5.7}$$

Since, by definition, the sequence of real numbers $\{\|\mathbf{u}_k - \mathbf{u}\|\}$ converges to 0, it follows from (2.26) that

$$0 \leq \lim_{k\to\infty} |p_i(\mathbf{u}_k) - p_i(\mathbf{u})| \leq \lim_{k\to\infty} \|\mathbf{u}_k - \mathbf{u}\| = 0;$$

that is, the sequence $\{p_i(\mathbf{u}_k)\}$ converges to $p_i(\mathbf{u})$. Thus, $\{\mathbf{u}_k\}$ converges componentwise to $\mathbf{u}$.

To prove the converse, suppose that the sequence $\{\mathbf{u}_k\}$ converges componentwise to $\mathbf{u}$. Then, by definition, for each index $i$ with $1 \leq i \leq n$, $\lim_{k\to\infty} p_i(\mathbf{u}_k - \mathbf{u}) = 0$. But then by the addition and product properties of convergent real sequences, namely (2.23) and (2.24), respectively, it follows that

$$\lim_{k\to\infty}\left[\left(p_1\left(\mathbf{u}_k-\mathbf{u}\right)\right)^2+\cdots+\left(p_n\left(\mathbf{u}_k-\mathbf{u}\right)\right)^2\right]=0.$$

This last assertion means precisely that $\lim_{k\to\infty}\|\mathbf{u}_k-\mathbf{u}\|^2=0$, and hence, by the continuity of the square root function and (2.37), $\lim_{k\to\infty}\|\mathbf{u}_k-\mathbf{u}\|=0$; that is, the sequence $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$.

Definition: A point $\mathbf{a}\in\mathbb{R}^n$ is a *limit point* of $S$ if, $\forall r>0$, $(B_r(\mathbf{a})\setminus\mathbf{a})\cap S\neq\emptyset$. In this setting, equivalent terms for limit point are *cluster point* and *accumulation point*.

Definition: $\{\mathbf{a}_k\}$ is a *Cauchy sequence* if, for a given $\epsilon>0$, $\exists\, N\in\mathbb{N}$ such that $\forall n,m\geq N$, $\|\mathbf{a}_m-\mathbf{a}_n\|<\epsilon$.

As expected, (3.60) generalizes:

Theorem: Sequence $\{\mathbf{a}_k\}$ converges iff $\{\mathbf{a}_k\}$ is a Cauchy sequence.

Definition: As in the univariate case, the series $\sum_{k=1}^{\infty}\mathbf{a}_k$ is convergent if the sequence of partial sums, $\{\mathbf{s}_p\}$, where $\mathbf{s}_p=\sum_{k=1}^{p}\mathbf{a}_k$, is convergent.

Consider the function $f:\mathbb{N}\to I$, $I=(0,1]$, given by $f(k)=1/k$. Observe that $I$ is neither open nor closed. Clearly, $\lim_{k\to\infty}a_k=0$, and $0\notin I$. However, $\lim_{k\to\infty}a_k$ is contained in the closure of $I$, which is the closed set $[0,1]$. With this concept in mind, the following basic result of analysis should appear quite reasonable.

Theorem:

$$\text{The set } C\subset\mathbb{R}^n \text{ is } \textit{closed} \text{ iff it contains all its limit points.} \tag{5.8}$$

Definition: Given a mapping $F:A\to\mathbb{R}^m$, $A\subset\mathbb{R}^n$, and an index $i$, $1\leq i\leq m$, we define the function $F_i:A\to\mathbb{R}$ to be the composition of $F:A\to\mathbb{R}^m$ with the $i$th *component projection*, where the latter is the function from $A$ to $\mathbb{R}$ that returns the $i$th value of $F$, $1\leq i\leq m$. We call the function $F_i:A\to\mathbb{R}$ the $i$th component function of the mapping $F:A\to\mathbb{R}^m$. Thus,

$$F(\mathbf{u})=\left(F_1(\mathbf{u}),\ldots,F_m(\mathbf{u})\right),\qquad\text{for }\mathbf{u}\in A, \tag{5.9}$$

and the mapping $F:A\to\mathbb{R}^m$ is said to be represented by its component functions as

$$F=(F_1,\ldots,F_m):A\to\mathbb{R}^m. \tag{5.10}$$

For example, let $\mathcal{O}$ be the set of all nonzero points in $\mathbb{R}^n$. Define the mapping $F:\mathcal{O}\to\mathbb{R}^n$ by $F(\mathbf{u})=\mathbf{u}/\|\mathbf{u}\|^2$, $\mathbf{u}\in\mathcal{O}$. Then the representation of the mapping in component functions is

$$F(\mathbf{u})=\left(u_1/\|\mathbf{u}\|^2,\ldots,u_n/\|\mathbf{u}\|^2\right),\quad \mathbf{u}\in\mathcal{O}.$$

For $n=3$, this component representation can be written as

$$F(x,y,z)=\left(\frac{x}{x^2+y^2+z^2},\frac{y}{x^2+y^2+z^2},\frac{z}{x^2+y^2+z^2}\right),\quad (x,y,z)\in\mathcal{O}.$$

We now turn to limits of multivariate functions.

Definition: Let $\mathbf{f} : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a mapping, and let $\mathbf{x}_0 \in \overline{A}$ be a point in the closure of $A$. Then $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{b}$ if, $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$\|\mathbf{x} - \mathbf{x}_0\| < \delta, \;\; \mathbf{x} \in A \quad \Longrightarrow \quad \|\mathbf{f}(\mathbf{x}) - \mathbf{b}\| < \epsilon. \tag{5.11}$$

We can state limit result (5.11) also in terms of sequences. We do so for $m = 1$, and allow the reader to state the general $m$ case. This generalizes the result in (2.22). We use the notation $\mathrm{dist}(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|$.

Theorem: For $A \subset \mathbb{R}^n$, let $\mathbf{x}_*$ be a limit point of $A$. For a function $f : A \to \mathbb{R}$ and $\ell \in \mathbb{R}$, the following two assertions are equivalent:

i. $\lim_{\mathbf{x}\to\mathbf{x}_*} f(\mathbf{x}) = \ell$. That is, $\forall \{\mathbf{x}_k\} \subset A \setminus \{\mathbf{x}_*\}$,

$$\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}_* \quad \Longrightarrow \quad \lim_{k\to\infty} f(\mathbf{x}_k) = \ell. \tag{5.12}$$

ii. $\forall \epsilon > 0$, $\exists \delta$ such that

$$|f(\mathbf{x}) - \ell| < \epsilon \quad \text{if } \mathbf{x} \text{ is in } A \setminus \{\mathbf{x}_*\} \text{ and } \mathrm{dist}(\mathbf{x}, \mathbf{x}_*) < \delta. \tag{5.13}$$

A proof can be found in Fitzpatrick, Advanced Calculus (2009, Thm 13.7).

Analogous to the result in (2.23), in which $n = m = 1$, we have the following.

Theorem: Let $\mathbf{f}, \mathbf{g} : A \subset \mathbb{R}^n \to \mathbb{R}^m$. Assume for $\mathbf{x}_0 \in A$ that $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{g}(\mathbf{x})$ exist (which means, exists in $\mathbb{R}^m$). Then, for constant values $k_1, k_2 \in \mathbb{R}$, limits satisfy linearity and homogeneity, i.e., mixing the two,

$$\lim_{\mathbf{x}\to\mathbf{x}_0} (k_1\mathbf{f} + k_2\mathbf{g})(\mathbf{x}) = k_1 \lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) + k_2 \lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{g}(\mathbf{x}), \tag{5.14}$$

which means functions with limits at $\mathbf{x}_0 \in A$ form a vector space. In the $m = 1$ case,

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \big(k_1 f(\mathbf{x}) + k_2 g(\mathbf{x})\big) = k_1 \lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) + k_2 \lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}). \tag{5.15}$$

Theorem: Let $\mathbf{f}, \mathbf{g} : A \subset \mathbb{R}^n \to \mathbb{R}^m$. Analogous to limits of products of functions, each of which has a limit, we have

$$\lim_{\mathbf{x}\to\mathbf{x}_0} (\mathbf{f} \cdot \mathbf{g})(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) \cdot \lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{g}(\mathbf{x}), \tag{5.16}$$

where, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i$ is the dot product of $\mathbf{x}$ and $\mathbf{y}$, as in (4.1).

Theorem: Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$. If $\mathbf{f} : A \to \mathbb{R}^m$ and $\mathbf{g} : B \to \mathbb{R}^p$ such that $\mathbf{f}(A) \subset B$, then the composite function $\mathbf{g} \circ \mathbf{f}$ is well-defined. If $\mathbf{y}_0 := \lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x})$ and $\lim_{\mathbf{y}\to\mathbf{y}_0} \mathbf{g}(\mathbf{y})$ both exist, then $\lim_{\mathbf{x}\to\mathbf{x}_0} (\mathbf{g} \circ \mathbf{f})(\mathbf{x}) = \lim_{\mathbf{y}\to\mathbf{y}_0} \mathbf{g}(\mathbf{y})$.

Theorem: Let $\mathbf{f} : A \to \mathbb{R}^m$ with $A \subset \mathbb{R}^n$. Paralleling the univariate case (2.37), mapping $\mathbf{f}$ is *continuous* at $\mathbf{a} \in A$ if

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}\left(\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{x}\right) = \mathbf{f}(\mathbf{a}). \tag{5.17}$$

Equivalently, and as proven in the univariate case in §2.1, $\mathbf{f}$ is continuous at $\mathbf{a} \in A$ if, for a given $\epsilon > 0$, $\exists \delta > 0$ such that, if $\|\mathbf{x} - \mathbf{a}\| < \delta$ and $\mathbf{x} \in A$, then $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})\| < \epsilon$. We state this important result yet more explicitly: From Fitzpatrick (2009, Thm 11.11), who uses the notation $\mathrm{dist}(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|$:

Theorem: Let $A$ be a subset of $\mathbb{R}^n$ that contains the point $\mathbf{u}$. Then the following two assertions about a mapping $F : A \to \mathbb{R}^m$ are equivalent:

i. Mapping $F : A \to \mathbb{R}^m$ is continuous at the point $\mathbf{u}$; that is, for a sequence $\{\mathbf{u}_k\} \subset A$,

$$\lim_{k\to\infty} \mathrm{dist}\left(\mathbf{u}_k, \mathbf{u}\right) = 0 \implies \lim_{k\to\infty} \mathrm{dist}\left(F\left(\mathbf{u}_k\right), F(\mathbf{u})\right) = 0. \tag{5.18}$$

ii. For each positive number $\epsilon$ there is a positive number $\delta$ such that, for a point $\mathbf{v}$ in $A$,

$$\mathrm{dist}(\mathbf{v}, \mathbf{u}) < \delta \implies \mathrm{dist}(F(\mathbf{v}), F(\mathbf{u})) < \epsilon. \tag{5.19}$$

Definition: If $\mathbf{f}$ is continuous at every point in its domain $A$, then $\mathbf{f}$ is said to be *continuous*, and we write $\mathbf{f} \in \mathcal{C}^0$ or, more accurately, $\mathbf{f} \in \mathcal{C}^0(A)$.

Definition: Mapping $\mathbf{f}$ is *uniformly continuous* on subset $S \subset A$ if: for a given $\epsilon > 0$, $\exists \delta > 0$ such that, if $\mathbf{x}, \mathbf{y} \in S$, and $\|\mathbf{x} - \mathbf{y}\| < \delta$, then $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| < \epsilon$.

Theorem: Let $\mathbf{f}, \mathbf{g} : A \subset \mathbb{R}^n \to \mathbb{R}^m$, and $k_1, k_2 \in \mathbb{R}$. Similar to the above results (5.14) and (5.16) on limits, if $\mathbf{f}$ and $\mathbf{g}$ are continuous at $\mathbf{x}_0 \in A$, then so are $k_1\mathbf{f} + k_2\mathbf{g}$ and $\mathbf{f} \cdot \mathbf{g}$ at $\mathbf{x}_0$. If $\mathbf{f}$ and $\mathbf{g}$ are continuous (meaning, as stated above, continuous at all points in their domain), then $k_1\mathbf{f} + k_2\mathbf{g}$ is continuous. This means that continuous functions (with the same domain and range) form a vector space.

Just as we have the Componentwise Convergence Criterion for the convergence of sequences in Euclidean space, we also have the following simple, useful criterion for the continuity of a mapping.

Theorem (The Componentwise Continuity Criterion): Let $A$ be a subset of $\mathbb{R}^n$ that contains the point $\mathbf{u}$ and consider the mapping

$$F = (F_1, \ldots, F_m) : A \to \mathbb{R}^m.$$

Then the mapping $F : A \to \mathbb{R}^m$ is continuous at $\mathbf{u}$ if and only if each of its component functions $F_i : A \to \mathbb{R}$ is continuous at $\mathbf{u}$. In short,

$$F : A \to \mathbb{R}^m \textbf{ is continuous at } \mathbf{u} \iff F_i : A \to \mathbb{R} \textbf{ is continuous at } \mathbf{u}. \tag{5.20}$$

> Proof: This result follows immediately from the Componentwise Convergence Criterion (5.6) since, if $\{\mathbf{u}_k\}$ is a sequence in $A$ that converges to the point $\mathbf{u}$, then the image sequence $\{F(\mathbf{u}_k)\}$ converges to $F(\mathbf{u})$ if and only if for each index $i$ with $1 \le i \le m$, the sequence $\{F_i(\mathbf{u}_k)\}$ converges to $F_i(\mathbf{u})$.

Again paralleling the univariate case, we need to address continuity of the composition of (two) functions.

Definition: Given a mapping $F : A \to \mathbb{R}^m$, if $B$ is a subset of the domain $A$, the *image of the set $B$ under the mapping $F : A \to \mathbb{R}^m$*, denoted by $F(B)$, is defined as

$$F(B) \equiv \{\mathbf{v} \text{ in } \mathbb{R}^m \mid \mathbf{v} = F(\mathbf{u}) \quad \text{for some } \mathbf{u} \text{ in } B\}.$$

Theorem: Let $A$ be a subset of $\mathbb{R}^n$ that contains the point $\mathbf{u}$. Suppose that the mapping $G : A \to \mathbb{R}^m$ is continuous at the point $\mathbf{u}$. Let $B$ be a subset of $\mathbb{R}^m$ with $G(A) \subseteq B$ and suppose that the mapping $H : B \to \mathbb{R}^k$ is continuous at the point $G(\mathbf{u})$. Then the composition $H \circ G : A \to \mathbb{R}^k$ is continuous at $\mathbf{u}$.

Proof: Fitzpatrick (2009, Thm 11.5) Let $\{\mathbf{u}_k\}$ be a sequence in $A$ that converges to the point $\mathbf{u}$. Since the mapping $G : A \to \mathbb{R}^m$ is continuous at $\mathbf{u}$, it follows that the image sequence $\{G(\mathbf{u}_k)\}$ converges to $G(\mathbf{u})$. But then $\{G(\mathbf{u}_k)\}$ is a sequence in $B$ that converges to the point $G(\mathbf{u})$. The continuity of the mapping $H : B \to \mathbb{R}^k$ at the point $G(\mathbf{u})$ implies that the sequence $\{H(G(\mathbf{u}_k))\}$ converges to $H(G(\mathbf{u}))$; that is, the sequence $\{(H \circ G)(\mathbf{u}_k)\}$ converges to $(H \circ G)(\mathbf{u})$.

Theorem (Continuity of the norm): Define the function $f : \mathbb{R}^n \to \mathbb{R}$ by $f(\mathbf{u}) = \|\mathbf{u}\|$, for $\mathbf{u} \in \mathbb{R}^n$. Then the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous.

Proof: Recall the definition of projection functions in (5.5), and observe that, with $p_i(\mathbf{u}) = u_i$, $i = 1, \ldots, n$,

$$f = h \circ (p_1 p_1 + \cdots + p_n p_n) : \mathbb{R}^n \to \mathbb{R},$$

where $h(x) = \sqrt{x}$ for $x \geq 0$. Since products, sums, and compositions of continuous maps are again continuous, it follows that the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous. Thus, for $\mathbf{x}_*$ a point in $\mathbb{R}^n$,

$$\lim_{\mathbf{x} \to \mathbf{x}_*} \|\mathbf{x}\| = \|\mathbf{x}_*\| . \tag{5.21}$$

Definition: Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$. We say that $f$ has a *relative maximum* at $\mathbf{a} \in A$ if there exists an $n$-ball $U \subset A$ such that $f(\mathbf{a}) \geq f(\mathbf{x})$ for all $\mathbf{x} \in U$. The definition for a relative minimum is analogous.

Theorem: Similar to (2.59) in the univariate case, if $A \subset \mathbb{R}^n$ is closed and bounded, and $f : A \to \mathbb{R}$ is continuous, then $f$ takes on minimum and maximum values. That is,

$$\exists \mathbf{a}, \mathbf{b} \in A \textbf{ such that, } \forall \mathbf{x} \in A\textbf{,}\ f(\mathbf{a}) \leq f(\mathbf{x}) \leq f(\mathbf{b}). \tag{5.22}$$

We now consider some examples of determining the existence of limits in the multivariate setting. This is often more involved than in the univariate case. As an example of the latter, let $f(x) = 1/x$ for $x \in D = \mathbb{R} \setminus \{0\}$. It is easy to see that $\lim_{x \to 0} f(x)$ does not exist, though one-sided limits do exist in $\mathbb{X}$. Similar phenomena exist in the multivariate case. The next example, perhaps common to every textbook, illustrates an idea of how to show that the limit at a particular point does *not* exist. More work and different ideas, to prove that the limit exists, are required. This will be explored in Examples 5.4 and 5.5.

**Example 5.2** *Let $f : A \to \mathbb{R}$ with $A = \mathbb{R}^2 \setminus \mathbf{0}$ and $f(\mathbf{x}) = x_1 x_2 / (x_1^2 + x_2^2)$. To see that $\lim_{\mathbf{x} \to \mathbf{0}} f(\mathbf{x})$ does not exist, set $x_2(x_1) = k x_1$ for some fixed $k \in \mathbb{R}$ so that $\lim_{x_1 \to 0} x_2(x_1) = 0$ and $f(\mathbf{x}) = f(x_1, x_2(x_1)) = f(x_1, k x_1) = k x_1^2 / (x_1^2 + k^2 x_1^2) = k/(1 + k^2)$. Thus, along the line $x_2 = k x_1$, $\lim_{\mathbf{x} \to \mathbf{0}} f(\mathbf{x}) = k/(1 + k^2)$, i.e., it depends on the choice of $k$, showing that $\lim_{\mathbf{x} \to \mathbf{0}} f(\mathbf{x})$ depends on the path that $\mathbf{x}$ takes towards zero. Thus, $\lim_{\mathbf{x} \to \mathbf{0}} f(\mathbf{x})$ cannot exist.*

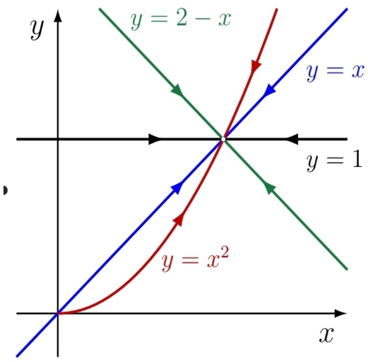
*Another way to see this is to first observe that the sequence $\{(1/k, 1/k)\}$ converges to the point $(0, 0)$; and as $f(1/k, 1/k) = 1/2$ for each $k \in \mathbb{N}$, it follows that the image sequence $\{f(1/k, 1/k)\}$ converges to $1/2$. On the other hand, the sequence $\{(1/k, 0)\}$ also converges to the point $(0, 0)$, and as $f(1/k, 0) = 0$ for each $k \in \mathbb{N}$, it follows that the image sequence $\{f(1/k, 0)\}$ converges to 0. Thus, $\lim_{\mathbf{x} \to \mathbf{0}} f(\mathbf{x})$ does not exist.* ■

**Example 5.3** *As in Miklavcic, An Illustrative Guide to Multivariable and Vector Calculus (2020, p. 56), consider*

$$\lim_{(x,y)\to(1,1)} \frac{x-y}{x-1}.$$

*This is of the form* $0/0$*, so the function is undefined at* $(1,1)$*. Evaluating the limit by approaching the point* $(1,1)$ *along four different paths, we obtain*

$$\begin{aligned}
y = x : \lim_{(1,1)} \frac{x-y}{x-1} &= \lim_{(1,1)} \frac{0}{x-1} = 0. \\
y = 2-x : \lim_{(1,1)} \frac{x-y}{x-1} &= \lim_{(1,1)} \frac{2(x-1)}{x-1} = 2. \\
y = x^2 : \lim_{(1,1)} \frac{x-y}{x-1} &= \lim_{(1,1)} \frac{x-x^2}{x-1} = -1. \\
y = 1 : \lim_{(1,1)} \frac{x-y}{x-1} &= \lim_{(1,1)} \frac{x-1}{x-1} = 1,
\end{aligned}$$




*as shown in the accompanying figure. The fact that at least two paths result in a different limit indicate that the function is not continuous at* $(1,1)$*.* ■

**Example 5.4** *Consider the following limit:*

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}.$$

*Define* $f(x,y) = x^3/(x^2+y^2)$ *if* $(x,y) \neq (0,0)$*. By observing that, for* $k \in \mathbb{N}$*, the sequence* $\{(0,1/k)\}$ *converges to* $(0,0)$ *and that* $f(0,1/k) = 0$ *for each index* $k$*, we see that the only possible value of the limit is 0. To verify that the limit is indeed 0, it is necessary to make some estimates of the size of* $f(x,y)$*. Indeed, if* $x \neq 0$*, then*

$$\left|\frac{x^3}{x^2+y^2}\right| \leq \left|\frac{x^3}{x^2}\right| = |x|,$$

*and, therefore,*

$$\left|\frac{x^3}{x^2+y^2}\right| \leq |x| \quad \text{if } (x,y) \neq (0,0),$$

*since this estimate also clearly holds if* $x = 0$ *and* $y \neq 0$*. Now suppose that the sequence* $\{(x_k,y_k)\}$ *converges to* $(0,0)$ *with each* $(x_k,y_k) \neq (0,0)$*. Then the sequence* $\{x_k\}$ *converges to 0, so from the preceding estimate and the comparison test for convergent sequences (2.249), it follows that the image sequence* $\{f(x_k,y_k)\}$ *converges to 0. Thus,*

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0.$$

*More generally, let* $m$ *and* $n$ *be natural numbers. One can show that the limit*

$$\lim_{(x,y)\to(0,0)} \frac{x^n y^m}{x^2+y^2}$$

*exists if and only if* $m+n > 2$*.* ■

**Example 5.5** *As in Miklavcic (2020, p. 58), consider*

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - x^2 y}{x^2 + y^2 + xy}.$$

*Note that the function is undefined at the origin. First we evaluate the limit along a few simple paths. Along $y = 0$ and $x = 0$, respectively, we have*

$$\lim_{(0,0)} \frac{x^3 - x^2 y}{x^2 + y^2 + xy} = \lim_{(0,0)} \frac{x^3}{x^2} = \lim_{x\to 0} x = 0; \quad \lim_{(0,0)} \frac{x^3 - x^2 y}{x^2 + y^2 + xy} = \lim_{y\to 0} \frac{0}{y^2} = 0.$$

*We get the same result along any straight line $y = kx$. If the limit exists, it must be 0. This analysis with straight lines is not adequate to declare that the limit exists. Consider an arbitrary curve $r = f(\theta) > 0$, where $x = r\cos\theta$, $y = r\sin\theta$, and let $r \to 0$. Figure 39 shows a typical case.*

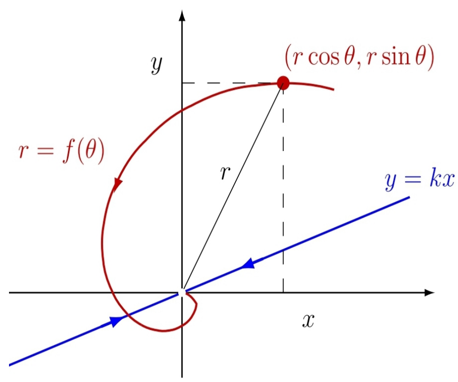



Figure 39: Spiral paths all lead to 0

*Substitute the polar functions for $x$ and $y$ in the definition of the function limit to get*

$$\begin{aligned}\left|\frac{x^3 - x^2 y}{x^2 + y^2 + xy} - 0\right| &= \left|\frac{r^3\cos^3\theta - r^3\cos^2\theta\sin\theta}{r^2\cos^2\theta + r^2\cos^2\theta + r^2\cos\theta\sin\theta}\right| \\ &= \left|\frac{r^3\cos^2\theta(\cos\theta - \sin\theta)}{r^2(1 + \cos\theta\sin\theta)}\right| = r\cos^2\theta\frac{|\cos\theta - \sin\theta|}{|1 + \sin\theta\cos\theta|}.\end{aligned}$$

*It is actually sufficient to stop here: The denominator is not zero and the numerator is bounded and proportional to $r$, which converges to zero. Using basic trigonometry results,*

$$\begin{aligned}\left|\frac{x^3 - x^2 y}{x^2 + y^2 + xy} - 0\right| &= r\cos^2\theta\frac{\sqrt{2}|\cos\theta\cos(\pi/4) - \sin\theta\sin(\pi/4)|}{\left|1 + \frac{1}{2}2\sin\theta\cos\theta\right|} \\ &\le r\sqrt{2}\frac{|\cos(\theta + \pi/4)|}{\left|1 + \frac{1}{2}\sin(2\theta)\right|} \le r\frac{\sqrt{2}}{1/2} = 2\sqrt{2}r \to 0 \text{ as } r \to 0.\end{aligned}$$

*Thus, given $\epsilon > 0$, however, small, we can find a $\delta$, as a function of $\epsilon$, e.g., $\delta = \epsilon/(2\sqrt{2})$, such that*

$$\left|\frac{x^3 - x^2 y}{x^2 + y^2 + xy} - 0\right| < \epsilon \quad \text{whenever} \quad r < \delta.$$

*Given that we have invoked an arbitrary curve whose sole requirement is to pass through the limit point (the origin) the result is general, the limit exists, and is indeed 0.* ■

The intermediate value theorem (IVT) given in (2.60) for univariate functions can be generalized in an immediate way as follows. Let $f : A \to \mathbb{R}$ be continuous on subset $S \subset A \subset \mathbb{R}^n$, where $S$ is a closed box in $\mathbb{R}^n$, e.g., the set $[a, b] \times [c, d] \subset \mathbb{R}^2$. Notice that, for $n = 1$, $S$ is a closed, bounded interval, as was required in the IVT for univariate functions. Assume $\mathbf{a}, \mathbf{b} \in S$. Let $\alpha = f(\mathbf{a})$ and $\beta = f(\mathbf{b})$. Given a number $\gamma$ with $\alpha < \gamma < \beta$, $\exists \mathbf{c} \in S$ such that $f(\mathbf{c}) = \gamma$.

However, for $n > 1$, other specifications for $S$ are possible such that the IVT holds. Characterizing the IVT in this case entails introducing the concepts of connectedness, polygonally connected, and regions. See, e.g., Trench (2013, pp. 295-7) and, particularly, Petrovic, Advanced Calculus: Theory and Practice, 2nd ed., 2020, §10.6, for clear, detailed presentations.

Recall that continuity alone is not sufficient for the IVT for $n = 1$. As an example, let domain $D = \mathbb{R} \setminus \{0\}$, and $f : D \to \mathbb{R}$ be given by $f(x) = -1$ for $x < 0$ and $f(x) = 1$ for $x > 0$. Function $f$ is continuous on $D$, but there is no $x_0$ such that $f(x_0)$ takes on a value strictly between $-1$ and 1. The next example below illustrates the same idea for $n = 2$.

A set $A \subset \mathbb{R}^n$ is said to be *polygonally connected* if, for any two points $P, Q \in A$ there exists a polygonal line that connects them. More precisely, there exists a positive integer $n$ and points $P_0 = P, P_1, \ldots, P_n = Q$ all in $A$, such that each line segment $P_i P_{i+1}, 0 \leq i \leq n-1$, completely lies within $A$. Note that a closed box in $\mathbb{R}^n$ is polygonally connected. The set $A = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$ is polygonally connected.

The $n$-dimensional IVT, as stated and proved in Petrovic, p. 327, is: Let $f$ be a continuous function on a polygonally connected domain $A \subset \mathbb{R}^n$, and let $P, Q \in A$. If $f(P) < 0$ and $f(Q) > 0$ then there exists $M \in A$ such that $f(M) = 0$.

**Example 5.6** *(Petrovic, p. 324) (A continuous function that does not have the IVP) Prove that the function*

$$f(x, y) = \begin{cases} -1, & \text{if } x^2 + y^2 < 1, \\ 2, & \text{if } x^2 + y^2 > 4, \end{cases}$$

*is continuous, but there is no point* $(c_1, c_2)$ *such that* $f(c_1, c_2) = 0$.

*Solution. The domain of $f$ consists of the open unit disk and the outside of the disk centered at the origin and of radius 2. It is obvious that $f$ is continuous at every point of its domain. Further, $f(0, 0) = -1$ and $f(2, 3) = 2$ (because $2^2 + 3^2 = 13 > 4$). Yet, there is no point $(c_1, c_2)$ such that $f(c_1, c_2) = 0$.* ■

We end this section by stating the following “Structure of Open Sets” results, as they are fundamental in analysis, but we will not make use of them:

Theorem: If $U$ is an open subset of $\mathbb{R}$, then there exists a finite or countable collection $\{I_j\}$ of pairwise disjoint open intervals such that $U = \bigcup_n I_j$. See, e.g., Terrell Theorem 4.1.6; or Stoll, Thm 2.2.20, for proof.

This is extended to $\mathbb{R}^n$ as follows, from, e.g., Terrell, p. 519; or Heil, Lemma 1.13.17.

Theorem (Structure of Open Sets in $\mathbb{R}^n$): Every open set in $\mathbb{R}^n, n \geq 1$, can be expressed as a countable union of nonoverlapping closed cubes.

## 5.2 Partial Derivatives and the Gradient

> In this section we deal with functions of one variable. The multivariable case in which $f : \mathbb{R}^n \to \mathbb{R}$ offers no new ideas, only new notation.
>
> (Charles Pugh, Real Mathematical Analysis, 2nd edition, 2015, p. 406)

This opening quote from Pugh, in his §6.6, regards the Lebesgue integral, and indeed, one of the luxuries of that integral is that the multivariate case is very easy to handle, once the framework in the univariate case is established. The reason I include this quote here is that, when it comes to differentiation for multivariate functions, it is *not* the case that "only new notation" is required; and in fact, quite some new ideas and concepts emerge. We will see some new ideas in this section, but notably in subsequent sections, such as for directional derivatives and the multivariate Mean Value Theorem.

Let $f : A \to \mathbb{R}$ with $A \subset \mathbb{R}^n$ an open set. For every $\mathbf{x} = (x_1, \ldots, x_n) \in A$ and for each $i = 1, 2, \ldots, n$, the *partial derivative* of $f$ with respect to $x_i$ is defined as

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}, \tag{5.23}$$

if the limit exists. Because the remaining $n-1$ variables in $\mathbf{x}$ are held constant, the partial derivative is conceptually identical to the Newton quotient (2.63) for univariate functions. This can be more compactly written by defining $\mathbf{e}_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ to be the $n$-length vector with a one in the $i$th position, and zero elsewhere, so that

$$(D_i f)(\mathbf{x}) := \frac{\partial f}{\partial x_i}(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{e}_i) - f(\mathbf{x})}{h}. \tag{5.24}$$

As indicated, a popular and useful alternative notation for the partial derivative is $D_i f(\mathbf{x})$ or, better, $(D_i f)(\mathbf{x})$, with the advantage that the name of the $i$th variable (in this case $x_i$) does not need to be explicitly mentioned. This is termed the partial derivative of $f$ with respect to the $i$th variable, at $\mathbf{x}$.

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$. Then the function $f : \mathcal{O} \to \mathbb{R}$ is said to have *first-order partial derivatives* provided that, for each index $i$ with $1 \le i \le n$, the function $f : \mathcal{O} \to \mathbb{R}$ has a partial derivative with respect to its $i$th component, at every point in $\mathcal{O}$.

If each of the $n$ partial derivatives exists at $\mathbf{x}$, then the *gradient* of $f$ at $\mathbf{x}$, denoted $(\operatorname{grad} f)(\mathbf{x})$ (and rhyming with sad and glad), or $(\nabla f)(\mathbf{x})$, is the row vector of all partial derivatives:

$$(\nabla f)(\mathbf{x}) = (\operatorname{grad} f)(\mathbf{x}) := (D_1 f(\mathbf{x}), \ldots, D_n f(\mathbf{x})). \tag{5.25}$$

Some further insight into partial derivatives is gained by using a *parametrized path*, allowing for a consideration that parallels the univariate case. In particular, recall from (2.63) and (2.65) that, for $f : D \subset \mathbb{R} \to \mathbb{R}$ differentiable on $D^o$ (the interior of $D$), and a point $x \in D^o$, the derivative of $f$ at $x$ is the Newton quotient

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}. \tag{5.26}$$

Below, we produce equation (5.27), which is a direct analog of this.

The following presentation is based on Fitzpatrick (2009, §13.2). It is of use to review the concept of line segment in $\mathbb{R}^n$ from Example 5.1. Own notes are in blue color.

Recall the MVT (2.94), which states: If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on $(a,b)$, then $\exists c\in(a,b)$ such that $f(b)-f(a)=f'(c)(b-a)$. Consider a real-valued function of several real variables $f:\mathbb{R}^n\to\mathbb{R}$, together with two points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Suppose that we want to compare $f(\mathbf{u})$ with $f(\mathbf{v})$. When $n=1$ and the function $f:\mathbb{R}\to\mathbb{R}$ is differentiable, we can use the MVT to compare these two values. When $n>1$, the following restriction procedure is natural. Look at the parametrized segment from $\mathbf{u}$ to $\mathbf{v}$—that is, the parametrized path $\gamma:[0,1]\to\mathbb{R}^n$ defined by

$$\gamma(t)=\mathbf{u}+t(\mathbf{v}-\mathbf{u})=t\mathbf{v}+(1-t)\mathbf{u},\quad \text{for}\quad 0\le t\le 1.$$

Then consider the composition of the function $f:\mathbb{R}^n\to\mathbb{R}$ with this parametrized path, which is the function $\psi:[0,1]\to\mathbb{R}$ defined by

$$\psi(t)=f(\mathbf{u}+t(\mathbf{v}-\mathbf{u})),\quad \text{for}\quad 0\le t\le 1,$$

with $\psi(0)=f(\mathbf{u})$ and $\psi(1)=f(\mathbf{v})$. Thus, to compare $f(\mathbf{u})$ with $f(\mathbf{v})$ is to compare $\psi(0)$ with $\psi(1)$. If we can determine that $\psi:[0,1]\to\mathbb{R}$ is continuous and that $\psi:(0,1)\to\mathbb{R}$ is differentiable, then we can apply the Mean Value Theorem for functions of a single variable to compare $f(\mathbf{u})$ with $f(\mathbf{v})$. Thus, it is necessary to investigate the properties of the function $f:\mathbb{R}^n\to\mathbb{R}$ that will allow us to conclude that the above auxiliary function $\psi:[0,1]\to\mathbb{R}$ is continuous; to conclude that $\psi:(0,1)\to\mathbb{R}$ is differentiable; and to compute $\psi':(0,1)\to\mathbb{R}$.

We can regard $\psi:[0,1]\to\mathbb{R}$ as being the restriction of the function $f:\mathbb{R}^n\to\mathbb{R}$ to the line segment between the points $\mathbf{u}$ and $\mathbf{v}$, together with the placing of a coordinate system on this line segment. In the case where $n=2$, the graph of $\psi:[0,1]\to\mathbb{R}$ is obtained by intersecting the graph of $f:\mathbb{R}^2\to\mathbb{R}$ with the plane that is parallel to the $z$-axis and contains the segment joining $\mathbf{u}$ and $\mathbf{v}$. For this reason, we refer to the function $\psi:[0,1]\to\mathbb{R}$ as a *section of the function* $f:\mathbb{R}^n\to\mathbb{R}$.

In order to analyze the differentiability of the function $\psi:(0,1)\to\mathbb{R}$ at the point $t_0$, we change variables by setting $\mathbf{x}=\mathbf{u}+t_0(\mathbf{v}-\mathbf{u})$, $\mathbf{p}=\mathbf{v}-\mathbf{u}$, and $s=t-t_0$; then

$$\frac{\psi(t)-\psi(t_0)}{t-t_0}=\frac{f(\mathbf{x}+s\mathbf{p})-f(\mathbf{x})}{s},$$

and taking limits and noting $s\to 0$ is the same as $t\to t_0$, we have the analog of (5.26),

$$\psi'(t_0)=\lim_{t\to t_0}\frac{\psi(t)-\psi(t_0)}{t-t_0}=\lim_{s\to 0}\frac{f(\mathbf{x}+s\mathbf{p})-f(\mathbf{x})}{s},\tag{5.27}$$

provided that the limit exists. The strategy of looking at sections of a function, together with (5.27), motivates the introduction of the following concept of a partial derivative.

Definition: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$, and let $i$ be an index with $1\le i\le n$. A function $f:\mathcal{O}\to\mathbb{R}$ is said to *have a partial derivative with respect to its ith component at the point* $\mathbf{x}$ provided that the limit

$$\lim_{t\to 0}\frac{f(\mathbf{x}+t\mathbf{e}_i)-f(\mathbf{x})}{t}\tag{5.28}$$

exists, where, instead of generic point $\mathbf{p}$ in (5.27), $\mathbf{e}_i$ is the $i$th unit vector, i.e., the vector whose $i$th component is 1 and whose other components are 0. If this limit exists, then we denote its value by $\partial f/\partial x_i(\mathbf{x})$, and call it the *partial derivative of* $f : \mathcal{O} \to \mathbb{R}$ *with respect to the* $i$*th component, at the point* $\mathbf{x}$.

The geometric meaning of $\partial f/\partial x_i(\mathbf{x})$ is as follows: Choose a positive number $r$ such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$ and consider the section defined by $\psi(t; i, \mathbf{x}) = \psi(t) = f(\mathbf{x} + t\mathbf{e}_i)$, for $|t| < r$. Then $f : \mathcal{O} \to \mathbb{R}$ has a partial derivative with respect to its $i$th component at the point $\mathbf{x}$ precisely when there is a tangent line to the graph of this section at the point on the graph corresponding to $t = 0$, at which point the slope of this tangent is the number

$$\psi'(0) = \frac{\partial f}{\partial x_i}(\mathbf{x}),$$

or, in more detail, and of value for directional derivatives below,

$$\psi'(0) = \frac{d}{dt}\psi(t)\bigg|_{t=0} = \frac{d}{dt} f(\mathbf{x} + t\mathbf{e}_i)\bigg|_{t=0} = \frac{\partial f}{\partial x_i}(\mathbf{x}) = (D_i f)(\mathbf{x}). \tag{5.29}$$

Thus, the existence of $\partial f/\partial x_i(\mathbf{x})$ is equivalent to the differentiability of a function of a single real variable, so we can immediately use the single-variable differentiation results to obtain addition, product, and quotient rules for partial derivatives.

Recall from (2.73) that, in the univariate case, if $f$ is differentiable at $a$, then $f$ is continuous at $a$; and if $f$ is differentiable on its entire domain $D$, then $f$ is continuous on $D$. This is not necessarily the case for $n > 1$: A function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ that has first-order partial derivatives at all points in the (interior of the) domain need not be continuous. The next example gives a case in point. The key is that, while $\forall \mathbf{x} \in D$, $(\operatorname{grad} f)(\mathbf{x})$ exists, it is not continuous on all of $D$.

**Example 5.7** *Recall the bivariate function from Example 5.2,*

$$f(x, y) = \begin{cases} xy/\left(x^2 + y^2\right), & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0), \end{cases}$$

*in which we showed that* $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})$ *does not exist. Repeating from Example 5.2, observe that the sequence* $\{(1/k, 1/k)\}$ *converges to* $(0, 0)$ *and that* $f(1/k, 1/k) = 1/2$ *for each index* $k$*, so the image sequence* $\{f(1/k, 1/k)\}$ *converges to* $1/2$*. But* $1/2 \neq f(0, 0)$*. Thus, the function* $f$ *is not continuous at the point* $(0, 0)$*.*

*Using* (5.28) *with* $(x, y) = (0, 0)$,

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0 - 0}{h} = 0;$$

*and, from symmetry, the same is true for* $(D_2 f)(0, 0)$*. Thus,* $(\operatorname{grad} f)(0, 0)$ *exists, and equals* $(0, 0)$.

*For* $(x, y) \neq (0, 0)$*, there is a neighborhood of* $(x, y)$ *on which the restriction of* $f : \mathbb{R}^2 \to \mathbb{R}$ *is a quotient of polynomials whose denominator does not vanish. Thus, on that neighborhood,* $\partial f/\partial x(x, y)$ *and* $\partial f/\partial y(x, y)$ *exist; moreover, a short computation yields*

$$\frac{\partial f}{\partial x}(x, y) = \frac{y^3 - x^2 y}{\left(x^2 + y^2\right)^2} \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = \frac{x^3 - y^2 x}{\left(x^2 + y^2\right)^2}.$$

*Thus, the function* $f$ *has first-order partial derivatives at every point in the plane* $\mathbb{R}^2$*. However, these two derivatives are not continuous at* $(0, 0)$*. To see this, compare their limiting behavior for sequences* $(0, 1/k)$ *and* $(1/k, 0)$.

*Below, we will see that this lack of continuity in the partial derivatives means we cannot conclude that $f$ is continuous at all points in its domain; and indeed, this is true for this function $f$.* ■

## 5.3 Differentiability and Tangent Maps

> One thing you will observe about all these books – they use pictures to convey the mathematical ideas. Beware of books that don't.
>
> Charles Pugh, Real Mathematical Analysis, 2nd edition, 2015, p. 467

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and let the function $f : \mathcal{O} \to \mathbb{R}$ be such that it has first-order partial derivatives. If in addition, $\operatorname{grad} f$ (i.e., each of $(D_i f)(\mathbf{x}) : \mathcal{O} \to \mathbb{R}$, $1 \le i \le n$) is continuous, then we will see below that $f$ is continuous. Since this additional assumption will play an important part later, it is useful to name it:

Definition: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$. Then a function $f : \mathcal{O} \to \mathbb{R}$ is said to be *continuously differentiable*, provided that it has first-order partial derivatives for all $\mathbf{x} \in \mathcal{O}$, and such that each partial derivative $(D_i f)(\mathbf{x}) : \mathcal{O} \to \mathbb{R}$ is continuous, $1 \le i \le n$.

We also state now the fundamental result on continuity, albeit whose proof needs to wait until other results are proven. This result is shown below in (5.49), and then, with a different, more sophisticated proof, using the multivariate Mean Value Theorem in §5.5, on page 330.

Theorem (Continuity): Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Then the function $f : \mathcal{O} \to \mathbb{R}$ is continuous.

**Example 5.8** *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(0,0) = 0$ and*

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad (x,y) \neq (0,0).$$

*To assess continuity, we compute*

$$\lim_{x \to 0} f(x,0) = \lim_{x \to 0} \frac{x^2}{x^2} = \lim_{x \to 0} 1 = 1, \qquad \lim_{y \to 0} f(0,y) = \lim_{y \to 0} \frac{-y^2}{y^2} = \lim_{y \to 0} -1 = -1,$$

*showing that $f$ is not continuous at $(0,0)$, and there is no way to redefine its value at $(0,0)$ to make it continuous. From the contrapositive of the continuity theorem, we know that at least one of the first order partial derivatives is not continuous on the whole domain.* ■

**Example 5.9** *Now consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(0,0) = 0$ and*

$$f(x,y) = xy \frac{x^2 - y^2}{x^2 + y^2}, \quad (x,y) \neq (0,0).$$

*We compute*

$$\lim_{x \to 0} f(x,0) = \lim_{x \to 0} 0 = 0, \qquad \lim_{y \to 0} f(0,y) = \lim_{y \to 0} 0 = 0,$$

*i.e., "so far so good", and continue to check: Letting $y = ax$ for some $a \in \mathbb{R}$,*

$$\lim_{x \to 0} f(x, ax) = \lim_{x \to 0} a x^2 \frac{x^2 (1 - a^2)}{x^2 (1 + a^2)} = a \frac{(1 - a^2)}{(1 + a^2)} \lim_{x \to 0} x^2 = 0,$$

*so that the limit of the function on every linear path to* $(0,0)$ *is zero. Further, nonlinear paths would need to be inspected, but we omit these, and presume that* $f$ *is indeed continuous at* $(0,0)$*. With* $f$ *being a ratio of polynomials for* $(x,y) \neq (0,0)$*, it is thus continous at all points in its domain* $\mathbb{R}^2$*. The continuity theorem is only an "if then", so that we cannot conclude that the first order partial derivatives are continous. Instead, we have to check. Straightforward calculation shows that*

$$\frac{\partial f}{\partial x} = \frac{yx^4 - y^5 + 4x^2y^3}{\left(x^2+y^2\right)^2}, \qquad \frac{\partial f}{\partial y} = \frac{x^5 - xy^4 - 4x^3y^2}{\left(x^2+y^2\right)^2}, \qquad (x,y) \neq (0,0).$$

*To assess their continuity, note first that*

$$\lim_{x\to 0} \frac{\partial f}{\partial x}(x,0) = \lim_{x\to 0} \frac{0}{x^4} = \lim_{x\to 0} 0 = 0, \qquad \lim_{y\to 0} \frac{\partial f}{\partial x}(0,y) = \lim_{y\to 0} -y = 0,$$

*so that, if the limit of* $(\partial f/\partial x)(x,y)$ *as* $(x,y) \to (0,0)$ *exists, it must be zero. Let* $y = ax$ *for some* $a \in \mathbb{R}$*, so that*

$$\lim_{x\to 0} \frac{\partial f}{\partial x}(x,ax) = \lim_{x\to 0} \frac{ax^5 - a^5x^5 + 4a^3x^5}{x^4\left(1+a^2\right)^2} = \frac{a\left(4a^2 - a^4 + 1\right)}{\left(1+a^2\right)^2} \lim_{x\to 0} x = 0,$$

*confirming that, at least along all lines approaching* $(0,0)$*, the limit of* $\partial f/\partial x$ *is zero. One could and should check other, nonlinear, paths to be sure. Next, observe that, using* (5.28) *with* $(x,y) = (0,0)$*,*

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

*Thus, it is presumable (we did not check all paths) that* $\partial f/\partial x$ *is continuous at zero, and thus, being a ratio of polynomials for* $(x,y) \neq (0,0)$*, is continuous everywhere on the domain* $\mathbb{R}^2$*.*

*Similarly for* $\partial f/\partial y$*,*

$$\lim_{x\to 0} \frac{\partial f}{\partial y}(x,0) = \lim_{x\to 0} \frac{x^5}{x^4} = 0, \qquad \lim_{y\to 0} \frac{\partial f}{\partial y}(0,y) = \lim_{y\to 0} \frac{0}{y^4} = \lim_{y\to 0} 0 = 0,$$

*and, setting* $y = ax$ *for some* $a \in \mathbb{R}$*,*

$$\lim_{x\to 0} \frac{\partial f}{\partial y}(x,ax) = \lim_{x\to 0} \frac{x^5 - a^4x^5 - 4x^5a^2}{x^4\left(1+a^2\right)^2} = \frac{\left(1 - a^4 - 4a^2\right)}{\left(1+a^2\right)^2} \lim_{x\to 0} x = 0.$$

*Also, using* (5.28) *with* $(x,y) = (0,0)$*,*

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

*Assuming the adequacy of this inspection that* $\operatorname{grad} f$ *is continuous, the above theorem implies that* $f$ *is continuous, which we confirmed above.* ■

In the following, let $f : A \to \mathbb{R}$ with $A \subset \mathbb{R}^n$ an open set and such that $(\operatorname{grad} f)(\mathbf{x})$ exists $\forall \mathbf{x} \in A$. Recall from (2.321) and (2.330) that, for $n = 1$, i.e., $f : A \subset \mathbb{R} \to \mathbb{R}$, the *tangent to the curve* at the point $(x_0, y_0)$, for $x_0 \in A \subset \mathbb{R}$ and $y_0 = f(x_0)$ is the (non-vertical) line

$T(x) = y_0 + f'(x_0)(x - x_0)$. This is the best linear approximation to $f$ in a neighborhood of $x_0$ such that $f(x_0) = T(x_0)$, and, from (2.330) and (2.331), satisfies

$$\lim_{x \to x_0} \frac{f(x) - T(x)}{x - x_0} = 0.$$

For $n = 2$, envisioning a thin, flat board resting against a sphere in 3-space, we seek a (non-vertical) plane in $\mathbb{R}^3$ that is "tangent" to $f$ at a given point, say $(x_0, y_0, z_0)$, for $(x_0, y_0) \in A$ and $z_0 = f(x_0, y_0)$. A plane is linear in both $x$ and $y$, so its equation is, recalling (4.8),

$$z = z_0 + s(x - x_0) + t(y - y_0),$$

where $s$ and $t$ need to be determined.

When restricted to the plane $y = y_0$, the surface $f$ is just the curve $z = g(x) := f(x, y_0)$ in $\mathbb{R}^2$, and the plane we seek is just the line $z = z_0 + s(x - x_0)$. This is the $n = 1$ case previously discussed, so the tangent to the curve $g(x)$ at $x_0$ is the line $z = z_0 + g'(x_0)(x - x_0)$, i.e., $s = D_1 f(x_0, y_0)$. Similarly, $t = D_2 f(x_0, y_0)$. This gives rise to the definition, in the $n = 2$ case, of the *tangent plane* of $f$ at $(x_0, y_0, z_0)$:

Definition: For function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and $(x_0, y_0) \in D^o$, the tangent plane at $(x_0, y_0, z_0)$, where $z_0 = f(x_0, y_0)$, is the linear function

$$T(x, y) = f(x_0, y_0) + (D_1 f(x_0, y_0))(x - x_0) + (D_2 f(x_0, y_0))(y - y_0) \tag{5.30}$$

that satisfies

$$f(x_0, y_0) = T(x_0, y_0), \quad \text{and} \quad \lim_{(x,y) \to (x_0, y_0)} \frac{f(x, y) - T(x, y)}{\|(x, y) - (x_0, y_0)\|} = 0. \tag{5.31}$$

The reason for (the more strenuous condition of) dividing $f(x, y) - T(x, y)$ in (5.31) by

$$\|(x, y) - (x_0, y_0)\| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

is discussed below.

Let $\mathbf{v}_1$ be the vector in the $xz$ plane, on the slice in 3D space $y = y_0$, originating from the origin, with endpoint $(1, 1 \times D_1 f(x_0, y_0))$, and thus parallel to the tangent line $L_1$ given by $z = z_0 + (D_1 f)(x_0, y_0)(x - x_0)$. Vector $\mathbf{v}_2$ in the $yz$ plane, and line $L_2$, are similarly defined. Motivated by the quote at the beginning of this subsection regarding the importance of pictures for conveying mathematical ideas, we illustrate these lines and vectors in Figures 40 and 41, taken from Miklavcic's excellent and accessible *An Illustrative Guide to Multivariable and Vector Calculus* (2020).

Notice that $\mathbf{v}_1$ and $\mathbf{v}_2$ cannot be parallel to each other. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ define a tangent plane, $T$. Capitalizing on our discussion of the cross product, from (4.80) or (4.85), this tangent plane, for $n = 2$, has normal vector

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 0 & \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} \\ 0 & 1 & \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} \end{vmatrix} = -\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \mathbf{e}_1 - \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \mathbf{e}_2 + \mathbf{e}_3. \tag{5.32}$$

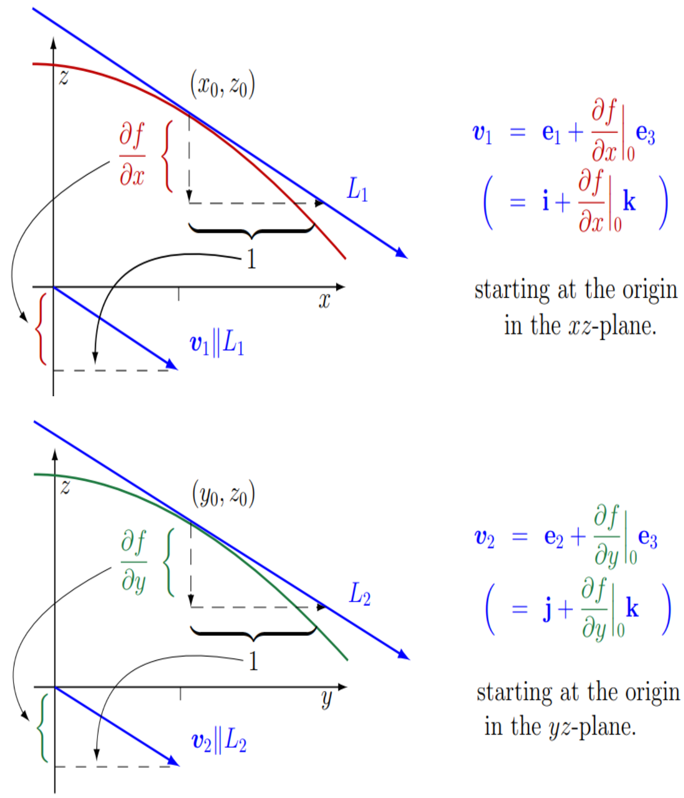



Figure 40: Top: A tangent vector and line in the $x$-direction. In the cross section parallel to the $xz$-plane, a vector parallel to line $L_1$ is $\mathbf{v}_1$. Bottom: A tangent vector and line in the $y$-direction. In the cross section parallel to the $yz$-plane, a vector parallel to line $L_2$ is $\mathbf{v}_2$. Taken from Miklavcic, p. 65.

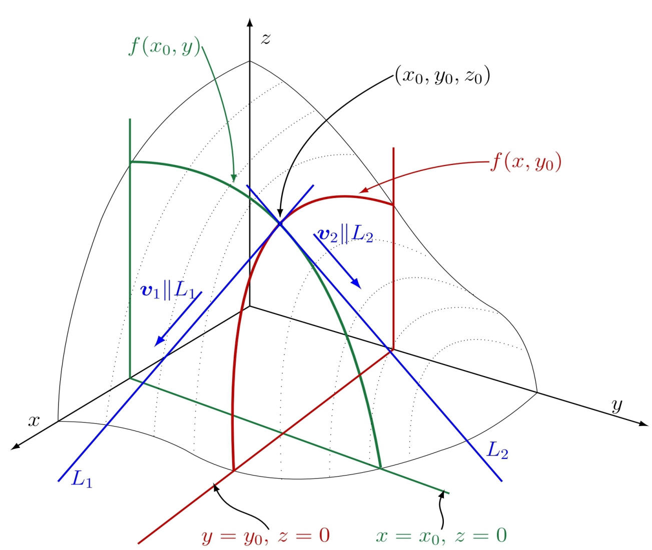



Figure 41: The two components of the tangent plane for a function from $\mathbb{R}^2$ to $\mathbb{R}$. Taken from Miklavcic, p. 66

The plane defined by $L_1$ and $L_2$ satisfies the following:

1. It is tangent to the surface $z = f$ at $(x_0, y_0, z_0)$.
2. It is spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.
3. It has the same normal as the normal to the graph of $z = f(x, y, z)$ at $\mathbf{x}_0 = (x_0, y_0, z_0)$.

Its equation can be found from the scalar vector product $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$, i.e.,

$$z - z_0 = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} (y - y_0), \tag{5.33}$$

which, obviously and by necessity, agrees with (5.30).

This discussion of the $n = 2$ case, notably (5.30) and (5.31), motivates the following definition of differentiability and the tangent map.

Definition: For $n \in \mathbb{N}$, let $f : A \to \mathbb{R}$ for $A \subset \mathbb{R}^n$ and let $\mathbf{x}_0 = (x_{01}, x_{02}, \ldots, x_{0n})'$ be an interior point of $A$. The function $f$ is said to be *differentiable at* $\mathbf{x}_0$ if

1. $(\text{grad}\, f)(\mathbf{x_0})$ exists, and
2. there exists a *tangent map* $T : \mathbb{R}^n \to \mathbb{R}$ of $f$ at $\mathbf{x}_0$, such that

$$f(\mathbf{x}_0) = T(\mathbf{x}_0) \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{x}_0} \frac{f(\mathbf{x}) - T(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|} = 0, \tag{5.34}$$

where ($\mathbf{x}$ is also a column vector and)

$$T(\mathbf{x}) = f(\mathbf{x}_0) + (\text{grad}\, f)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0). \tag{5.35}$$

Remarks:

1. With $\mathbf{x}$ and $\mathbf{x}_0$ column vectors, and grad being a row vector, the latter term in (5.35) is matrix multiplication of a column and row vector, yielding a scalar.

2. Formally, the definition does not include the above definition of $T$ in terms of the gradient. However, if the tangent map of $f$ at $\mathbf{x}_0$ exists, and is restricted to being a tangent plane, i.e., linear, as in (5.30) for the $n = 2$ case, then it is unique, and given by (5.35).

We can write (5.35) as

$$T(\mathbf{x}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} (D_i f)(\mathbf{x}_0)(x_i - x_{0i}) \tag{5.36}$$

$$=: f(\mathbf{x}_0) + \mathrm{d}f(\mathbf{x}_0, \mathbf{x} - \mathbf{x}_0), \tag{5.37}$$

where the term $\mathrm{d}f(\mathbf{x}_0, \mathbf{x} - \mathbf{x}_0)$ defined in (5.37) is the *total differential of* $f$ *at* $\mathbf{x}_0$, i.e.,

$$\mathrm{d}f(\mathbf{x}, \mathbf{h}) = (\text{grad}\, f)(\mathbf{x}) \cdot \mathbf{h}. \tag{5.38}$$

If $f$ is differentiable at all points of $A$, then $f$ is said to be *differentiable (on A)*.

We now need to explain why the limit condition for the tangent map in (5.31) for the $n = 2$ case, and, more generally, in (5.34), divides by $\|\mathbf{x} - \mathbf{x}_0\|$. To do so, we review the $n = 1$ case: Recall from (2.330) and (2.331) with $x_0 = c$ that, for $f : D \subset \mathbb{R} \to \mathbb{R}$, $x_0 \in D^o$ (the interior of $D$), and $f$ such that, $\forall x \in D^o$, $\exists f''(x)$,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + r(x), \quad r(x) = \frac{1}{2} f''(\zeta)(x - x_0)^2, \tag{5.39}$$

with

$$\lim_{x \to x_0} \frac{r(x)}{|x - x_0|} = \frac{1}{2} f''(\zeta) \lim_{x \to x_0} \frac{(x - x_0)^2}{|x - x_0|} = \frac{1}{2} f''(\zeta) \lim_{x \to x_0} |x - x_0| = 0. \tag{5.40}$$

This means $f(x_0) + f'(x_0)(x - x_0)$ is an affine linear approximation to $f(x)$ with the property that, not only is the error term $r(x)$ such that $\lim_{x \to x_0} r(x) = 0$, but also the limit of $r(x)$ after dividing by the linear quantity that itself goes to zero, is zero.

> Remark: Another way of explaining the stronger condition of dividing by $\|\mathbf{x} - \mathbf{x}_0\|$ in (5.34) is given at the end of this subsection, in the excerpt from Lang's book. He also starts with the $n = 1$ case, as we did above. We include this "redundancy" because of the importance of this tangent map criterion; and because it often helps to see the same idea presented in slightly different ways. Furthermore, the tangent map is a first-order approximation to the function, and below, in §5.8, we will define and use $k$th order approximations, these resulting from the Taylor series expansion of the function. The reader can quickly peak ahead and look at equation (5.127), and also the expression of (5.34) in terms of what is called the First-Order Approximation Theorem, in (5.128). Indeed, the first half of §5.8 is yet another description and development of (5.34), taken from Fitzpatrick's book. Its equation (5.151) gives the generalization of error term $r(x)$ in (5.39) to the general $n$ case. The second half of that subsection details the second-order approximation, and its relevance for determining minima and maxima of functions, generalizing the univariate results (2.105) and (2.106).

The obvious generalization of (5.39) and (5.40) to the $n = 2$ case of the differentiability condition (5.34) is, for $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and $(x_0, y_0) \in D^o$, there exists a real function $r(x)$ such that, $\forall (x, y) \in D^o$,

$$f(x, y) = f(x_0, y_0) + (D_1 f(x_0, y_0))(x - x_0) + (D_2 f(x_0, y_0))(y - y_0) + r(x, y), \tag{5.41}$$

with

$$\lim_{(x,y) \to (x_0,y_0)} \frac{r(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0. \tag{5.42}$$

For general $n$, with $f : D \subset \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{x}, \mathbf{x}_0 \in D^o$, and using definition (5.35),

$$f(\mathbf{x}) = T(\mathbf{x}) + r(\mathbf{x}) = f(\mathbf{x}_0) + (\operatorname{grad} f)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + r(\mathbf{x}), \tag{5.43}$$

with, from (5.34),

$$r(\mathbf{x}) = f(\mathbf{x}) - T(\mathbf{x}), \qquad \lim_{\mathbf{x} \to \mathbf{x}_0} \frac{r(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|} = 0. \tag{5.44}$$

The latter term in (5.44) obviously implies

$$\lim_{\mathbf{x} \to \mathbf{x}_0} r(\mathbf{x}) = 0. \tag{5.45}$$

Theorem: If a function $f : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}$ for open domain $\mathcal{O}$ is differentiable at $\mathbf{x}_0 \in \mathcal{O}$, then it is continuous at $\mathbf{x}_0$.

> Proof: Recall the equivalent definitions of continuity in (5.12) and (5.13). The result is very clear for $n = 2$ using $\mathbf{x}_0 = (x_0, y_0)$ in (5.41) and (5.42), so that, with $\mathbf{x} = (x, y)$,
>
> $$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0). \tag{5.46}$$
>
> The general $n$ case follows directly from (5.43) and (5.44).

Observe how the above definition of differentiability at a point $\mathbf{x}_0$ in the (interior of the) domain requires not just the existence of $(\operatorname{grad} f)(\mathbf{x}_0)$, but also the tangent map (5.34). This dual condition implies that we could have existence of $(\operatorname{grad} f)(\mathbf{x}_0)$, but not (5.34). This is true, as the following example shows.

**Example 5.10** *(Petrovic, p. 344) Prove that the function*

$$f(x,y) = \begin{cases} x + y, & \text{if } x = 0 \text{ or } y = 0 \\ 1, & \text{otherwise} \end{cases}$$

*has partial derivatives at $(0,0)$ but there is no tangent map at this point.*

*Solution. Using (5.28) with $(x,y) = (0,0)$,*

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1,$$

*and, similarly, $(\partial f / \partial y)(0,0) = 1$. However, $f$ is not continuous at $(0,0)$. Indeed, $f(0,0) = 0$, but for any $n \in \mathbb{N}$, $f(1/n, 1/n) = 1$ so $\lim f(1/n, 1/n) = 1$. This is bad news because, if we substitute $(x_0, y_0) = (0,0)$ and $(x,y) = (1/n, 1/n)$ in (5.41) and assume (5.42), we would get*

$$f\left(\frac{1}{n}, \frac{1}{n}\right) = 0 + 1\left(\frac{1}{n} - 0\right) + 1\left(\frac{1}{n} - 0\right) + r\left(\frac{1}{n}, \frac{1}{n}\right)$$

*The left side equals 1, but the right side converges to 0. Thus, (5.41) and (5.42) do not hold, there is no tangent map at this point, and $f$ is not differentiable.* ■

In light of the previous example, in which $f$ was not continuous, one might ask: If $(\operatorname{grad} f)(\mathbf{x}_0)$ exists, and $f$ is continuous, then does the tangent map (5.34) exist? The answer is no, as the next example shows.

**Example 5.11** *(Petrovic, p. 345) Prove that the function*

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2 + y^2}, & \text{if } x^2 + y^2 > 0, \\ 0, & \text{if } (x,y) = (0,0), \end{cases}$$

*has partial derivatives at $(0,0)$, and it is continuous at $(0,0)$, but (5.31) does not hold.*

*Solution. This time $f$ is continuous. It is easy to see that this is true at any point different from the origin. For the continuity at the origin, we will show that*

$$\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^2 + y^2} = 0. \tag{5.47}$$

*Using the arithmetic-geometric mean inequality* $|2xy| \le x^2 + y^2$, *we obtain that*

$$0 \le \left|\frac{x^2 y}{x^2+y^2}\right| \le \left|\frac{x^2 y}{2xy}\right| = \frac{|x|}{2},$$

*which implies* (5.47) *via the Squeeze Theorem* (2.9). *So,* $f$ *is continuous. Also, the partial derivatives at* $(0,0)$ *exist: Using* (5.28) *with* $(x,y)=(0,0)$,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0,$$

*and, similarly,* $(\partial f/\partial y)(0,0) = 0$. *However,* (5.31) *does not hold. Otherwise, we would have* $f(x,y) = r(x,y)$, *and it would follow that*

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{(x)^2+(y)^2}} = 0.$$

*In particular, taking once again* $(x,y) = (1/n, 1/n)$, *we would obtain that*

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)^2 \frac{1}{n}}{\left(\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)^2\right)^{3/2}} = 0,$$

*but this is incorrect because the limit on the left side is* $1/(2\sqrt{2})$. ■

Sufficient conditions for $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $D$ open, in order for the tangent map (5.37) to exist at $\mathbf{x}_0 \in D$, are:

(i) the existence of $\operatorname{grad} f$ on $D$; and (ii) continuity of $\operatorname{grad} f$ at $\mathbf{x}_0$.

These are fulfilled if $f$ is continuously differentiable, i.e., $\operatorname{grad} f$ (exists and) is continuous (on $D$), denoted $f \in \mathcal{C}^1(D)$. The proof, from Petrovic, p. 345, is for $n = 2$, with the general $n \in \mathbb{N}$ case clear in principle.

Theorem: Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ with $D$ open. Suppose that partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist in $D$ and that they are continuous at $(x_0, y_0) \in D$. Then

$$f \textbf{ is differentiable at } (x_0, y_0). \tag{5.48}$$

Proof: We will start with the equality

$$f(x,y) - f(x_0,y_0) = [f(x,y) - f(x_0,y)] + [f(x_0,y) - f(x_0,y_0)].$$

The existence of partial derivatives allows us to apply the Mean Value Theorem to each pair above. We obtain that

$$f(x,y) - f(x_0,y_0) = \frac{\partial f}{\partial x}(z,y)(x-x_0) + \frac{\partial f}{\partial y}(x_0,w)(y-y_0),$$

for some real numbers $z$ (between $x$ and $x_0$) and $w$ (between $y$ and $y_0$). We will write

$$\frac{\partial f}{\partial x}(z,y) = \frac{\partial f}{\partial x}(x_0,y_0) + \alpha, \quad \frac{\partial f}{\partial y}(x_0,w) = \frac{\partial f}{\partial y}(x_0,y_0) + \beta,$$

and the continuity of partial derivatives at $(x_0, y_0)$ implies that, when $(x, y) \to (x_0, y_0)$, $\alpha, \beta \to 0$. Therefore,

$$\begin{aligned} f(x,y) - f(x_0, y_0) &= \left( \frac{\partial f}{\partial x}(x_0, y_0) + \alpha \right)(x - x_0) + \left( \frac{\partial f}{\partial y}(x_0, y_0) + \beta \right)(y - y_0) \\ &= \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) + \alpha (x - x_0) + \beta (y - y_0). \end{aligned}$$

This is precisely the form (5.41), and the result will follow if we can show (5.42), i.e.,

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\alpha (x - x_0) + \beta (y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

Notice that

$$\frac{|x - x_0|}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}, \frac{|y - y_0|}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \le 1.$$

It follows that

$$0 \le \left| \frac{\alpha (x - x_0) + \beta (y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \right| \le |\alpha| + |\beta| \to 0, \quad (x, y) \to (x_0, y_0).$$

Note that, as stated above, the theorem provides a set of sufficient conditions. It is thus not true that differentiability of $f$, i.e., the existence of grad $f$ and the existence of the tangent map (5.37), implies that any or all of the $(D_i f)$ are continuous.

The proof for the $n = 3$ case starts the same, and requires writing $f(x, y, z) - f(x_0, y_0, z_0)$ as the appropriate sum of three terms. This sum is given below in (5.76) and (5.77). The rest of the proof is then the same. Although cumbersome, one could attempt to write the relevant sum expansion of $f(\mathbf{x}) - f(\mathbf{x}_0)$ for the general $n$ case.

Theorem (Continuity Theorem): For $f : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}$,

$$f \in \mathcal{C}^1(\mathcal{O}) \implies f \in \mathcal{C}^0(\mathcal{O}), \quad \textbf{i.e., } f \textbf{ is continuous on } \mathcal{O}. \tag{5.49}$$

Proof: This results from combining the theorems (5.46) and (5.48) (and assuming the latter holds for general $n$).

In §5.5, page 330, we give the proof of this result using the multivariate MVT.

We now turn to differentiability of basic functions of two differentiable functions, namely additivity and homogeneity; the dot product (5.16); and, as the range of $f$ and $g$ are $\mathbb{R}$, the quotient.

Let $f, g : D \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable, and let $\mathbf{x}, \mathbf{x}_0 \in D^o$. From (5.43) and (5.34), but adding appropriate subscripts to $T$ and $r$,

$$f(\mathbf{x}) = T_f(\mathbf{x}) + r_f(\mathbf{x}) = f(\mathbf{x}_0) + (\operatorname{grad} f)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + r_f(\mathbf{x}), \tag{5.50}$$

$$g(\mathbf{x}) = T_g(\mathbf{x}) + r_g(\mathbf{x}) = g(\mathbf{x}_0) + (\operatorname{grad} g)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + r_g(\mathbf{x}), \tag{5.51}$$

with $r_f(\mathbf{x}) = f(\mathbf{x}) - T_f(\mathbf{x})$, $r_g(\mathbf{x}) = g(\mathbf{x}) - T_g(\mathbf{x})$, and

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{r_f(\mathbf{x})}{\|\mathbf{x}-\mathbf{x}_0\|} = 0, \qquad \lim_{\mathbf{x}\to\mathbf{x}_0} \frac{r_g(\mathbf{x})}{\|\mathbf{x}-\mathbf{x}_0\|} = 0. \tag{5.52}$$

Theorem: Under the above conditions on $f, g$, and with $k_1, k_2 \in \mathbb{R}$,

$$k_1 f + k_2 g \textbf{ is differentiable,} \tag{5.53}$$

with

$$\operatorname{grad}(k_1 f + k_2 g) = k_1 \operatorname{grad}(f) + k_2 \operatorname{grad}(g). \tag{5.54}$$

Notice differentiable functions form a vector space.

> Proof: For homogeneity, analogous to the linearity property of differentiation in the univariate case, we see that, from (5.23), with $(kf)(\mathbf{x}) = kf(\mathbf{x})$, $(D_i(kf))(\mathbf{x}) = k(D_i f)(\mathbf{x})$. Thus, from (5.25), $(\nabla(kf))(\mathbf{x}) = (\operatorname{grad}(kf))(\mathbf{x}) = k\,(D_1 f(\mathbf{x}), \ldots, D_n f(\mathbf{x}))$, and, simply multiplying (5.50) by $k$,
>
> $$(kf)(\mathbf{x}) = kf(\mathbf{x}) = kf(\mathbf{x}_0) + k(\operatorname{grad} f)(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0) + kr_f(\mathbf{x}).$$
>
> With $N_{\mathbf{x}} = \|\mathbf{x}-\mathbf{x}_0\|$, the linearity property (5.15) implies $\lim_{\mathbf{x}\to\mathbf{x}_0} kr_f(\mathbf{x})/N_{\mathbf{x}} = k\lim_{\mathbf{x}\to\mathbf{x}_0} r_f(\mathbf{x})/N_{\mathbf{x}} = 0$, so that $(kf)$ is differentiable: $\operatorname{grad}(kf)$ exists and is $k \operatorname{grad} f$; and the tangent map exists and is $T_{(kf)} = kT_f$.
>
> For linearity, adding (5.50) and (5.51), and using the univariate result (2.68), we have $(\operatorname{grad}(f+g))(\mathbf{x}_0) = (\operatorname{grad} f)(\mathbf{x}_0) + (\operatorname{grad} g)(\mathbf{x}_0)$ and $r_{f+g}(\mathbf{x}) = r_f(\mathbf{x}) + r_g(\mathbf{x})$. It remains to show that $\lim_{\mathbf{x}\to\mathbf{x}_0} r_{f+g}(\mathbf{x})/N_{\mathbf{x}} = 0$. But this again follows from (5.52) and the linearity property of limits (5.15).

Theorem: Let $f, g : A \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{a} \in A^o$. Then $f \cdot g$ is differentiable at $\mathbf{a}$, and

$$\big(\operatorname{grad}(f \cdot g)\big)(\mathbf{a}) = (\operatorname{grad} f)(\mathbf{a}) \cdot g(\mathbf{a}) + f(\mathbf{a}) \cdot (\operatorname{grad} g)(\mathbf{a}). \tag{5.55}$$

If, in addition, $g(\mathbf{a}) \neq 0$, then the function $f/g$ is differentiable at $\mathbf{a}$, and

$$\left(\operatorname{grad}\left(\frac{f}{g}\right)\right)(\mathbf{a}) = \frac{(\operatorname{grad} f)(\mathbf{a}) \cdot g(\mathbf{a}) - f(\mathbf{a}) \cdot (\operatorname{grad} g)(\mathbf{a})}{[g(\mathbf{a})]^2}. \tag{5.56}$$

> Proof: For convenience, denote $(\operatorname{grad} f)(\mathbf{x}) = (D_1 f(\mathbf{x}), \ldots, D_n f(\mathbf{x}))$ as $(A_1, \ldots, A_n)$. Likewise, let $(\operatorname{grad} g)(\mathbf{x}) = (B_1, \ldots, B_n)$. For (5.55), multiplying (5.50) and (5.51), and with $\mathbf{x} \in A^o$,
>
> $$\begin{aligned} f(\mathbf{x})g(\mathbf{x}) = f(\mathbf{a})g(\mathbf{a}) &+ \sum_{i=1}^n \left(A_i g(\mathbf{a}) + B_i f(\mathbf{a})\right)(x_i - a_i) + \sum_{i=1}^n \sum_{j=1}^n A_i B_j (x_i - a_i)(x_j - a_j) \\ &+ r_f(\mathbf{x})g(\mathbf{x}) + r_g(\mathbf{x})f(\mathbf{x}) - r_f(\mathbf{x})r_g(\mathbf{x}). \end{aligned}$$
>
> Consider the last three remainder terms. From the basic multiplicity property of limits and the differentiability of $f$; in particular (5.52),
>
> $$\lim_{\mathbf{x}\to\mathbf{a}} \frac{r_f(\mathbf{x})g(\mathbf{x})}{\|\mathbf{x}-\mathbf{a}\|} = \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) \cdot \lim_{\mathbf{x}\to\mathbf{a}} \frac{r_f(\mathbf{x})}{\|\mathbf{x}-\mathbf{a}\|} = g(\mathbf{a}) \cdot 0 = 0,$$

and likewise for the $r_g(\mathbf{x})f(\mathbf{x})$ term. For the $r_f(\mathbf{x})r_g(\mathbf{x})$ term, from (5.45),

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{r_f(\mathbf{x})r_g(\mathbf{x})}{\|\mathbf{x}-\mathbf{a}\|} = \lim_{\mathbf{x}\to\mathbf{a}} r_f(\mathbf{x}) \cdot \lim_{\mathbf{x}\to\mathbf{a}} \frac{r_g(\mathbf{x})}{\|\mathbf{x}-\mathbf{a}\|} = 0 \cdot 0.$$

For the double sum in the above product expression, note that each of its $n^2$ terms satisfies

$$0 \leq \frac{|x_i - a_i|\,|x_j - a_j|}{\|\mathbf{x}-\mathbf{a}\|} \leq |x_i - a_i|\,,$$

seen by multiplying by $\|\mathbf{x}-\mathbf{a}\|$ and using (5.7). Taking limits and using the Squeeze Theorem (2.9) implies that each term, and thus the double sum, divided by $\|\mathbf{x}-\mathbf{a}\|$, converges to zero.

Thus, we can define $r_{f\cdot g}$ to be the sum of the last four components in the above product expression for $f(\mathbf{x})g(\mathbf{x})$. Using (5.50) and (5.51) as analogies, the single-term sum in the above expression for $f(\mathbf{x})g(\mathbf{x})$ must be $(\mathrm{grad}(f\cdot g))(\mathbf{a})(\mathbf{x}-\mathbf{a})$. This can be expressed as

$$\begin{aligned}
(\mathrm{grad}(f\cdot g))(\mathbf{a}) &= \begin{bmatrix} A_1 g(\mathbf{a}) + B_1 f(\mathbf{a}) & A_2 g(\mathbf{a}) + B_2 f(\mathbf{a}) & \dots & A_n g(\mathbf{a}) + B_n f(\mathbf{a}) \end{bmatrix} \\
&= g(\mathbf{a}) \begin{bmatrix} A_1 & A_2 & \dots & A_n \end{bmatrix} + f(\mathbf{a}) \begin{bmatrix} B_1 & B_2 & \dots & B_n \end{bmatrix} \\
&= g(\mathbf{a})(\mathbf{D}f)(\mathbf{a}) + f(\mathbf{a})(\mathbf{D}g)(\mathbf{a}).
\end{aligned}$$

For the quotient result (5.56), see Petrovic, p. 349.

For some optional reading and enrichment into this material, we give the presentation from Lang (1987, §III.3). it nicely shows the extension of (2.66) (copied here as (5.57)) to the $n > 1$ case. Recall the fundamental lemma of differentiation (2.66):

> Let $f : A \subset \mathbb{R} \to \mathbb{R}$ be differentiable at the point $x$. Then there exists a function $\eta$ defined on an interval about zero such that
>
> $$f(x+h) - f(x) = \big[f'(x) + \eta(h)\big] \cdot h, \tag{5.57}$$
>
> and $\eta$ is continuous at zero, with $\eta(0) = 0$.

Let $f$ be a function defined on an open set $U$. Let $X$ be a point of $U$. For all vectors $H$ such that $\|H\|$ is small (and $H \neq O$), the point $X + H$ also lies in the open set. However, we cannot form a quotient

$$\frac{f(X+H) - f(X)}{H},$$

because it is meaningless to divide by a vector. In order to define what we mean for a function $f$ to be differentiable, we must therefore find a way that does not involve dividing by $H$. We reconsider the case of functions of one variable. Let us fix a number $x$. We had defined the derivative to be

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Let

$$\varphi(h) = \frac{f(x+h) - f(x)}{h} - f'(x).$$

Then $\varphi(h)$ is not defined when $h = 0$, but $\lim_{h\to 0} \varphi(h) = 0$. We can write

$$f(x+h) - f(x) = f'(x)h + h\varphi(h).$$

This relation has meaning so far only when $h \neq 0$. However, we observe that if we define $\varphi(0)$ to be 0, then the preceding relation is obviously true when $h = 0$ (because we just get $0 = 0$). Notice $\varphi$ is $\eta$ from (5.57).

Let

$$g(h) = \varphi(h), \quad \text{if} \quad h > 0, \qquad g(h) = -\varphi(h), \quad \text{if} \quad h < 0.$$

Then we have shown that, if $f$ is differentiable, then there exists a function $g$ such that

$$f(x+h) - f(x) = f'(x)h + |h|g(h), \tag{5.58}$$

and $\lim_{h\to 0} g(h) = 0$. Conversely, suppose that there exists a number $a$ and a function $g(h)$ such that

$$f(x+h) - f(x) = ah + |h|g(h) \tag{5.59}$$

and

$$\lim_{h\to 0} g(h) = 0.$$

We find for $h \neq 0$,

$$\frac{f(x+h) - f(x)}{h} = a + \frac{|h|}{h} g(h).$$

Taking the limit as $h$ approaches 0, we observe that

$$\lim_{h\to 0} \frac{|h|}{h} g(h) = 0.$$

Hence, the limit of the Newton quotient exists and is equal to $a$. Hence $f$ is differentiable, and its derivative $f'(x)$ is equal to $a$.

Therefore, the existence of a number $a$ and a function $g$ satisfying (5.59) could have been used as the definition of differentiability in the case of functions of one variable. The great advantage of (5.58) is that no $h$ appears in the denominator. It is this relation that will suggest to us how to define differentiability for functions of several variables, and how to prove the chain rule for them.

Let us begin with two variables. We let $X = (x, y)$ and $H = (h, k)$. Then the notion corresponding to $x+h$ in one variable is here $X + H = (x+h, y+k)$. We wish to compare the values of a function $f$ at $X$ and $X + H$, i.e. we wish to investigate the difference

$$f(X + H) - f(X) = f(x+h, y+k) - f(x, y).$$

Definition: We say that $f$ is *differentiable* at $X$ if the partial derivatives

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}$$

exist, and if there exists a function $g$ (defined for small $H$) such that $\lim_{H\to O} g(H) = 0$ and

$$f(x+h, y+k) - f(x, y) = \frac{\partial f}{\partial x} h + \frac{\partial f}{\partial y} k + \|H\| g(H). \tag{5.60}$$

Observe that this is precisely what we have above, namely equations (5.41) and (5.42).

We view the term

$$\frac{\partial f}{\partial x}h + \frac{\partial f}{\partial y}k$$

as an approximation to $f(X+H) - f(X)$, depending in a particularly simple way on $h$ and $k$. If we use the abbreviation grad $f = \nabla f$, then (5.60) can be written

$$f(X+H) - f(X) = \nabla f(x) \cdot H + \|H\| g(H).$$

As with grad $f$, one must read $(\nabla f)(X)$ and not the meaningless $\nabla(f(X))$ since $f(X)$ is a number for each value of $X$, and thus it makes no sense to apply $\nabla$ to a number. The symbol $\nabla$ is applied to the function $f$, and $(\nabla f)(X)$ is the value of $\nabla f$ at $X$.

We now consider a function of $n$ variables. Let $f$ be a function defined on an open set $U$. Let $X$ be a point of $U$. If $H = (h_1, \ldots, h_n)$ is a vector such that $\|H\|$ is small enough, then $X + H$ will also be a point of $U$ and so $f(X+H)$ is defined. Note that

$$X + H = (x_1 + h_1, \ldots, x_n + h_n)\,.$$

This is the generalization of the $x + h$ with which we dealt previously in one variable, or the $(x+h, y+k)$ in two variables. For three variables, we already run out of convenient letters, so we may as well write $n$ instead of 3.

Definition: We say that $f$ is differentiable at $X$ if, first, all the partial derivatives $D_i f(X)$ exist, $i = 1, \ldots, n$; and, second, if there exists a function $g$ (defined for small $H$) such that $\lim_{H \to O} g(H) = 0$, also written $\lim_{\|H\| \to 0} g(H) = 0$, whereby

$$f(X+H) - f(X) = D_1 f(X) h_1 + \cdots + D_n f(x) h_n + \|H\| g(H).$$

We say that $f$ is *differentiable in the open set* $U$ if it is differentiable at every point of $U$, so that the above relation holds for every point $X \in U$. In view of the definition of the gradient, we can rewrite our fundamental relation in the form

$$f(X+H) - f(X) = (\operatorname{grad} f(X)) \cdot H + \|H\| g(H).$$

The term $\|H\| g(H)$ has an order of magnitude smaller than the previous term involving the dot product. This is one advantage of the present notation. We know how to handle the formalism of dot products and are accustomed to it, and its geometric interpretation. This will help us later in interpreting the gradient geometrically.

As an example, suppose that we consider values for $H$ pointing only in the direction of the standard unit vectors. In the case of two variables, consider for instance $H = (h, 0)$. Then for such $H$, the condition for differentiability reads:

$$f(X+H) = f(x+h, y) = f(x, y) + \frac{\partial f}{\partial x}h + |h| g(H).$$

In higher dimensional space, let $E_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the $i$th unit vector. Let $H = hE_i$ for some number $h$, so that $H = (0, \ldots, 0, h, 0, \ldots, 0)$. Then for such $H$,

$$f(X+H) = f(X + hE_i) = f(X) + \frac{\partial f}{\partial x_i}h + |h| g(H),$$

and, therefore, if $h \neq 0$, we obtain

$$\frac{f(X+H) - f(X)}{h} = D_i f(X) + \frac{|h|}{h} g(H).$$

Because of the special choice of $H$, we can divide by the number $h$, but we are not dividing by the vector $H$.

I now give a different (and far less efficient; and possibly faulty) proof than the one above, around (5.46), for the $n = 2$ case, that differentiability of $f : A \subset \mathbb{R}^2 \to \mathbb{R}$, $A$ open, implies $f$ is continuous (on $A$). The goal was to use the $\epsilon$-$\delta$ formulation of continuity in (5.19) instead of the (in this case, far easier) sequential limit formulation (5.18). The right way to do the proof is shown in (5.97), which is for general $n$ and $m$. The odd thing about my attempt is that it appears to show $\lim_{(\delta_x,\delta_y)\to(0,0)} R_{x,y} = 0$, where $R_{x,y}$ is given in (5.64), which is the requirement in (5.42). In other words, it appears as though only the existence of grad $f$ was required, as opposed to also requiring existence of the tangent map. Again recall the fundamental lemma of differentiation (2.66):

> Let $f : A \subset \mathbb{R} \to \mathbb{R}$ be differentiable at the point $x$. Then there exists a function $\eta$ defined on an interval about zero such that
>
> $$f(x+h) - f(x) = \left[f'(x) + \eta(h)\right] \cdot h, \tag{5.61}$$
>
> and $\eta$ is continuous at zero, with $\eta(0) = 0$.

"Proof": Let $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ be differentiable on the open set $A$. We wish to show that $f$ is also continuous on $A$. For $(x, y) \in A$, let $z = f(x, y)$ and let $\delta_x$ and $\delta_y$ represent very small, positive quantities such that $(x + \delta_x, y + \delta_y) \in A$. Let $\delta_z$ be such that $z + \delta_z = f(x + \delta_x, y + \delta_y)$, i.e.,

$$\begin{aligned}
\delta_z &= f(x+\delta_x, y+\delta_y) - f(x,y) \\
&= f(x+\delta_x, y+\delta_y) - f(x, y+\delta_y) + f(x, y+\delta_y) - f(x,y) \\
&= \frac{f(x+\delta_x, y+\delta_y) - f(x, y+\delta_y)}{\delta_x}\delta_x + \frac{f(x, y+\delta_y) - f(x,y)}{\delta_y}\delta_y.
\end{aligned} \tag{5.62}$$

Using (5.61), with $\delta_x$ playing the role of $h$ in (5.61); and, separately, $\delta_y$ playing the role of $h$ in (5.61), (5.62) can be written as

$$\begin{aligned}
\delta_z &= \left(\frac{\partial f(x, y+\delta_y)}{\partial x} + \eta_x\right)\delta_x + \left(\frac{\partial f(x,y)}{\partial y} + \eta_y\right)\delta_y \\
&= \frac{\partial f(x, y+\delta_y)}{\partial x}\delta_x + \frac{\partial f(x,y)}{\partial y}\delta_y \;+\; \eta_x\delta_x \;+\; \eta_y\delta_y,
\end{aligned} \tag{5.63}$$

where the existence of $D_1 f$ and $D_2 f$ was assumed via differentiability, and which is of the form (5.41) with $r(\mathbf{x}) = \eta_x\delta_x + \eta_y\delta_y$. With

$$N_{x,y} := \sqrt{\delta_x^2 + \delta_y^2}, \quad \text{and} \quad R_{x,y} := \left|\frac{\eta_x\delta_x + \eta_y\delta_y}{N_{x,y}}\right|, \tag{5.64}$$

and using the triangle inequality,

$$0 \le \lim_{(\delta_x,\delta_y)\to(0,0)} R_{x,y} \le \lim_{(\delta_x,\delta_y)\to(0,0)} |\eta_x| \frac{|\delta_x|}{N_{x,y}} + |\eta_y| \frac{|\delta_y|}{N_{x,y}} < \lim_{(\delta_x,\delta_y)\to(0,0)} |\eta_x| + |\eta_y|, \tag{5.65}$$

because $0 < |\delta_x|/N_{x,y} < 1$ and $0 < |\delta_y|/N_{x,y} < 1$. The continuity of the $\eta$ function and that $\eta(0) = 0$ implies $\lim_{\delta_x\to 0}\eta_x = \lim_{\delta_y\to 0}\eta_y = 0$. Thus, the Squeeze Theorem (2.9) implies the lhs limit in (5.65) is zero; which obviously implies

$$\lim_{(\delta_x,\delta_y)\to(0,0)} (\eta_x\delta_x + \eta_y\delta_y) = 0.$$

That is, $\exists\delta_x > 0$ and $\exists\delta_y > 0$ such that $\delta_z$ in (5.63) can be made arbitrarily close to zero. Thus, as $(x, y) \in A$ was arbitrary, $f$ is continuous on $A$.

## 5.4 Higher Order Partial Derivatives

We now turn to second-order partial derivatives, using for this subsection the presentation in Fitzpatrick. Further comments I have added for clarity are indicated in blue color.

Given an open subset $\mathcal{O}$ of $\mathbb{R}^n$ and an index $i$ with $1 \le i \le n$, if the function $f : \mathcal{O} \to \mathbb{R}$ has a partial derivative with respect to its $i$th component at each point in $\mathcal{O}$, then the function $\partial f/\partial x_i : \mathcal{O} \to \mathbb{R}$ is defined and we can ask whether this new function itself has first-order partial derivatives. Fix an index $j$ with $1 \le j \le n$. If the function $\partial f/\partial x_i : \mathcal{O} \to \mathbb{R}$ has a partial derivative with respect to its $j$th component at the point $\mathbf{x}$ in $\mathcal{O}$, we write

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}) \quad \text{to denote} \quad \frac{\partial}{\partial x_j}\left[\frac{\partial f}{\partial x_i}\right](\mathbf{x}).$$

When $n = 2$ or 3, and points are labeled without subscripts, we use a more suggestive notation for second partial derivatives; e.g., $\partial^2 f/\partial x \partial y$, etc..

Definition: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and consider a function $f : \mathcal{O} \to \mathbb{R}$:

i. The function $f : \mathcal{O} \to \mathbb{R}$ is said to have second-order partial derivatives provided that it has first-order partial derivatives and that, for each index $i$ with $1 \le i \le n$, the function $\partial f/\partial x_i : \mathcal{O} \to \mathbb{R}$ also has first-order partial derivatives.

ii. The function $f : \mathcal{O} \to \mathbb{R}$ is said to have continuous second-order partial derivatives provided that it has second-order partial derivatives and that, for each pair of indices $i$ and $j$ with $1 \le i \le n$ and $1 \le j \le n$, the function $\partial^2 f/\partial x_i \partial x_j : \mathcal{O} \to \mathbb{R}$ is continuous.

So, if $f$ is continuously differentiable, then it means its first derivatives (exist and) are continuous. The Continuity Theorem (5.49) then implies $f$ is continuous. Apply this to the second derivatives: If second derivatives (exist and) are continuous, then applying the previous result, first derivatives are continuous, i.e., the function $f$ is continuously differentiable.

We now state and prove a fundamental result regarding exchange of the partial derivative operator. Some books refer to this as Clairaut's, or Schwarz's, Theorem. We give the proof as in Fitzpatrick, but the reader can also see, e.g., Lang (1997, p. 372) or Protter and Morrey (1991, p. 179). Before proceeding, we note that a set of weaker conditions is adequate for the theorem. This is detailed, with references, in `https://math.stackexchange.com/questions/98514`, notably Apostol's Mathematical Analysis (2nd edition, 1974, p. 360): If $D_r\mathbf{f}, D_k\mathbf{f}$ and $D_{k,r}\mathbf{f}$ are continuous in an $n$-ball $B(\mathbf{c})$, then $D_{r,k}\mathbf{f}(\mathbf{c})$ exists and equals $D_{k,r}\mathbf{f}(\mathbf{c})$.

Theorem (Fitzpatrick, 13.10): Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has continuous second-order partial derivatives. For any two indices $i$ and $j$ with $1 \le i \le n$ and $1 \le j \le n$ and any point $\mathbf{x}$ in $\mathcal{O}$,

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}). \tag{5.66}$$

To prove this, we first require the following lemma:

Lemma (Fitzpatrick, 13.11): Let $\mathcal{U}$ be an open subset of the plane $\mathbb{R}^2$ that contains the point $(x_0, y_0)$ and suppose that the function $f : \mathcal{U} \to \mathbb{R}$ has second-order partial derivatives (not necessarily continuous). Then there are points $(x_1, y_1)$ and $(x_2, y_2)$ in $\mathcal{U}$ at which

$$\frac{\partial^2 f}{\partial x \partial y}\left(x_1, y_1\right) = \frac{\partial^2 f}{\partial y \partial x}\left(x_2, y_2\right).$$

Proof: Since $\mathcal{U}$ is open, we can choose a positive number $r$ such that if we define the intervals of real numbers $I$ and $J$ by $I = (x_0 - 2r, x_0 + 2r)$ and $J = (y_0 - 2r, y_0 + 2r)$, then the rectangle $I \times J$ is contained in $\mathcal{U}$. The idea of the proof is to express

$$f(x_0 + r, y_0 + r) - f(x_0 + r, y_0) - f(x_0, y_0 + r) + f(x_0, y_0)$$

as a difference in two different ways: First as the difference

$$[f(x_0 + r, y_0 + r) - f(x_0 + r, y_0)] - [f(x_0, y_0 + r) - f(x_0, y_0)], \tag{5.67}$$

(Below, this is the same as $\phi(x_0 + r) - \phi(x_0)$.), and then as the difference

$$[f(x_0 + r, y_0 + r) - f(x_0, y_0 + r)] - [f(x_0 + r, y_0) - f(x_0, y_0)]. \tag{5.68}$$

Then we use the Mean Value Theorem for functions of a single real variable to express (5.67) and (5.68) as second-order partial derivatives of the function $f : \mathcal{U} \to \mathbb{R}$.

First we analyze the difference (5.67). Define the auxiliary function $\varphi : I \to \mathbb{R}$ by

$$\varphi(x) = f(x, y_0 + r) - f(x, y_0) \qquad \text{for } x \text{ in } I.$$

Since $f : \mathcal{U} \to \mathbb{R}$ has a partial derivative with respect to its first component, the function $\varphi : I \to \mathbb{R}$ is differentiable. Thus, we can apply the Mean Value Theorem to the restriction of the function $\varphi : I \to \mathbb{R}$ to the closed interval $[x_0, x_0 + r]$ to select a point $x_1$ in the open interval $(x_0, x_0 + r)$ such that

$$\frac{\varphi(x_0 + r) - \varphi(x_0)}{r} = \varphi'(x_1);$$

that is, (Below, this is $\alpha(y_0 + r) - \alpha(y_0)$.)

$$\frac{\varphi(x_0 + r) - \varphi(x_0)}{r} = \frac{\partial f}{\partial x}(x_1, y_0 + r) - \frac{\partial f}{\partial x}(x_1, y_0). \tag{5.69}$$

With this point $x_1$ fixed, define another auxiliary function $\alpha : J \to \mathbb{R}$ by

$$\alpha(y) = \frac{\partial f}{\partial x}(x_1, y), \quad y \in J.$$

We can apply the Mean Value Theorem to the restriction of the function $\alpha : J \to \mathbb{R}$ to the closed interval $[y_0, y_0 + r]$ to select a point $y_1$ in the open interval $(y_0, y_0 + r)$ such that

$$\frac{\alpha(y_0 + r) - \alpha(y_0)}{r} = \frac{\partial^2 f}{\partial y \partial x}(x_1, y_1), \quad x_1 \in (x_0, x_0 + r), \;\; y_1 \in (y_0, y_0 + r). \tag{5.70}$$

From (5.69) and (5.70), we obtain

$$\varphi(x_0 + r) - \varphi(x_0) = r^2 \frac{\partial^2 f}{\partial y \partial x}(x_1, y_1).$$

However, $\varphi(x_0 + r) - \varphi(x_0)$ equals the difference (5.67), and hence we have

$$\begin{aligned}&[f(x_0 + r, y_0 + r) - f(x_0 + r, y_0)] - [f(x_0, y_0 + r) - f(x_0, y_0)] \\ &\quad = r^2 \frac{\partial^2 f}{\partial y \partial x}(x_1, y_1).\end{aligned} \tag{5.71}$$

In order to analyze the difference (5.68), we now repeat the same argument applied to the auxiliary function $\psi : J \to \mathbb{R}$ defined by

$$\psi(y) = f(x_0 + r, y) - f(x_0, y) \quad y \in J.$$

From this it will follow that we can select a point $(x_2, y_2)$ in the rectangle $I \times J$ such that

$$\begin{aligned}&[f(x_0 + r, y_0 + r) - f(x_0, y_0 + r)] - [f(x_0 + r, y_0) - f(x_0, y_0)]\\ &= r^2 \frac{\partial^2 f}{\partial x \partial y}(x_2, y_2).\end{aligned} \tag{5.72}$$

From the equality of the left-hand sides of (5.71) and (5.72) follows the equality of the right-hand sides, so the lemma is proved.

Proof of Fitzpatrick Theorem 13.10: We prove the theorem when $n = 2$ and leave the general case to the reader. Let $(x_0, y_0)$ be a point in $\mathcal{O}$. Choose a positive number $r$ such that the open ball $\mathcal{B}_r(x_0, y_0)$ is contained in $\mathcal{O}$. Let $k$ be a natural number. Then we can apply the lemma with $\mathcal{U} = \mathcal{B}_{r/k}(x_0, y_0)$ and select points $(x_k, y_k)$ and $(u_k, v_k)$ in $\mathcal{B}_{r/k}(x_0, y_0)$ at which

$$\frac{\partial^2 f}{\partial x \partial y}(x_k, y_k) = \frac{\partial^2 f}{\partial y \partial x}(u_k, v_k). \tag{5.73}$$

But, by assumption, the function $\partial^2 f/\partial x \partial y : \mathcal{O} \to \mathbb{R}$ is continuous at $(x_0, y_0)$, as is the function $\partial^2 f/\partial y \partial x : \mathcal{O} \to \mathbb{R}$. Since the sequences $\{(x_k, y_k)\}$ and $\{(u_k, v_k)\}$ both converge to the point $(x_0, y_0)$, it follows from (5.17) that

$$\lim_{k\to\infty} \left[\frac{\partial^2 f}{\partial x \partial y}(x_k, y_k)\right] = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0), \quad \lim_{k\to\infty} \left[\frac{\partial^2 f}{\partial y \partial x}(u_k, v_k)\right] = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0).$$

In view of these two equations, and taking limits in (5.73), we arrive at (5.66).

Observe that, in Lemma 13.11, we required only that the function $f : \mathcal{O} \to \mathbb{R}$ have second-order partial derivatives. On the other hand, in Theorem 13.10, we required that the second-order partial derivatives be continuous. This extra assumption is necessary. The following is an example of a function $f : \mathcal{O} \to \mathbb{R}$ that has second-order partial derivatives, and yet we do not have equality of $\partial^2 f/\partial x \partial y$ and $\partial^2 f/\partial y \partial x$ at all points.

**Example 5.12** *(Fitzpatrick, p. 361) Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by*

$$f(x, y) = \begin{cases} xy\,(x^2 - y^2) / (x^2 + y^2), & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

*Calculations show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ has second-order partial derivatives but that*

$$\frac{\partial^2 f}{\partial y \partial x}(0, 0) = -1 \quad \text{while} \quad \frac{\partial^2 f}{\partial x \partial y}(0, 0) = 1,$$

*and thus not equal.* ■

According to https://en.m.wikipedia.org/wiki/Symmetry_of_second_derivatives#Requirement_of_continuity, the previous example is from Peano, and can be found in, among other books, Apostol's Mathematical Analysis (2nd edition, 1974, pp. 358–359).

**Example 5.13** *(https://math.stackexchange.com/questions/956095) Consider the function $f(x,y) := \frac{xy^3}{x^2+y^2}$; and $f(0,0) := (0,0)$. You can check that this function is continuous and differentiable for all $(x,y)$. We have*

$$\frac{\partial f}{\partial y}(x,y) = \frac{3xy^2}{x^2+y^2} - \frac{2xy^4}{(x^2+y^2)^2},$$

*and at $y = 0$ we have $\frac{\partial f}{\partial y}(x,0) = 0$, thus $\frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$. On the other hand,*

$$\frac{\partial f}{\partial x}(x,y) = \frac{y^3}{x^2+y^2} - \frac{2x^2y^3}{(x^2+y^2)^2},$$

*and, hence, $\frac{\partial f}{\partial x}(0,y) = y$. Thus $\frac{\partial^2 f}{\partial y \partial x}(0,0) = 1$, and $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$.* ■

The next example shows a different kind of application of Clairaut's theorem (5.66).

**Example 5.14** *(Bóna and Shabanov, Concepts in Calculus III, p. 153) Find $f(x,y,z)$ if $f'_x = yz + 2x = F_1$, $f'_y = xz + 3y^2 = F_2$, and $f'_z = xy + 4z^3 = F_3$; or show that it does not exist.*

*Solution: The integrability conditions $(F_1)'_y = (F_2)'_x$, $(F_1)'_z = (F_3)'_x$, and $(F_2)'_z = (F_3)'_z$ are satisfied (their verification is left to the reader). So $f$ exists. Taking the antiderivative with respect to $x$ in the first equation, one finds*

$$f'_x = yz + 2x \quad \Longrightarrow \quad f(x,y,z) = \int (yz+2x)dx = xyz + x^2 + g(y,z),$$

*where $g(y,z)$ is arbitrary. The substitution of $f$ into the second equations yields*

$$\begin{aligned} f'_y = xz + 3y^2 &\Longrightarrow xz + g'_y(y,z) = xz + 3y^2 \\ &\Longrightarrow g'_y(y,z) = 3y^2 \\ &\Longrightarrow g(y,z) = \int 3y^2 dy = y^3 + h(z) \\ &\Longrightarrow f(x,y,z) = xyz + x^2 + y^3 + h(z), \end{aligned}$$

*where $h(z)$ is arbitrary. The substitution of $f$ into the third equation yields*

$$\begin{aligned} f'_z = xy + 4z^3 &\Longrightarrow xy + h'(z) = xy + 4z^3 \\ &\Longrightarrow h'(z) = 4z^3 \\ &\Longrightarrow h(z) = z^4 + c \\ &\Longrightarrow f(x,y,z) = xyz + x^2 + y^3 + z^4 + c, \end{aligned}$$

*where $c$ is a constant.* ■

## 5.5 Directional Derivatives and the Multivariate MVT

This subsection, taken nearly verbatim from Fitzpatrick (2009, §13.3), is core material. For example, directional derivatives are fundamental in understanding multivariate function optimization. Whether minimizing a cost function or a financial risk measure; or maximizing an economic utility function or the statistical likelihood associated with a data set, optimization is perhaps the best example of the power and necessity of understanding this material. That holds, perhaps obviously, for gradient- and Hessian-based optimization algorithms, notably the very popular BFGS algorithm, and stochastic gradient descent in large-scale models in machine learning, but also for methods that are applicable for functions that are not differentiable, and perhaps not even continuous, such as evolutionary algorithms (differential evolution, genetic programming), tabu search, particle swarm, simulated annealing and, arguably the best of them all, CMA-ES, the latter also deeply intertwined with probability and statistical theory.[29]

A further goal of working through the material is to get to the proof that, if $f : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable on its open domain, then it is continuous. We have seen this result already in (5.49), though the proof of one of its parts, (5.48), was only heuristically determined for $n > 2$.

Further comments I have added for clarity are indicated in blue color.

Lemma (Fitzpatrick, 13.14, The Mean Value Lemma): Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and let $i$ be an index with $1 \leq i \leq n$. Suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has a partial derivative with respect to its $i$th component at each point in $\mathcal{O}$. Let $\mathbf{x}$ be a point in $\mathcal{O}$ and let $a$ be a real number such that the segment between the points $\mathbf{x}$ and $\mathbf{x} + a\mathbf{e}_i$ lies in $\mathcal{O}$. Then there is a number $\theta$ with $0 < \theta < 1$ such that

$$f(\mathbf{x} + a\mathbf{e}_i) - f(\mathbf{x}) = \frac{\partial f}{\partial x_i}(\mathbf{x} + \theta a\mathbf{e}_i)\, a. \tag{5.74}$$

Proof: Since $\mathcal{O}$ is open in $\mathbb{R}^n$, we can select an open interval of real numbers $I$ that contains the numbers 0 and $a$ such that, for each $t$ in $I$, the point $\mathbf{x} + t\mathbf{e}_i$ belongs to $\mathcal{O}$.

It is very useful to first review the parametrized path formulation in the beginning of §5.2, notably equations (5.27) and (5.28).

Define the function $\phi : I \to \mathbb{R}$ by $\phi(t; i, \mathbf{x}) = \phi(t) = f(\mathbf{x} + t\mathbf{e}_i)$ for each $t$ in $I$. Then the partial differentiability of the function $f : \mathcal{O} \to \mathbb{R}$ with respect to its $i$th component implies that, at each point $t$ in $I$,

$$\phi'(t) = \frac{\partial f}{\partial x_i}(\mathbf{x} + t\mathbf{e}_i).$$

It follows that the function $\phi : I \to \mathbb{R}$ is differentiable. Thus, we can apply the Mean Value Theorem for functions of a single variable to the restriction of the function $\phi : I \to \mathbb{R}$ to the closed interval $[0, a]$ to obtain a point $\theta$ with $0 < \theta < 1$ such that

$$\phi(a) - \phi(0) = \phi'(\theta a)a,$$

which, in view of the definition of the function $\phi : I \to \mathbb{R}$ and the calculation of $\phi'(t)$, can be rewritten as (5.74).

[29] For an introduction to CMA-ES, see Paolella, Fundamental Statistics, 2018, §4.4; and the Wikipedia entry; both of which contain Matlab codes. The latter also discusses recent extensions of the baseline construct.

Proposition (Fitzpatrick, 13.15, The Mean Value Proposition): Let $\mathbf{x}$ be a point in $\mathbb{R}^n$ and let $r$ be a positive number. Suppose that the function $f : \mathcal{B}_r(\mathbf{x}) \to \mathbb{R}$ has first-order partial derivatives. Then if the point $\mathbf{x}+\mathbf{h}$ belongs to $\mathcal{B}_r(\mathbf{x})$, there are points $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n \in \mathcal{B}_r(\mathbf{x})$ such that

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{z}_i), \tag{5.75}$$

For $h = (0, \ldots, a, 0, \ldots, 0)$, with value $a$ in the $i$th position, this is (5.74), with $z_i = x + \theta a \mathbf{e}_i = (x_1, x_2, \ldots, x_i + \theta a, x_{i+1}, \ldots, x_n)$.

and

Note with my $h$ and $z_i$ just given, and recalling $0 < \theta < 1$, $\|x - z_i\| = \theta|a| < |a| = \|h\|$, in agreement with:

$$\|\mathbf{x} - \mathbf{z}_i\| < \|\mathbf{h}\| \quad \text{for each index } i \text{ with } 1 \le i \le n.$$

Proof: We prove the result with $n = 3$. From this, it will be clear that the general result is also true. The trick is to expand the difference $f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})$. We have

$$\begin{aligned} f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) =& f(x_1+h_1, x_2+h_2, x_3+h_3) - f(x_1, x_2, x_3) \\ =& f(x_1+h_1, x_2+h_2, x_3+h_3) - f(x_1+h_1, x_2+h_2, x_3) \\ &+ f(x_1+h_1, x_2+h_2, x_3) - f(x_1+h_1, x_2, x_3) \\ &+ f(x_1+h_1, x_2, x_3) - f(x_1, x_2, x_3). \end{aligned} \tag{5.76}$$

The 2nd (3rd, 4th) line in (5.76) shows changes in the 3rd (2nd, 1st) component, respectively.

We apply the previous Mean Value Lemma (Fitzpatrick, 13.14) to each of these differences to find numbers $\theta_1$, $\theta_2$, and $\theta_3$ in the open interval $(0, 1)$ with

$$\begin{aligned} f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) =& \frac{\partial f}{\partial x_3}(x_1+h_1, x_2+h_2, x_3+\theta_3 h_3) h_3 \\ &+ \frac{\partial f}{\partial x_2}(x_1+h_1, x_2+\theta_2 h_2, x_3) h_2 \\ &+ \frac{\partial f}{\partial x_1}(x_1+\theta_1 h_1, x_2, x_3) h_1. \end{aligned} \tag{5.77}$$

In (5.77), use the substitutions

$$\begin{aligned} \mathbf{z}_1 &= (x_1+\theta_1 h_1, x_2, x_3), \\ \mathbf{z}_2 &= (x_1+h_1, x_2+\theta_2 h_2, x_3), \\ \mathbf{z}_3 &= (x_1+h_1, x_2+h_2, x_3+\theta_3 h_3), \end{aligned}$$

and the result follows. For the above norm inequality, recall $\theta_1, \theta_2, \theta_3 \in (0, 1)$, so that

$$\begin{aligned} \|\mathbf{x} - \mathbf{z}_1\| &= \|\mathbf{z}_1 - \mathbf{x}\| = \|(\theta_1 h_1,\ 0,\ 0)\| = \theta_1|h_1| < \|\mathbf{h}\|, \\ \|\mathbf{x} - \mathbf{z}_2\| &= \|\mathbf{z}_2 - \mathbf{x}\| = \|(h_1,\ \theta_2 h_2,\ 0)\| < \|\mathbf{h}\|, \\ \|\mathbf{x} - \mathbf{z}_3\| &= \|\mathbf{z}_3 - \mathbf{x}\| = \|(h_1,\ h_2,\ \theta_3 h_3)\| < \|\mathbf{h}\|. \end{aligned}$$

We now turn to an analysis of the limit in (5.28) when the point $\mathbf{e}_i$ is replaced by a general nonzero point $\mathbf{p}$ in $\mathbb{R}^n$.

Definition: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$. Consider a function $f : \mathcal{O} \to \mathbb{R}$ and a nonzero point $\mathbf{p}$ in $\mathbb{R}^n$. If the limit exists, we define

$$(D_{\mathbf{p}}f)(\mathbf{x}) = \frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}) = \lim_{t\to 0} \frac{f(\mathbf{x}+t\mathbf{p}) - f(\mathbf{x})}{t} \tag{5.78}$$

to be the directional derivative of the function $f : \mathcal{O} \to \mathbb{R}$ in the direction $\mathbf{p}$ at the point $\mathbf{x}$. Observe that, if $\mathbf{p} = \mathbf{e}_i$, then (5.78) is equivalent to $(D_i f)(\mathbf{x})$. Figure 42 illustrates this for a bivariate function at point $\mathbf{x} = (x_0, y_0)$ in the direction of vector $\mathbf{u} = (a, b)$.

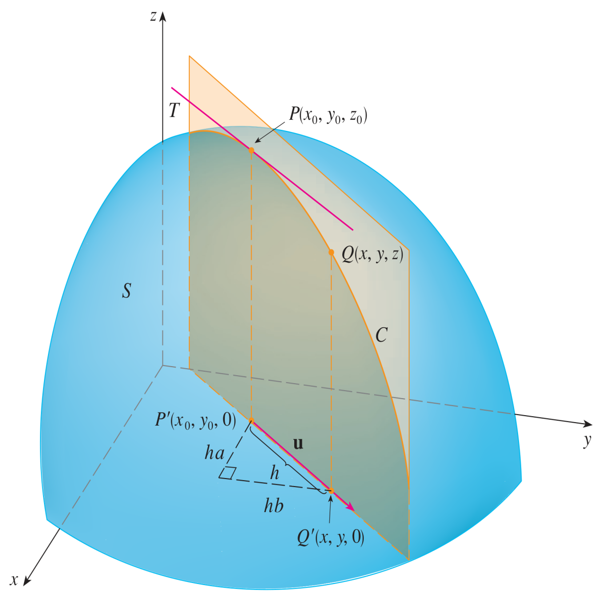



Figure 42: The slope of the tangent line $T$ to slice $C$ at the point $P$ is the rate of change of $z = f(x, y)$ in the direction of vector $\mathbf{u} = (a, b)$, with $h = 1$ corresponding to $\|\mathbf{u}\| = 1$. From Stewart, Multivariate Calculus, 8th ed., 2016, p. 987.

**Example 5.15** *(DePree and Swartz, Introduction to Real Analysis, 1988, p. 102) Even if a function has directional derivatives in all directions at a point, it may still fail to be continuous there. Consider the function*

$$f(x,y) = \begin{cases} xy^2/\left(x^2+y^4\right), & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

*For* $\mathbf{p} = (a, b)$ *with* $a \neq 0$,

$$\lim_{t\to 0} \frac{f(\mathbf{0}+t\mathbf{p}) - f(\mathbf{0})}{t} = \lim_{t\to 0} \frac{ab^2}{a^2+t^2b^4} = \frac{b^2}{a} = (D_{\mathbf{p}}f)(\mathbf{0});$$

*if* $a = 0$, $D_{\mathbf{p}}f(\mathbf{0}) = 0$. *Thus* $f$ *has directional derivatives in all directions at* $\mathbf{0}$. *However,* $f$ *is not continuous there, for on the curve* $y^2 = x$, $f$ *has the value* $1/2$. ■

The terminology *directional derivative* is standard but is somewhat misleading, because the directional derivative depends not only on the direction of $\mathbf{p}$, but also on its length. As one possible way to see this, recall (5.29): $(D_{\mathbf{p}}f)(\mathbf{x})$ is the derivative of $\gamma(t) = f(\mathbf{x} + t\mathbf{p})$ at 0. If we allow $\|\mathbf{p}\| \neq 1$, we can define $\phi(t) = f(\mathbf{x} + t\mathbf{p}/\|\mathbf{p}\|)$. Then

$$\gamma'(t) = \frac{d}{dt}f(\mathbf{x} + t\mathbf{p}) = \frac{d}{dt}f\left(\mathbf{x} + (t\|\mathbf{p}\|)\frac{\mathbf{p}}{\|\mathbf{p}\|}\right) = \|\mathbf{p}\|\phi'(t),$$

which results in different directional derivatives along the same direction. Another, immediate way to see this is from (5.79) below.

Theorem (Fitzpatrick, 13.16, The Directional Derivative Theorem): Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Then for each point $\mathbf{x}$ in $\mathcal{O}$ and each nonzero point $\mathbf{p}$ in $\mathbb{R}^n$, the function $f : \mathcal{O} \to \mathbb{R}$ has a directional derivative in the direction $\mathbf{p}$ at the point $\mathbf{x}$ that is given by the formula

$$\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}) = \sum_{i=1}^{n} p_i \frac{\partial f}{\partial x_i}(\mathbf{x}), \qquad \mathbf{p} = (p_1, \ldots, p_n). \tag{5.79}$$

Proof: Since $\mathcal{O}$ is an open subset of $\mathbb{R}^n$, we can choose a positive number $r$ such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$. Then from the previous Mean Value Proposition (Fitzpatrick, 13.15), we see that, if $t$ is any number with $|t|\|\mathbf{p}\| < r$, then there are $n$ points $\mathbf{z}_1, \ldots, \mathbf{z}_n \in \mathcal{B}_r(\mathbf{x})$ such that

Recall in the statement of Prop. 13.15, "if the point $\mathbf{x} + \mathbf{h}$ belongs to $\mathcal{B}_r(\mathbf{x})$". Thus, $\mathbf{x} + t\mathbf{p} \in \mathcal{B}_r(\mathbf{x})$. Also, perhaps obviously, $|t|\|\mathbf{p}\|$ has the same direction as $\mathbf{p}$, for $t \neq 0$.

$$f(\mathbf{x} + t\mathbf{p}) - f(\mathbf{x}) = \sum_{i=1}^{n} tp_i \frac{\partial f}{\partial x_i}(\mathbf{z}_i) \tag{5.80}$$

and

$$\|\mathbf{z}_i - \mathbf{x}\| \leq |t|\|\mathbf{p}\| \quad \text{ for each index } i \text{ with } 1 \leq i \leq n. \tag{5.81}$$

We can rewrite (5.80) as

$$\frac{f(\mathbf{x} + t\mathbf{p}) - f(\mathbf{x})}{t} = \sum_{i=1}^{n} p_i \frac{\partial f}{\partial x_i}(\mathbf{z}_i) \quad \text{ for } t \neq 0. \tag{5.82}$$

Since $\partial f/\partial x_i : \mathcal{O} \to \mathbb{R}$ is continuous at the point $\mathbf{x}$ for each index $i$ with $1 \leq i \leq n$, it follows from (5.81) and (5.82) that and the definition of directional derivative

$$\lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{p}) - f(\mathbf{x})}{t} = \lim_{t \to 0} \sum_{i=1}^{n} p_i \frac{\partial f}{\partial x_i}(\mathbf{z}_i) = \sum_{i=1}^{n} p_i \frac{\partial f}{\partial x_i}(\mathbf{x}).$$

This proves formula (5.79).

In view of formula (5.79), we introduce the following definition.

Definition: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has first-order partial derivatives at $\mathbf{x}$. (Not necessarily continuous.) We define the gradient of the function $f : \mathcal{O} \to \mathbb{R}$ at the point $\mathbf{x}$, denoted by $\nabla f(\mathbf{x})$, to be the point in $\mathbb{R}^n$ given by and as previously defined in (5.25)

$$\nabla f(\mathbf{x}) = \left( \frac{\partial f}{\partial x_1}(\mathbf{x}), \frac{\partial f}{\partial x_2}(\mathbf{x}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{x}) \right). \tag{5.83}$$

Through the identification of points in $\mathbb{R}^n$ with vectors, $\nabla f(\mathbf{x})$ is often referred to as the gradient vector or derivative vector. Using the scalar product and the gradient, formula (5.79) can be compactly written as (Or as a regular matrix (here, vector) product, with the convention that $\nabla f$ is a row vector, and $\mathbf{p}$ is a column vector.)

$$\frac{d}{dt}[f(\mathbf{x}+t\mathbf{p})]\bigg|_{t=0} = \frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}) = \langle \nabla f(\mathbf{x}), \mathbf{p} \rangle. \tag{5.84}$$

It is also useful to observe a slight extension of (5.84): Replacing the point $\mathbf{x}$ in the latter two quantities; and not the first! with the point $\mathbf{x}+t\mathbf{p}$, it follows that and note that the left hand sides of (5.84) and (5.85) are not the same![30]

$$\frac{d}{dt}[f(\mathbf{x}+t\mathbf{p})] = \langle \nabla f(\mathbf{x}+t\mathbf{p}), \mathbf{p} \rangle, \tag{5.85}$$

provided that the segment between $\mathbf{x}$ and $\mathbf{x}+t\mathbf{p}$ lies in $\mathcal{O}$.

For further clarity, let $\phi(t;\mathbf{x},\mathbf{p}) = \phi(t) = f(\mathbf{x}+t\mathbf{p})$, so that, from definition (5.78),

$$\phi'(t) = \frac{d}{dt} f(\mathbf{x}+t\mathbf{p}) = \frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}+t\mathbf{p}),$$

and (5.84) is

$$\phi'(0) = \frac{d}{dt}[f(\mathbf{x}+t\mathbf{p})]\bigg|_{t=0} = \frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}).$$

Summarizing, from result (5.79) and using notation from (5.83),

$$\phi'(t) = \frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}+t\mathbf{p}) = \langle \nabla f(\mathbf{x}+t\mathbf{p}), \mathbf{p} \rangle, \qquad \phi'(0) = \frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}) = \langle \nabla f(\mathbf{x}), \mathbf{p} \rangle.$$

Theorem (Fitzpatrick, 13.17, The Mean Value Theorem): Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Because we need the Directional Derivative Theorem. If the segment joining the points $\mathbf{x}$ and $\mathbf{x}+\mathbf{h}$ lies in $\mathcal{O}$, then there is a number $\theta$ with $0 < \theta < 1$ such that $\phi(1) - \phi(0) =$

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = \langle \nabla f(\mathbf{x}+\theta\mathbf{h}), \mathbf{h} \rangle. \tag{5.86}$$

> Proof: Since $\mathcal{O}$ is an open subset of $\mathbb{R}^n$, we can select an open interval of real numbers $I$, which contains the numbers 0 and 1, such that $\mathbf{x}+t\mathbf{h}$ belongs to $\mathcal{O}$ for each $t$ in $I$. Define
>
> $$\phi(t) = f(\mathbf{x}+t\mathbf{h}) \quad \text{for each } t \text{ in } I.$$

[30] In (5.85), Fitzpatrick also included that $0 \le t \le 1$, which does not seem correct. His subsequent statement "provided that the segment between $\mathbf{x}$ and $\mathbf{x}+t\mathbf{p}$ lies in $\mathcal{O}$" is what is required.

Using the slight generalization of the Directional Derivative Theorem stated as formula (5.85), we see that

$$\phi'(t) = \langle \nabla f(\mathbf{x}+t\mathbf{h}), \mathbf{h}\rangle \quad \text{for each } t \text{ in } I. \tag{5.87}$$

Thus, we can apply the Mean Value Theorem for functions of a single real variable to the restriction of the function $\phi : I \to \mathbb{R}$ to the closed interval $[0,1]$ in order to select a number $\theta$ with $0 < \theta < 1$ such that

$$\phi(1) - \phi(0) = \phi'(\theta).$$

Using (5.87) and the definition of $\phi : [0,1] \to \mathbb{R}$, it is clear that this formula is a restatement of (5.86).

In the case where $\mathbf{p}$ is a point in $\mathbb{R}^n$ of norm 1, a directional derivative in the direction $\mathbf{p}$ can be interpreted as a rate of change. To see this, let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Then if the point $\mathbf{p}$ is of norm 1 and $t$ is a positive real number,

$$t = \|t\mathbf{p}\|$$

so, if $t$ is positive and sufficiently small, so as to ensure that $x + t\mathbf{p}$ is in $\mathcal{O}$,

$$\frac{f(\mathbf{x}+t\mathbf{p}) - f(\mathbf{x})}{t} = \frac{f(\mathbf{x}+t\mathbf{p}) - f(\mathbf{x})}{\|t\mathbf{p}\|}.$$

In view of this, if the norm of $\mathbf{p}$ is 1, then it is reasonable to call $\partial f/\partial \mathbf{p}(\mathbf{x})$ as defined in (5.78) the rate of change of the function $f : \mathcal{O} \to \mathbb{R}$ in the direction $\mathbf{p}$ at the point $\mathbf{x}$.

Corollary (Fitzpatrick, 13.18): Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. If $\nabla f(\mathbf{x}) \neq \mathbf{0}$, then the direction of norm 1 at the point $\mathbf{x}$ in which the function $f : \mathcal{O} \to \mathbb{R}$ is increasing the fastest is the direction $\mathbf{p}_0$ defined by

$$\mathbf{p}_0 = \frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}. \tag{5.88}$$

Proof: Using formula (5.84) and the Cauchy-Schwarz inequality (1.22) or (4.20), it follows that, if $\mathbf{p}$ is any point in $\mathbb{R}^n$ of norm 1, then

$$\left|\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x})\right| = |\langle \nabla f(\mathbf{x}), \mathbf{p}\rangle| \leq \|\nabla f(\mathbf{x})\| \cdot \|\mathbf{p}\| = \|\nabla f(\mathbf{x})\|. \tag{5.89}$$

On the other hand, if $\mathbf{p}_0$ is defined by (5.88), then $\mathbf{p}_0$ has norm 1, and using (5.84), it follows that

$$\frac{\partial f}{\partial \mathbf{p}_0}(\mathbf{x}) = \langle \nabla f(\mathbf{x}), \mathbf{p}_0\rangle = \left\langle \nabla f(\mathbf{x}), \frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}\right\rangle = \|\nabla f(\mathbf{x})\|.$$

This calculation, together with inequality (5.89), implies that, if $\mathbf{p}$ has norm 1, then

$$\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}) \leq \frac{\partial f}{\partial \mathbf{p}_0}(\mathbf{x}).$$

**Example 5.16** *Define*

$$f(x, y) = e^{x^2-y^2} \quad \textit{for } (x, y) \in \mathbb{R}^2.$$

*The function* $f : \mathbb{R}^2 \to \mathbb{R}$ *is continuously differentiable. A short calculation shows that*

$$\frac{\partial f}{\partial x}(1, 1) = 2 \quad \textit{and} \quad \frac{\partial f}{\partial y}(1, 1) = -2.$$

*Thus,* $\nabla f(1, 1) = (2, -2)$, *so the direction in which the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *is increasing the fastest at the point* $(1, 1)$ *is given by the vector* $(1/\sqrt{2}, -1/\sqrt{2})$. ■

We are finally in a position to provide a definitive proof that $f \in \mathcal{C}^1(\mathcal{O}) \Rightarrow f \in \mathcal{C}^0(\mathcal{O})$, for general $n$, as stated in (5.49).

Theorem (Fitzpatrick, 13.20): Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable, i.e., $f \in \mathcal{C}^1(\mathcal{O})$. Then $f$ is continuous.

> Proof: Let $\mathbf{x}$ be a point in $\mathcal{O}$. We need to show that the function $f : \mathcal{O} \to \mathbb{R}$ is continuous at $\mathbf{x}$. We directly apply the sequential definition of continuity (5.18). First, since $\mathbf{x}$ is an interior point of $\mathcal{O}$, we can select a positive number $r$ such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$. Let $\{\mathbf{x}_k\}$ be a sequence in $\mathcal{B}_r(\mathbf{x})$ that converges to $\mathbf{x}$. For each natural number $k$, set $\mathbf{h}_k = \mathbf{x}_k - \mathbf{x}$ and apply the Mean Value Theorem (5.86) to select a number $\theta_k$ with $0 < \theta_k < 1$ such that
>
> $$f(\mathbf{x}_k) - f(\mathbf{x}) = f(\mathbf{x} + \mathbf{h}_k) - f(\mathbf{x}) = \langle \nabla f(\mathbf{x} + \theta_k \mathbf{h}_k), \mathbf{h}_k \rangle. \tag{5.90}$$
>
> Now observe that
>
> $$\lim_{k\to\infty} \mathbf{h}_k = \mathbf{0} \quad \text{and} \quad \lim_{k\to\infty} [\mathbf{x} + \theta_k \mathbf{h}_k] = \mathbf{x}.$$
>
> Since $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable, it follows that (Each component of the gradient, and thus the entire vector itself, is continuous.)
>
> $$\lim_{k\to\infty} \nabla f(\mathbf{x} + \theta_k \mathbf{h}_k) = \nabla f(\mathbf{x}).$$
>
> Thus, since (5.90) holds for every index $k$, we conclude that
>
> $$\lim_{k\to\infty} [f(\mathbf{x}_k) - f(\mathbf{x})] = \langle \nabla f(\mathbf{x}), \mathbf{0} \rangle = 0,$$
>
> which means that the image sequence $\{f(\mathbf{x}_k)\}$ converges to $f(\mathbf{x})$.

Corollary: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has continuous second-order partial derivatives. Then the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable.

> Proof: For each index $i$ with $1 \le i \le n$, the function $\partial f/\partial x_i : \mathcal{O} \to \mathbb{R}$ is continuously differentiable, and hence, by Theorem (Fitzpatrick, 13.20), it is continuous. This is precisely what it means for the function $f : \mathbb{R}^n \to \mathbb{R}$ to be continuously differentiable.

## 5.6 The Jacobian and the Chain Rule

We now turn to multivariate functions of the form $\mathbf{f} : A \subset \mathbb{R}^n \to \mathbb{R}^m$, and note that we use bold face for the function name in order to indicate that $m > 1$.[31]

### 5.6.1 The Jacobian

Consider a multivariate function $\mathbf{f} : A \subset \mathbb{R}^n \to \mathbb{R}^m$ with $A$ an open set, where $\mathbf{f}$ is such that $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$, $f_i : A \to \mathbb{R}$, $i = 1, \ldots, m$, for all $\mathbf{x} = (x_1, \ldots, x_n)' \in A$, recalling the component notation from (5.9) and (5.10). If each partial derivative, $\partial f_i(\mathbf{x}_0)/\partial x_j$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, exists, then the *total derivative* of $\mathbf{f}$ at $\mathbf{x}_0 \in A$ is the $m \times n$ matrix

$$\mathbf{f}'(\mathbf{x}_0) := \mathbf{J}_\mathbf{f}(\mathbf{x}_0) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}_0) \end{pmatrix} = \begin{pmatrix} (\operatorname{grad} f_1)(\mathbf{x}_0) \\ \vdots \\ (\operatorname{grad} f_m)(\mathbf{x}_0) \end{pmatrix}, \tag{5.91}$$

also referred to as the *Jacobian matrix* of $\mathbf{f}$ at $\mathbf{x}_0$.[32] When $m = 1$, the total derivative is just the gradient (5.25). Analogous to the $m = 1$ case from (5.37), let

$$\mathbf{T}(\mathbf{x}) := \mathbf{f}(\mathbf{x}_0) + \mathbf{J}_\mathbf{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) =: \mathbf{f}(\mathbf{x}_0) + \mathbf{Df}(\mathbf{x}_0, \mathbf{x} - \mathbf{x}_0), \tag{5.92}$$

where $\mathbf{Df}(\mathbf{x}_0, \mathbf{h})$, defined on the rhs, is the total differential, analogous to the $m = 1$ case in (5.38), $\mathrm{d}f(\mathbf{x}, \mathbf{h}) = (\operatorname{grad} f)(\mathbf{x}) \cdot \mathbf{h}$.

**Warning**: Some authors (such as Petrovic; see below in equation (5.103); and Fitzpatrick; see (5.166)) refer to $\mathbf{J}_\mathbf{f}(\mathbf{x}_0)$ as $\mathbf{Df}(\mathbf{x}_0)$, which conflicts with our usage of $\mathbf{Df}$ in (5.92). Notice, at least, that the total differential $\mathbf{Df}(\mathbf{x}_0, \mathbf{h})$ takes two arguments, whereas the Jacobian $\mathbf{J}_\mathbf{f}(\mathbf{x}_0)$, also called the **total derivative** (Petrovic), or **derivative matrix** (Fitzpatrick), and its common equivalent notation $\mathbf{Df}(\mathbf{x}_0)$, take only one argument, and thus are distinguishable in context.

One resolution is, within this document, to convert all occurrences of the latter to the $\mathbf{J}_\mathbf{f}$ notation (or vice-versa). We opted not to do this, because both notations are popular in the literature, and it is best the reader becomes aware of this.

[31] The notes in this subsection, and §5.7, were assembled from me (for my math appendix in Fundamental Probability) over 20 years ago, and came from a compilation of several books. These included Trench (2003), the most recent version of which is Trench (Introduction to Real Analysis, Free Hyperlinked Edition 2.04, December, 2013); Lang (Undergraduate Analysis, 2nd ed., 1997); Hubbard and Hubbard (Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach, now in its 5th edition, 2015), and Protter and Morrey (A First Course in Real Analysis, 2nd edition, 1991).

[32] After the prolific Carl Gustav Jacob Jacobi (1804–1851), who made contributions in several branches of mathematics, including the study of functional determinants. Though the theory goes back (at least) to Cauchy in 1815, Jacobi's 1841 memoir *De determinantibus functionalibus* had the first modern definition of determinant, and the first use of the word Jacobian was by Sylvester in 1853. Jacobi is also remembered as an excellent teacher who introduced the "seminar method" for teaching the latest advances in math (whereby students present and discuss current articles and papers.)

We state two equivalent definitions of differentiability.

Definition: Let $\mathbf{f} = (f_1, f_2, \ldots, f_m) : A \subset \mathbb{R}^n \to \mathbb{R}^m$, with $A$ open, and let $\mathbf{x}_0 \in A$. Function $\mathbf{f}$ is said to be *differentiable at* $\mathbf{x}_0 \in A$ if each $f_i$, $i = 1, \ldots, m$, is differentiable at $\mathbf{x}_0 \in A$. Differentiability of $f_i : A \subset \mathbb{R}^n \to \mathbb{R}$ is defined in (5.34).

Definition: Let $\mathbf{f} = (f_1, f_2, \ldots, f_m) : A \subset \mathbb{R}^n \to \mathbb{R}^m$, with $A$ open, and let $\mathbf{x}_0 \in A$. Function $\mathbf{f}$ is said to be *differentiable at* $\mathbf{x}_0 \in A$ if there exists a *tangent map* $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ of $\mathbf{f}$ at $\mathbf{x}_0$, such that

$$\mathbf{f}(\mathbf{x}_0) = \mathbf{T}(\mathbf{x}_0) \quad \text{and} \quad \lim_{\mathbf{x} \to \mathbf{x}_0} \frac{\|\mathbf{f}(\mathbf{x}) - \mathbf{T}(\mathbf{x})\|}{\|\mathbf{x} - \mathbf{x}_0\|} = 0, \tag{5.93}$$

where $\mathbf{T}(\mathbf{x})$ is given in (5.92).

In the second definition, differentiability of $\mathbf{f}$ is defined analogously to the $m = 1$ case in (5.34). Tangent map $\mathbf{T}$ is an $m \times n$ matrix, called the (total) derivative of $\mathbf{f}$ at $\mathbf{x}_0$. As also stated for the $m = 1$ case just after (5.35), the actual definition does not include this latter specification of $\mathbf{T}(\mathbf{x})$, but rather only the existence of a tangent map and "a" total derivative matrix. The next theorem shows that this specification must in fact be (5.92) using the Jacobian matrix (5.91).

Theorem: $\mathbf{f}$ *is differentiable at* $\mathbf{x}_0 \in A$ *iff the Jacobian* $\mathbf{J}_\mathbf{f}(\mathbf{x}_0)$ *exists,* $\mathbf{f}(\mathbf{x}_0) = \mathbf{T}(\mathbf{x}_0)$*, and*

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{\|\mathbf{f}(\mathbf{x}) - \mathbf{T}(\mathbf{x})\|}{\|\mathbf{x} - \mathbf{x}_0\|} = 0. \tag{5.94}$$

Proof:

($\Rightarrow$) Assume $\mathbf{f}$ is differentiable at $\mathbf{x}_0 \in A$. Then, by definition, $(\operatorname{grad} f_i)(\mathbf{x}_0)$ exists, $i = 1, \ldots, m$, so that the form of (5.91) shows that $\mathbf{J}_\mathbf{f}(\mathbf{x}_0)$ exists. Next, differentiability of $\mathbf{f}$ means that there exists a tangent map of each $f_i$ at $\mathbf{x}_0$, given by, say, $T_i(\mathbf{x}) = f_i(\mathbf{x}_0) + (\operatorname{grad} f_i)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$, and, for each $i$, $f_i(\mathbf{x}_0) = T_i(\mathbf{x}_0)$. Thus, taking

$$\begin{aligned}\mathbf{T}(\mathbf{x}) = \begin{bmatrix} T_1(\mathbf{x}) \\ \vdots \\ T_m(\mathbf{x}) \end{bmatrix} &= \begin{bmatrix} f_1(\mathbf{x}_0) + (\operatorname{grad} f_1)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \\ \vdots \\ f_m(\mathbf{x}_0) + (\operatorname{grad} f_m)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \end{bmatrix} \\ &= \mathbf{f}(\mathbf{x}_0) + \mathbf{J}_\mathbf{f}(\mathbf{x}_0)\,(\mathbf{x} - \mathbf{x}_0)\,,\end{aligned} \tag{5.95}$$

it is clear that $\mathbf{T}(\mathbf{x}_0) = (f_1(\mathbf{x}_0), \ldots, f_m(\mathbf{x}_0))' = \mathbf{f}(\mathbf{x}_0)$. Lastly, from (5.34), differentiability of $\mathbf{f}$ implies that, for each $i = 1, \ldots, m$,

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f_i(\mathbf{x}) - T_i(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|} = 0 = \lim_{\mathbf{x} \to \mathbf{x}_0} \frac{|f_i(\mathbf{x}) - T_i(\mathbf{x})|}{\|\mathbf{x} - \mathbf{x}_0\|}. \tag{5.96}$$

Next, from the first inequality in (3.23), for $\mathbf{z} = (z_1, \ldots, z_m) \in \mathbb{R}^m$, $\|\mathbf{z}\| \leq |z_1| + \cdots + |z_m|$. Thus, with

$$\mathbf{z} = \frac{\mathbf{f}(\mathbf{x}) - \mathbf{T}(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|} = \frac{1}{\|\mathbf{x} - \mathbf{x}_0\|} \begin{bmatrix} f_1(\mathbf{x}) - T_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) - T_m(\mathbf{x}) \end{bmatrix},$$

it follows that

$$\|\mathbf{z}\| = \frac{\|\mathbf{f}(\mathbf{x}) - \mathbf{T}(\mathbf{x})\|}{\|\mathbf{x} - \mathbf{x}_0\|} \leq \sum_{i=1}^m \frac{|f_i(\mathbf{x}) - T_i(\mathbf{x})|}{\|\mathbf{x} - \mathbf{x}_0\|},$$

i.e., (5.94) follows from (5.96).

($\Leftarrow$) If $\mathbf{J_f}(\mathbf{x}_0)$ exists, then $(\text{grad}\, f_i)(\mathbf{x}_0)$ exists, $i = 1, \ldots, m$. From (5.95), if $\mathbf{T}(\mathbf{x}_0) = \mathbf{f}(\mathbf{x}_0)$, then $f_i(\mathbf{x}_0) = T_i(\mathbf{x}_0)$, $i = 1, \ldots, m$. Lastly, it trivially follows from the definition of the norm that

$$\frac{|f_i(\mathbf{x}) - T_i(\mathbf{x})|}{\|\mathbf{x} - \mathbf{x}_0\|} \leq \frac{\|\mathbf{f}(\mathbf{x}) - \mathbf{T}(\mathbf{x})\|}{\|\mathbf{x} - \mathbf{x}_0\|},$$

so that (5.96) follows from (5.94).

Paralleling the $m = 1$ case from §5.3, we have

Definition: $\mathbf{f} = (f_1, f_2, \ldots, f_m) : A \subset \mathbb{R}^n \to \mathbb{R}^m$, with $A$ open, is *differentiable on* $A$ if $\mathbf{f}$ is differentiable at each $\mathbf{x}_0 \in A$.

Definition: Let $\mathbf{f} = (f_1, f_2, \ldots, f_m) : A \subset \mathbb{R}^n \to \mathbb{R}^m$. If $\mathbf{f}$ is differentiable and all the partial derivatives of each $f_i$ are continuous, then $\mathbf{f}$ is *continuously differentiable*, and we write $\mathbf{f} \in \mathcal{C}^1(A^o)$, where $A^o$ is the interior of the domain $A$ of $\mathbf{f}$.

**Example 5.17** *Let* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \mapsto (ye^x, x^2y^3, -x)$. *Then, from* (5.91),

$$\mathbf{f}'(x, y) = \mathbf{J_f}(x, y) = \begin{pmatrix} ye^x & e^x \\ 2xy^3 & x^2 3y^2 \\ -1 & 0 \end{pmatrix},$$

*and* $\mathbf{f}$ *is continuously differentiable.* ■

In §2.3.1, we showed (the easy, standard proof) that, for $f : A \subset \mathbb{R} \to \mathbb{R}$, if $f$ is differentiable at the point $a \in A$, then $f$ is continuous at $a$. This was extended to $f : A \subset \mathbb{R}^n \to \mathbb{R}$ in (5.46). We now state and prove the result for $\mathbf{f} : A \subset \mathbb{R}^n \to \mathbb{R}^m$. This obviously subsumes the two mentioned special cases.

Theorem:

$$\textbf{If } \mathbf{f} \textbf{ is differentiable at } \mathbf{x}_0 \in A\textbf{, then } \mathbf{f} \textbf{ is continuous at } \mathbf{x}_0. \tag{5.97}$$

Proof: This hinges on the two most important inequalities in analysis. From (5.94), $\exists\, \delta^* > 0$ such that, for $\mathbf{x} \in A$, if $\|\mathbf{x} - \mathbf{x}_0\| < \delta^*$, then $\|\mathbf{f}(\mathbf{x}) - \mathbf{T}(\mathbf{x})\| < \|\mathbf{x} - \mathbf{x}_0\|$, where $\mathbf{T}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + \mathbf{J_f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$. Let $\mathbf{K} = \mathbf{J_f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$ so that $\mathbf{T}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + \mathbf{K}$. If $\|\mathbf{x} - \mathbf{x}_0\| < \delta^*$, then, from the triangle inequality (1.23),

$$\begin{aligned} \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\| &= \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0) - \mathbf{K} + \mathbf{K}\| = \|\mathbf{f}(\mathbf{x}) - \mathbf{T}(\mathbf{x}) + \mathbf{K}\| \\ &\leq \|\mathbf{f}(\mathbf{x}) - \mathbf{T}(\mathbf{x})\| + \|\mathbf{K}\| < \|\mathbf{x} - \mathbf{x}_0\| + \|\mathbf{K}\|. \end{aligned} \tag{5.98}$$

From (5.95), with row vector $\mathbf{w}_i = (w_{i1}, \ldots, w_{in}) := (\text{grad}\, f_i)(\mathbf{x}_0)$ and column vector $\mathbf{z}_i = (z_{i1}, \ldots, z_{in}) := (\mathbf{x} - \mathbf{x}_0)$,

$$\|\mathbf{K}\|^2 = \sum_{i=1}^m \big[ (\text{grad}\, f_i)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \big]^2 = \sum_{i=1}^m \left[\mathbf{w}_i \mathbf{z}_i\right]^2 = \sum_{i=1}^m \left( \sum_{j=1}^n w_{ij} z_{ij} \right)^2.$$

For each $i = 1, \ldots, m$, the Cauchy-Schwarz inequality (1.22) implies

$$\left( \sum_{j=1}^n w_{ij} z_{ij} \right)^2 \leq \left( \sum_{j=1}^n w_{ij}^2 \right) \left( \sum_{j=1}^n z_{ij}^2 \right) = \|(\text{grad}\, f_i)(\mathbf{x}_0)\|^2 \, \|(\mathbf{x} - \mathbf{x}_0)\|^2,$$

so that

$$\begin{aligned}\|\mathbf{K}\| &\leq \left(\sum_{i=1}^{m} \|(\operatorname{grad} f_i)(\mathbf{x}_0)\|^2 \|(\mathbf{x}-\mathbf{x}_0)\|^2\right)^{1/2} \\ &= \|(\mathbf{x}-\mathbf{x}_0)\| \left(\sum_{i=1}^{m} \|(\operatorname{grad} f_i)(\mathbf{x}_0)\|^2\right)^{1/2} =: \|(\mathbf{x}-\mathbf{x}_0)\| \, G. \end{aligned} \tag{5.99}$$

Thus, from (5.98) and (5.99),

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\| < \|\mathbf{x}-\mathbf{x}_0\| + \|\mathbf{K}\| < \|\mathbf{x}-\mathbf{x}_0\| \, (1+G). \tag{5.100}$$

Because we assume that $\mathbf{f}$ is differentiable at $\mathbf{x}_0$, $G$ is finite. Thus, for a given $\epsilon > 0$, we can find a $\delta > 0$ such that, if $\|\mathbf{x}-\mathbf{x}_0\| < \delta$ and $\mathbf{x} \in A$, then $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\| < \epsilon$. In particular, from (5.100), $\delta = \min(\delta^*, \epsilon/(1+G))$.

In (5.49) and page 330, we demonstrated that, if $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable on open domain $\mathcal{O}$, then $f$ is continuous. That is, for $m = 1$, $f \in \mathcal{C}^1(\mathcal{O}) \Rightarrow \mathcal{C}^0(\mathcal{O})$. We now state the result for the case of multivariate function $\mathbf{f} : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}^m$.

Theorem: If all the partial derivatives of $\mathbf{f} = (f_1, \ldots, f_m) : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}^m$ exist and are continuous on $\mathcal{O}$, then $\mathbf{f}$ is differentiable on $\mathcal{O}$ with derivative $\mathbf{J}_{\mathbf{f}}$.

See, e.g., Hubbard and Hubbard (2002, p. 159 and p. 680) for detailed proofs.

### 5.6.2 The Chain Rule

The next result is the chain rule, providing a big generalization to the univariate case (2.71). It is useful to review the definition of differentiability, as given either in (5.94); or below in (5.103) and (5.104).

Theorem: Let $\mathbf{f} : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{g} : B \subset \mathbb{R}^m \to \mathbb{R}^p$ with $A$ and $B$ open sets and $\mathbf{f}(A) \subset B$. If $\mathbf{f}$ is differentiable at $\mathbf{x} \in A$ and $\mathbf{g}$ is differentiable at $\mathbf{f}(\mathbf{x})$, then the composite function $\mathbf{g} \circ \mathbf{f}$ is differentiable at $\mathbf{x}$, with derivative

$$\mathbf{J}_{\mathbf{g}\circ\mathbf{f}}(\mathbf{x}) = \mathbf{J}_{\mathbf{g}}(\mathbf{f}(\mathbf{x})) \, \mathbf{J}_{\mathbf{f}}(\mathbf{x}). \tag{5.101}$$

We provide two proofs; one below, and another in §5.9.

**Example 5.18** *As in Example 5.17, let $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^3, (x,y) \mapsto (ye^x, x^2y^3, -x)$, and also let $\mathbf{g} : (\mathbb{R}_{>0} \times \mathbb{R}) \to \mathbb{R}^2, (x,y) \mapsto (\ln x, x+2y) = (g_1(x,y), g_2(x,y))$. The function $\mathbf{g}$ is continuously differentiable with derivative at $(x,y) \in (\mathbb{R}_{>0} \times \mathbb{R})$:*

$$\mathbf{g}'(x,y) = \mathbf{J}_{\mathbf{g}}(x,y) = \begin{bmatrix} 1/x & 0 \\ 1 & 2 \end{bmatrix}.$$

*Let $\mathbf{h} = \mathbf{f} \circ \mathbf{g}$. The composition $\mathbf{h}$ is continuously differentiable and its derivative at $(x,y) \in$*

$(\mathbb{R}_{>0} \times \mathbb{R})$ *is given by*

$$\begin{aligned}
\mathbf{h}'(x,y) &= \mathbf{J_h}(x,y) = \mathbf{J_f}(\mathbf{g}(x,y)) \cdot \mathbf{J_g}(x,y) \\
&= \begin{bmatrix} (x+2y)x & x \\ 2\ln(x)(x+2y)^3 & (\ln(x))^2 3(x+2y)^2 \\ -1 & 0 \end{bmatrix} \cdot \begin{pmatrix} 1/x & 0 \\ 1 & 2 \end{pmatrix} \\
&= \begin{bmatrix} (x+2y)+x & 2x \\ (1/x)2\ln x(x+2y)^3 + (\ln x)^2 3(x+2y)^2 & 2(\ln x)^2 3(x+2y)^2 \\ -1/x & 0 \end{bmatrix}.
\end{aligned}$$

*The reader is encouraged to calculate an expression for* $\mathbf{h}(x,y)$ *and compute* $\mathbf{J_h}$ *directly.* ■

For the chain rule (5.101), a special case of interest is $n = p = 1$, i.e., $\mathbf{f} = (f_1, \ldots, f_m) : A \subset \mathbb{R} \to \mathbb{R}^m$ and $g : B \subset \mathbb{R}^m \to \mathbb{R}$. Then for $x \in A$, $\mathbf{J}_g$ is a row vector and $\mathbf{J_f}$ is a column vector, so that (5.101) simplifies to (see also below, (5.114) and (5.115), for more detail)

$$\mathbf{J}_{g \circ \mathbf{f}}(x) = \sum_{i=1}^{m} \frac{\partial g}{\partial f_i}(\mathbf{f}(x)) \frac{df_i}{dx}(x), \tag{5.102}$$

where $\partial g / \partial f_i$ denotes the $i$th partial derivative of $g$.[33] A mnemonic version of this formula is, with $h = g \circ \mathbf{f}$,

$$\frac{dh}{dx} = \sum_{i=1}^{m} \frac{\partial h}{\partial f_i} \frac{df_i}{dx}.$$

**Example 5.19** *Assume that the United States GDP, denoted by $P$, is a continuously differentiable function of the capital, $C$, and the work force, $W$. Moreover, assume $C$ and $W$ are continuously differentiable functions of time, $t$. Then $P$ is a continuously differentiable function of $t$ and economists would write:*

$$\frac{dP}{dt} = \frac{\partial P}{\partial C}\frac{dC}{dt} + \frac{\partial P}{\partial W}\frac{dW}{dt},$$

*showing us how the change of $P$ can be split into a part due to the decrease or increase of $C$ and another due to the change of $W$.* ■

The following example is useful, as it shows what can go wrong when at least one of the functions is not differentiable. In particular, there is no tangent map of function $g$ at the point $(0,0)$. Recall Example 5.10 in which the function has partial derivatives at each point in its domain, but at $(0,0)$, there is no tangent map, and thus the function is not differentiable over its whole domain.

**Example 5.20** *(Counterexample to Example 5.19) Consider the following functions:*

$$g : \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto \begin{cases} \dfrac{x^3+y^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0), \end{cases}$$

[33] This notation is somewhat misleading, as $f_i$ in $\partial g / \partial f_i$ is a function itself. Keep in mind that, in "$\partial g / \partial f_i$", the $f_i$ could be replaced by, say, $y_i$, or any other name of variable as long as it can be easily inferred which partial derivative is meant.

*and* $\mathbf{f} : \mathbb{R} \to \mathbb{R}^2$*, given by* $t \mapsto (t, t)$*. The partial derivatives of* $g$ *exist on the whole domain, in particular, at* $(x, y) = (0, 0)$*: Using* (5.28),

$$\frac{\partial g}{\partial x}(0,0) = \lim_{x\to 0} \frac{g(x,0) - g(0,0)}{x - 0} = \lim_{x\to 0} \frac{x^3 + 0}{x(x^2 + 0)} = \lim_{x\to 0} 1 = 1 = \frac{\partial g}{\partial y}(0,0),$$

*but the partial derivatives of* $g$ *are not continuous. However,* $\mathbf{f}$ *is continuously differentiable with derivative*

$$\mathbf{f}' : \mathbb{R} \to \mathbb{R}^2, \;\; t \mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

*So, if the chain rule were applicable here, then the derivative of* $h := g \circ \mathbf{f}$ *at* $t = 0$ *is calculated to be*

$$h'(0) = \mathbf{J}_g(\mathbf{f}(0))\mathbf{J}_{\mathbf{f}}(0) = (1 \quad 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2.$$

*On the other hand, we can calculate* $h(t)$ *for* $t \in \mathbb{R}$ *directly as*

$$h(t) = \frac{t^3 + t^3}{t^2 + t^2} = \frac{2t^3}{2t^2} = t,$$

*so that* $h'(0) = 1$*. This demonstrates that the chain rule generally does not hold when one of the functions is not continuously differentiable.* ■

We include now an excerpt from Petrovic, pp. 355-6, repeating some of the above definitions, and including a proof of the Chain Rule (5.101), and an additional basic result. We will give yet another discussion of this topic in §5.9, and, there, yet another proof of the Chain Rule, which appears in (5.187). Recall the notational warning given above.

Definition: Let $\mathbf{f} = (f_1, f_2, \ldots, f_m)$ be a function defined in an open ball $A \subset \mathbb{R}^n$, with values in $\mathbb{R}^m$, and let $\mathbf{a} \in A$. Then $\mathbf{f}$ is differentiable at $\mathbf{a}$ if and only if there exists an $m \times n$ matrix $\mathbf{Df}(\mathbf{a})$, called the (total) derivative of $\mathbf{f}$ at $\mathbf{a}$, such that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + \mathbf{Df}(\mathbf{a})(\mathbf{x} - \mathbf{a}) + \mathbf{r}(\mathbf{x}), \quad \text{and} \tag{5.103}$$

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{\mathbf{r}(\mathbf{x})}{\|\mathbf{x} - \mathbf{a}\|} = \mathbf{0}. \tag{5.104}$$

If $\mathbf{f}$ is differentiable at every point of a set $A$, we say that it is differentiable on $A$.

We are making a standard identification between elements of the Euclidean space of dimension $n$, and $n \times 1$ matrices. For example, $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ can be viewed as a column matrix of dimension $m \times 1$. That means that (5.103) states an equality between matrices. If we read it row by row, we can conclude several things. First, the rows of $\mathbf{Df}$ are precisely the partial derivatives of the functions $f_1, f_2, \ldots, f_m$, so

$$\mathbf{Df}(\mathbf{a}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}.$$

Second, $\mathbf{f}$ is differentiable at $\mathbf{a}$ iff $f_i$ is differentiable at $\mathbf{a}$ for all $1 \le i \le m$.

All the expected rules for derivatives hold, as shown next, simply because they hold for each of the component functions. Recall, e.g., (5.55), which states that, for $f, g : A \subset \mathbb{R}^n \to \mathbb{R}$ differentiable at $\mathbf{a} \in A^o$, $f \cdot g$ is differentiable at $\mathbf{a}$, with

$$(\operatorname{grad}(f \cdot g))(\mathbf{a}) = g(\mathbf{a})(\operatorname{grad} f)(\mathbf{a}) + f(\mathbf{a})(\operatorname{grad} g)(\mathbf{a}), \tag{5.105}$$

which can be compared to part (d) of the next theorem.

Theorem: Let $\mathbf{f}, \mathbf{g} : A \subset \mathbb{R}^n \to \mathbb{R}^m$, $A$ open, and let $\mathbf{a} \in A$. Also, let $\alpha \in \mathbb{R}$ and $\varphi : A \subset \mathbb{R}^n \to \mathbb{R}$. If $\mathbf{f}$, $\mathbf{g}$, and $\varphi$ are differentiable at $\mathbf{a}$, then so are $\mathbf{f} + \mathbf{g}$, $\alpha\mathbf{f}$, $\varphi\mathbf{f}$, and $\mathbf{f} \cdot \mathbf{g}$:

(a) $\mathbf{D}(\alpha\mathbf{f})(\mathbf{a}) = \alpha\mathbf{D}\mathbf{f}(\mathbf{a})$;

(b) $\mathbf{D}(\mathbf{f} + \mathbf{g})(\mathbf{a}) = \mathbf{D}\mathbf{f}(\mathbf{a}) + \mathbf{D}\mathbf{g}(\mathbf{a})$;

(c) $\mathbf{D}(\varphi\mathbf{f})(\mathbf{a}) = \mathbf{f}(\mathbf{a})\mathbf{D}\varphi(\mathbf{a}) + \varphi(\mathbf{a})\mathbf{D}\mathbf{f}(\mathbf{a})$;

(d) $\mathbf{D}(\mathbf{f} \cdot \mathbf{g})(\mathbf{a}) = \mathbf{g}(\mathbf{a})'\mathbf{D}\mathbf{f}(\mathbf{a}) + \mathbf{f}(\mathbf{a})'\mathbf{D}\mathbf{g}(\mathbf{a})$.

Remarks and verification of (d):

1. The proofs of (a) and (b) follows easily from its $m = 1$ counterpart (5.53) and (5.54). Part (a) also follows from (c), as noted next.
2. Let $\varphi(\mathbf{a}) = \alpha$. Then $\mathbf{D}\varphi(\mathbf{a}) = \mathbf{0}$, of size $1 \times n$, and (c) implies (a).
3. The three objects in (c) must all be $m \times n$; and as $\mathbf{D}\varphi(\mathbf{a})$ is $1 \times n$, we require $\mathbf{f}(\mathbf{a})$ to be the column vector of size $m \times 1$. We apply the same to $\mathbf{g}(\mathbf{a})$.
4. For (d), note that $\mathbf{f} \cdot \mathbf{g}$ is a dot product, and thus a mapping from $A$ to $\mathbb{R}$. That means $\mathbf{D}(\mathbf{f} \cdot \mathbf{g})(\mathbf{a})$ must be $1 \times n$. The rhs indeed has this dimension. We verify the rhs next.
5. Denote, as usual, the component functions of $\mathbf{f}$ as $(f_1, \ldots, f_m)$; and likewise for $\mathbf{g}$. To see how (d) follows from (5.105), first note that, as $\mathbf{D}$ preserves linearity, and from (5.105),

$$\begin{aligned} \mathbf{D}(\mathbf{f} \cdot \mathbf{g})(\mathbf{a}) &= \mathbf{D}\left(\sum_{i=1}^{m} f_i(\mathbf{a})\, g_i(\mathbf{a})\right) = \sum_{i=1}^{m} \mathbf{D}\left(f_i(\mathbf{a})\, g_i(\mathbf{a})\right) \\ &= \sum_{i=1}^{m} \left[g_i(\mathbf{a})\, \nabla f_i(\mathbf{a}) + f_i(\mathbf{a})\, \nabla g_i(\mathbf{a})\right]. \end{aligned} \tag{5.106}$$

   Next, (d) states (with each $\nabla f_i(\mathbf{a})$ and $\nabla g_i(\mathbf{a})$ being of size $1 \times n$, $i = 1, \ldots m$)

$$\begin{aligned} \mathbf{D}(\mathbf{f} \cdot \mathbf{g})(\mathbf{a}) &= (g_1(\mathbf{a}), \ldots, g_m(\mathbf{a})) \begin{bmatrix} \nabla f_1(\mathbf{a}) \\ \nabla f_2(\mathbf{a}) \\ \vdots \\ \nabla f_m(\mathbf{a}) \end{bmatrix} + (f_1(\mathbf{a}), \ldots, f_m(\mathbf{a})) \begin{bmatrix} \nabla g_1(\mathbf{a}) \\ \nabla g_2(\mathbf{a}) \\ \vdots \\ \nabla g_m(\mathbf{a}) \end{bmatrix} \\ &= \big[g_1(\mathbf{a})\, \nabla f_1(\mathbf{a}) + \cdots + g_m(\mathbf{a})\, \nabla f_m(\mathbf{a}) \\ &\quad + f_1(\mathbf{a})\, \nabla g_1(\mathbf{a}) + \cdots + f_m(\mathbf{a})\, \nabla g_m(\mathbf{a})\big]. \end{aligned} \tag{5.107}$$

   The two formulations (5.106) and (5.107) are the same, thus confirming (d).
6. Let $\mathbf{1}_m = (1, 1, \ldots, 1)'$ of length $m$, so that $\mathbf{1}_m'\mathbf{f}(\mathbf{a}) = \sum_{i=1}^{m} f_i(\mathbf{a})$. Then $\mathbf{D}\big(\mathbf{1}_m'\mathbf{f}\big)(\mathbf{a})$ is given by part (d) by taking $\mathbf{g} = (g_1, \ldots, g_m) : A \subset \mathbb{R}^n \to \mathbb{R}^m$ such that each component $g_i$ is identical to one, in which case $\mathbf{D}\mathbf{g}(\mathbf{a}) = \mathbf{0}_{m \times n}$, $\mathbf{g}(\mathbf{a})' = (1, \ldots, 1)$, and

$\mathbf{D}\left(\mathbf{1}_m'\mathbf{f}\right)(\mathbf{a}) = \mathbf{g}(\mathbf{a})'\mathbf{Df}(\mathbf{a}) = \sum_{i=1}^m (\text{grad } f_i)(\mathbf{a})$. Observe this result follows from (5.53) and (5.54).

A bit more generally, let $\mathbf{g} : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be such that each component is identical, given by $\varphi(\mathbf{a}) = g_1(\mathbf{a}) = \cdots = g_m(\mathbf{a})$, where, as above, $\varphi : A \subset \mathbb{R}^n \to \mathbb{R}$. Then its $m \times n$ Jacobian matrix $\mathbf{Dg}(\mathbf{a})$ has identical rows, each being $\nabla\varphi(\mathbf{a})$. From (5.107),

$$\begin{aligned}\mathbf{D}(\mathbf{f}\cdot\mathbf{g})(\mathbf{a}) &= \mathbf{g}(\mathbf{a})'\mathbf{Df}(\mathbf{a}) + \mathbf{f}(\mathbf{a})'\mathbf{Dg}(\mathbf{a}) \\ &= \varphi(\mathbf{a})\sum_{i=1}^m (\text{grad } f_i)(\mathbf{a}) + \nabla\varphi(\mathbf{a})\sum_{i=1}^m f_i(\mathbf{a}).\end{aligned}$$

Before stating and proving the chain rule, it is useful to review the definition of differentiability, as given either in (5.94); or directly above in (5.103) and (5.104).

Theorem (The Chain Rule): Let $A$ be an open ball in $\mathbb{R}^n$, and let $\mathbf{f} : A \to \mathbb{R}^m$. Further, let $B$ be an open set in $\mathbb{R}^m$ that contains the range of $\mathbf{f}$, and let $\mathbf{g} : B \to \mathbb{R}^p$. If $\mathbf{f}$ is differentiable at $\mathbf{a} \in A$, and if $\mathbf{g}$ is differentiable at $\mathbf{f}(\mathbf{a})$, then the composition $\mathbf{g} \circ \mathbf{f}$ is differentiable at $\mathbf{a}$, and (notice that the right side represents a product of matrices)

$$\mathbf{D}(\mathbf{g}\circ\mathbf{f})(\mathbf{a}) = \mathbf{Dg}(\mathbf{f}(\mathbf{a}))\mathbf{Df}(\mathbf{a}). \tag{5.108}$$

Proof: Note $\mathbf{Dg}(\mathbf{f}(\mathbf{a}))$ is $p \times m$, and $\mathbf{Df}(\mathbf{a})$ is $m \times n$. From the differentiability of $\mathbf{f}$ at $\mathbf{a}$ and $\mathbf{g}$ at $\mathbf{f}(\mathbf{a})$, and with $\mathbf{b} = \mathbf{f}(\mathbf{a})$,

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + \mathbf{Df}(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \mathbf{r}_\mathbf{f}, \qquad \lim_{\mathbf{x}\to\mathbf{a}} \frac{\mathbf{r}_\mathbf{f}}{\|\mathbf{x}-\mathbf{a}\|} = \mathbf{0}, \tag{5.109}$$

$$\mathbf{g}(\mathbf{y}) = \mathbf{g}(\mathbf{b}) + \mathbf{Dg}(\mathbf{b})(\mathbf{y}-\mathbf{b}) + \mathbf{r}_\mathbf{g}, \qquad \lim_{\mathbf{y}\to\mathbf{b}} \frac{\mathbf{r}_\mathbf{g}}{\|\mathbf{y}-\mathbf{b}\|} = \mathbf{0}. \tag{5.110}$$

It follows from (5.109) that $\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{g}\big(\mathbf{f}(\mathbf{a}) + \mathbf{Df}(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \mathbf{r}_\mathbf{f}\big)$, so, with

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + \mathbf{Df}(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \mathbf{r}_\mathbf{f},$$

(5.110) implies that

$$\begin{aligned}\mathbf{g}(\mathbf{f}(\mathbf{x})) &= \mathbf{g}(\mathbf{f}(\mathbf{a})) + \mathbf{Dg}(\mathbf{f}(\mathbf{a}))\left(\mathbf{Df}(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \mathbf{r}_\mathbf{f}\right) + \mathbf{r}_\mathbf{g} \\ &= \mathbf{g}(\mathbf{f}(\mathbf{a})) + \mathbf{Dg}(\mathbf{f}(\mathbf{a}))\mathbf{Df}(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \mathbf{Dg}(\mathbf{f}(\mathbf{a}))\mathbf{r}_\mathbf{f} + \mathbf{r}_\mathbf{g}.\end{aligned}$$

Thus, it remains to show that (note the numerator and rhs are $p \times 1$)

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{\mathbf{Dg}(\mathbf{f}(\mathbf{a}))\mathbf{r}_\mathbf{f} + \mathbf{r}_\mathbf{g}}{\|\mathbf{x}-\mathbf{a}\|} = \mathbf{0}. \tag{5.111}$$

As $\mathbf{g}$ is differentiable, (5.103) implies $\mathbf{Dg}(\mathbf{f}(\mathbf{a}))$ exists and is finite. Thus, by definition of $\mathbf{r}_\mathbf{f}$ and (5.14), the first term $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{Dg}(\mathbf{f}(\mathbf{a}))\mathbf{r}_\mathbf{f} \,/\, \|\mathbf{x}-\mathbf{a}\| = \mathbf{0}$. Further,

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{y} = \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{a}} \big(\mathbf{f}(\mathbf{a}) + \mathbf{Df}(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \mathbf{r}_\mathbf{f}\big) = \mathbf{f}(\mathbf{a}) = \mathbf{b}.$$

Thus, from this and the latter part of (5.110), $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{r}_\mathbf{g}/\|\mathbf{y}-\mathbf{b}\| = \mathbf{0}$, i.e.,

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{\mathbf{r}_\mathbf{g}}{\|\mathbf{Df}(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \mathbf{r}_\mathbf{f}\|} = \mathbf{0}.$$

Now,

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{\mathbf{r_g}}{\|\mathbf{x}-\mathbf{a}\|} = \lim_{\mathbf{x}\to\mathbf{a}} \frac{\mathbf{r_g}}{\|\mathbf{Df}(\mathbf{a})(\mathbf{x}-\mathbf{a})+\mathbf{r_f}\|} \frac{\|\mathbf{Df}(\mathbf{a})(\mathbf{x}-\mathbf{a})+\mathbf{r_f}\|}{\|\mathbf{x}-\mathbf{a}\|} = \mathbf{0}, \tag{5.112}$$

because (with only the final inequality stated, without justification, in Petrovic) from the triangle inequality and the Generalized Cauchy-Schwarz Inequality (5.140),

$$\begin{aligned} \frac{\|\mathbf{Df}(\mathbf{a})(\mathbf{x}-\mathbf{a})+\mathbf{r_f}\|}{\|\mathbf{x}-\mathbf{a}\|} &\leq \frac{\|\mathbf{Df}(\mathbf{a})(\mathbf{x}-\mathbf{a})\|}{\|\mathbf{x}-\mathbf{a}\|} + \frac{\|\mathbf{r_f}\|}{\|\mathbf{x}-\mathbf{a}\|} \\ &\leq \|\mathbf{Df}(\mathbf{a})\| + \frac{\|\mathbf{r_f}\|}{\|\mathbf{x}-\mathbf{a}\|}, \end{aligned}$$

the second fraction in (5.112) is bounded, thus showing (5.111) and ending the proof.

### 5.6.3 The Mean Value Theorem (MVT)

Recall one of the highlights of §5.5 was the MVT (5.86). We now give another proof of the MVT for functions $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$. The proof is very short and easy, because it uses the chain rule, given in both (5.101) and (5.108).

We first review a special case of the chain rule that we will require. In fact, we have it already, in (5.102), but I prefer to redo things (different notation; same concept; good practice). Let $m = 1$ and $p = n$. The chain rule then reads: Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{g} : B \subset \mathbb{R} \to \mathbb{R}^n$ with $A$ and $B$ open sets and $f(A) \subset B$. If $f$ is differentiable at $\mathbf{x} \in A$ and $\mathbf{g}$ is differentiable at $y = f(\mathbf{x})$, then the composite function $(\mathbf{g} \circ f) : A \subset \mathbb{R}^n \to \mathbb{R}^n$ is differentiable at $\mathbf{x} \in A \subset \mathbb{R}^n$, with derivative the $n \times n$ Jacobian matrix

$$\mathbf{J}_{\mathbf{g}\circ f}(\mathbf{x}) = \mathbf{J_g}(f(\mathbf{x}))\, \mathbf{J}_f(\mathbf{x}). \tag{5.113}$$

Given the dimensions of $f$ and $\mathbf{g}$, we can also state the Jacobian of $(f \circ \mathbf{g}) : B \subset \mathbb{R} \to \mathbb{R}$, and now with $\mathbf{y} = \mathbf{g}(x)$ and $x \in B \subset \mathbb{R}$, it is the $1 \times 1$ matrix, i.e., scalar

$$J_{f\circ\mathbf{g}}(x) = \mathbf{J}_f(\mathbf{g}(x))\mathbf{J_g}(x), \tag{5.114}$$

this latter case being the one of relevance for the MVT, and the same (albeit with a change in notation) as (5.102). To spell things out for, e.g., $n = 3$, $f$ differentiable at $\mathbf{y} = (y_1, y_2, y_3) \in A$, $\mathbf{g}$ in terms of its component functions from notation (5.10) as $\mathbf{g} = (g_1, g_2, g_3)$, and $\mathbf{g}$ differentiable at $x \in B$, we have

$$\nabla f(\mathbf{y}) = \mathbf{J}_f(\mathbf{y}) = \begin{bmatrix} \frac{\partial f(\mathbf{y})}{\partial y_1} & \frac{\partial f(\mathbf{y})}{\partial y_2} & \frac{\partial f(\mathbf{y})}{\partial y_3} \end{bmatrix}, \quad \mathbf{J_g}(x) = \begin{bmatrix} \frac{\partial g_1(x)}{\partial x} \\ \frac{\partial g_2(x)}{\partial x} \\ \frac{\partial g_3(x)}{\partial x} \end{bmatrix},$$

and

$$\mathbf{J}_{f\circ\mathbf{g}}(x) = \mathbf{J}_f(\mathbf{y})\,\mathbf{J_g}(x) = \sum_{i=1}^{3} \frac{\partial f(\mathbf{y})}{\partial y_i} \frac{\partial g_i(x)}{\partial x}. \tag{5.115}$$

From Terrell, A Passage to Modern Analysis, 2019, p. 316, we have:

Theorem (Mean Value Theorem for Real Functions): Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ and suppose that $\mathbf{a}$ and $\mathbf{b}$ are interior points of $U$. If the line segment $l_{\mathbf{ab}}$ is contained in the interior of

$U$ and $f$ is continuous on $l_{\mathbf{ab}}$ and differentiable at all points of $l_{\mathbf{ab}}$ (except possibly at its endpoints $\mathbf{a}$ and $\mathbf{b}$), then there is a point $\mathbf{c} \in l_{\mathbf{ab}}$ such that

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}). \tag{5.116}$$

Proof: The curve $\mathbf{r} : [0,1] \to \mathbb{R}^n$ given by $\mathbf{r}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$ is continuous on $[0,1]$ and differentiable on $(0,1)$, and $\mathbf{r}'(t) = \mathbf{b} - \mathbf{a}$. Define the function $\phi : [0,1] \to \mathbb{R}^n$ by $\phi(t) = f(\mathbf{r}(t)) = f(\mathbf{a} + t(\mathbf{b} - \mathbf{a}))$. Then $\phi$ is continuous on $[0,1]$ and differentiable on $(0,1)$, and $\phi(0) = f(\mathbf{a})$, $\phi(1) = f(\mathbf{b})$. Since $f$ is differentiable at all points of $l_{\mathbf{ab}}$, the chain rule applies, and we have

$$\phi'(t) = \nabla f(\mathbf{a} + t(\mathbf{b} - \mathbf{a})) \cdot (\mathbf{b} - \mathbf{a}).$$

By the single variable mean value theorem, there is a $t_0 \in (0,1)$ such that $\phi(1) - \phi(0) = \phi'(t_0)(1-0) = \phi'(t_0)$, hence (5.116) holds with $\mathbf{c} = \mathbf{a} + t_0(\mathbf{b} - \mathbf{a})$.

Observe that (5.116) indeed agrees with (5.86).

## 5.7 Higher Order Derivatives and Taylor Series

Recall the notation and results in §5.4. We repeat the basic definitions here, and state some extensions.

Let $f : A \to \mathbb{R}$ with $A \subset \mathbb{R}^n$ an open set such that the partial derivatives $D_i f(\mathbf{x})$ are continuous at point $\mathbf{x} = (x_1, \ldots, x_n) \in A$, $i = 1, 2, \ldots, n$. As $D_i f$ is a function, its partial derivative may be computed, if it exists, i.e., we can apply the $D_j$ operator to $D_i f$ to get $D_j D_i f$, called the iterated partial derivative of $f$ with respect to $i$ and $j$. As we have shown in §5.4,

**If $D_i f$, $D_j f$, $D_i D_j f$ and $D_j D_i f$ exist and are continuous, then**

$$D_i D_j f = D_j D_i f. \tag{5.117}$$

This extends as follows. Let $D_i^k f$ denote $k$ applications of $D_i$ to $f$, e.g., $D_i^2 f = D_i D_i f$. Then, for nonnegative integers $k_i$, $i = 1, \ldots, n$, any iterated partial derivative operator can be written as $D_1^{k_1} \cdots D_n^{k_n}$, with $D_i^0 = 1$, i.e., the derivative with respect to the $i$th variable is not taken. The *order* of $D_1^{k_1} \cdots D_n^{k_n}$ is $\sum k_i$. Extending (5.117), if $D_i^j f$ is continuous for $i = 1, \ldots, n$ and $j = 0, \ldots, k_i$, then $D_1^{k_1} \cdots D_n^{k_n} f$ is the same for all possible orderings of the elements of $D_1^{k_1} \cdots D_n^{k_n}$.[34]

Let $f : I \to \mathbb{R}$, with $I \subset \mathbb{R}$ an open interval that contains points $x$ and $x + c$. From (2.325), the Taylor series expansion of $f(c + x)$ around $c$ can be expressed as

$$f(c + x) = \sum_{k=0}^{r} \frac{f^{(k)}(c)}{k!} x^k + \text{ remainder term}, \tag{5.118}$$

if $f^{(r+1)}$ exists, where the order of the expansion is $r$ (before we used $n$, as is standard convention, but now $n$ is the dimension of the domain of $f$, which is also standard.) This

[34] This latter statement does not seem to be adequate, comparing it to the two-variable case. In particular, it appears we also need that $D_1^{j_1} \cdots D_n^{j_n} f$ is continuous, for every combination of $0 \le j_1 \le k_1, \cdots, 0 \le j_n \le k_n$, and that, for each of the $n!$ possible orderings of the elements of $D_1^{j_1} \cdots D_n^{j_n}$. I am searching for a valid statement in this general case.

can be extended to function $f : \mathbb{R}^n \to \mathbb{R}$. We first consider the case with $n = 2$ and $r = 2$, which suffices for many useful applications.

Let $f : A \to \mathbb{R}$ with $A \subset \mathbb{R}^2$ an open set, and let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{c} = (c_1, c_2)$ be column vectors such that $(\mathbf{c} + t\mathbf{x}) \in A$ for $0 \le t \le 1$. Let $\boldsymbol{\ell} : I \to \mathbb{R}^2$ with $I = [0, 1]$ and $\boldsymbol{\ell}(t) = \ell(t; \mathbf{c}, \mathbf{x}) = \mathbf{c} + t\mathbf{x}$. Let $g : I \to \mathbb{R}$ be the composite univariate function defined by $g(t) = (f \circ \boldsymbol{\ell})(t) = f(\mathbf{c} + t\mathbf{x})$, so that $g(0) = f(\mathbf{c})$ and $g(1) = f(\mathbf{c} + \mathbf{x})$. Applying (5.118) to $g$ (setting $c = 0$ and $x = 1$ in (5.118)) gives

$$f(\mathbf{c} + \mathbf{x}) = g(1) = g(0) + g'(0) + \frac{g''(0)}{2} + + \frac{g'''(0)}{6} + \cdots + \frac{g^{(k)}(0)}{k!} + \cdots . \tag{5.119}$$

From the chain rule (5.114) for $(f \circ \boldsymbol{\ell})$ (here, $g$ is the composite function),

$$g'(t) = (\operatorname{grad} f)(\mathbf{c} + t\mathbf{x})\, \mathbf{x} = x_1 D_1 f(\mathbf{c} + t\mathbf{x}) + x_2 D_2 f(\mathbf{c} + t\mathbf{x}) =: f_1(\mathbf{c} + t\mathbf{x}), \tag{5.120}$$

where $f_1$ is the linear combination of differential operators applied to $f$ given by $f_1 := (x_1 D_1 + x_2 D_2) f$. Again from the chain rule, now applied to $f_1$, and using (5.117),

$$\begin{aligned} g''(t) &= (\operatorname{grad} f_1)(\mathbf{c} + t\mathbf{x})\, \mathbf{x} \\ &= x_1 D_1 f_1(\mathbf{c} + t\mathbf{x}) + x_2 D_2 f_1(\mathbf{c} + t\mathbf{x}) \\ &= x_1^2 (D_1^2 f)(\mathbf{c} + t\mathbf{x}) + 2x_1 x_2 (D_1 D_2 f)(\mathbf{c} + t\mathbf{x}) + x_2^2 (D_2^2 f)(\mathbf{c} + t\mathbf{x}) \\ &= \left[(x_1 D_1 + x_2 D_2)^2 f\right](\mathbf{c} + t\mathbf{x}) =: f_2(\mathbf{c} + t\mathbf{x}). \end{aligned}$$

This and (5.120) give $g'(0) = f_1(\mathbf{c})$ and $g''(0) = f_2(\mathbf{c})$, so that (5.119) yields

$$f(\mathbf{c} + \mathbf{x}) = f(\mathbf{c}) + f_1(\mathbf{c}) + \frac{1}{2} f_2(\mathbf{c}) + \cdots \tag{5.121}$$

$$\begin{aligned} &= f(\mathbf{c}) + x_1 D_1 f(\mathbf{c}) + x_2 D_2 f(\mathbf{c}) \\ &\quad + \frac{x_1^2}{2}(D_1^2 f)(\mathbf{c}) + x_1 x_2 (D_1 D_2 f)(\mathbf{c}) + \frac{x_2^2}{2}(D_2^2 f)(\mathbf{c}) + \cdots . \end{aligned} \tag{5.122}$$

If we "remove" the second coordinate used in $f$, writing $x$ instead of $\mathbf{x} = (x_1, x_2)$ and similar for $\mathbf{c}$, then (5.122) simplifies to

$$f(c + x) = f(c) + x D_1 f(c) + \frac{x^2}{2}(D_1^2 f)(c) + \cdots ,$$

which agrees with (5.118). From (5.119), expansion (5.121) can be continued with

$$\begin{aligned} g'''(t) &= \frac{d}{dt} f_2(\mathbf{c} + t\mathbf{x}) = (\operatorname{grad} f_2)(\mathbf{c} + t\mathbf{x})\, \mathbf{x} = x_1 D_1 f_2(\mathbf{c} + t\mathbf{x}) + x_2 D_2 f_2(\mathbf{c} + t\mathbf{x}) \\ &= x_1 D_1 \big(x_1^2 (D_1^2 f)(\mathbf{c} + t\mathbf{x}) + 2x_1 x_2 (D_1 D_2 f)(\mathbf{c} + t\mathbf{x}) + x_2^2 (D_2^2 f)(\mathbf{c} + t\mathbf{x})\big) \\ &\quad + x_2 D_2 \big(x_1^2 (D_1^2 f)(\mathbf{c} + t\mathbf{x}) + 2x_1 x_2 (D_1 D_2 f)(\mathbf{c} + t\mathbf{x}) + x_2^2 (D_2^2 f)(\mathbf{c} + t\mathbf{x})\big) \\ &= x_1^3 (D_1^3 f)(\mathbf{c} + t\mathbf{x}) + 3x_1^2 x_2 (D_1^2 D_2 f)(\mathbf{c} + t\mathbf{x}) \\ &\quad + 3x_1 x_2^2 (D_1 D_2^2 f)(\mathbf{c} + t\mathbf{x}) + x_2^3 (D_2^3 f)(\mathbf{c} + t\mathbf{x}) \\ &= \left[\, (x_1 D_1 + x_2 D_2)^3 f\right](\mathbf{c} + t\mathbf{x}) \\ &=: f_3(\mathbf{c} + t\mathbf{x}), \end{aligned}$$

and it seems natural to postulate that (5.121) takes the form

$$f(\mathbf{c} + \mathbf{x}) = \sum_{k=0}^{\infty} \frac{f_k(\mathbf{c})}{k!}, \tag{5.123}$$

where $f_k := (x_1 D_1 + x_2 D_2)^k f$, $k = 0, 1, \ldots$ . This is true if all the derivatives exist, and can be proven by induction. Note that $f_k$ can be expanded by the binomial theorem (1.34).

Expression (5.123) is for $n = 2$, although the extension to the case of general $n$ is the same, except that

$$f_k := (x_1 D_1 + \cdots + x_n D_n)^k f = (\nabla \mathbf{x})^k f,$$

where we use $\nabla$ to represent the operator that, when applied to $f$, returns the gradient, i.e., $\nabla := (D_1, \ldots, D_n)$, which is a row vector, and $\mathbf{x} = (x_1, \ldots, x_n)$ is a column vector. Note that $f_1(\mathbf{c})$ is the total differential. Now, $f_k$ is evaluated via the multinomial theorem (see the last part of §1.3), and each $f_k$ will have $\binom{k+n-1}{k}$ terms. With this notation, and assuming all relevant partial derivatives exist, for $f : A \to \mathbb{R}$ with $A \subset \mathbb{R}^n$ an open set,

$$f(\mathbf{c} + \mathbf{x}) = \sum_{k=0}^{r} \frac{\left[ (\nabla \mathbf{x})^k f \right](\mathbf{c})}{k!} + \text{ remainder term},$$

where it can be shown (see, e.g., Lang, 1997, §15.5) that the

$$\text{remainder term } = \frac{\left[ (\nabla \mathbf{x})^{r+1} f \right](\mathbf{c} + t\mathbf{x})}{(r+1)!}, \quad \text{for} \quad 0 \leq t \leq 1.$$

## 5.8 Local Approximation of Real-Valued Multivariate Functions

This section repeats some of the previous material—never a bad thing, unless the goal is absolute efficiency and terseness. It is based on (or, better, a near copy of) Ch. 14 of Fitzpatrick's Advanced Calculus, 2nd ed., 2009, this being a book I discovered after having initially wrote these notes over 20 years ago, and having used the excellent presentations in the books mentioned in the footnote at the beginning of §5.6. Fitzpatrick's presentation is admirable in its detail and clarity, and also covers a few relevant aspects not done in the previous sections. Another, more recent and excellent source of this material, is Petrovic's Advanced Calculus: Theory and Practice, 2nd ed., 2020. Finally, the (in parts magnificent) Terrell, A Passage to Modern Analysis, 2019, offers a slightly more advanced presentation of this material, as well as covering topics not in Fitzpatrick, e.g., Fourier series and Lebesgue integration.

Notes in blue are from me.

Suppose that $I$ is an open interval of real numbers and that the function $f : I \to \mathbb{R}$ is differentiable. By definition, this means that if $x$ is a point in $I$, then

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x).$$

If we rewrite the difference

$$\frac{f(x+h) - f(x)}{h} - f'(x) = \frac{f(x+h) - [f(x) + f'(x)h]}{h},$$

then the above definition of a derivative can be rewritten as

$$\lim_{h\to 0} \frac{f(x+h) - [f(x) + f'(x)h]}{h} = 0. \tag{5.124}$$

Recall (2.328) and (2.329) from the univariate Taylor series expression: Repeating these, if $k$ is a natural number and the function $f : I \to \mathbb{R}$ has continuous derivatives up to order $k+1$, then, for a point $x$ in $I$ and a perturbation $x + h$ that also belongs to $I$, there is a number $\theta$, $0 < \theta < 1$, such that

$$f(x+h) - \left[ f(x) + f'(x)h + \cdots + \left(\frac{1}{k!}\right) f^k(x)h^k \right] = \frac{f^{k+1}(x+\theta h)}{(k+1)!} \cdot h^{k+1}, \tag{5.125}$$

and, therefore, dividing by $h^k$,

$$\lim_{h\to 0} \frac{f(x+h) - \left[f(x) + f'(x)h + \cdots + (1/k!)f^k(x)h^k\right]}{h^k} = 0. \tag{5.126}$$

In what follows, we wish to establish results analogous to the approximation formulas inherent in (5.124) and, for $k = 2$, in (5.126), for functions of several real variables. It is useful to introduce the following definition.

Definition: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$. For a positive integer $k$, two functions $f : \mathcal{O} \to \mathbb{R}$ and $g : \mathcal{O} \to \mathbb{R}$ are said to be $k$th-order approximations of one another at the point $\mathbf{x}$, provided that

$$\lim_{\mathbf{h}\to \mathbf{0}} \frac{f(\mathbf{x}+\mathbf{h}) - g(\mathbf{x}+\mathbf{h})}{\|\mathbf{h}\|^k} = 0. \tag{5.127}$$

**Example 5.21** *Define* $f(h) = e^h$ *for each number* $h$. *Then* $f(0) = f'(0) = f''(0) = 1$. *From* (5.126) *at* $x = 0$,

$$\lim_{h\to 0} \frac{e^h - [1+h]}{h} = 0, \qquad \lim_{h\to 0} \frac{e^h - [1+h+(1/2)h^2]}{h^2} = 0.$$

*Thus, the first-degree Taylor polynomial* $p_1(h) = 1 + h$ *is a first-order approximation of* $f$ *at* $x = 0$, *while the second-degree Taylor polynomial* $p_2(h) = 1 + h + (1/2)h^2$ *is a second-order approximation of* $f$ *at* $x = 0$. ∎

The following theorem provides an extension to functions of several variables of the approximation formula (5.124). This is the same as (5.48). Also recall the dot or inner product notation $\langle \cdot, \cdot \rangle$ from (4.1); and note $\langle \nabla f(\mathbf{x}), \mathbf{h} \rangle$ is the total differential of $f$ at $\mathbf{x}$ from (5.38).

Theorem (The First-Order Approximation Theorem): Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Let $\mathbf{x}$ be a point in $\mathcal{O}$. Then

$$\lim_{\mathbf{h}\to \mathbf{0}} \frac{f(\mathbf{x}+\mathbf{h}) - [f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h} \rangle]}{\|\mathbf{h}\|} = 0. \tag{5.128}$$

Proof: Since $\mathbf{x}$ is an interior point of $\mathcal{O}$, we can choose a positive number $r$ such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$. Fix a nonzero point $\mathbf{h}$ in $\mathbb{R}^n$ with $\|\mathbf{h}\| < r$. Then the point $\mathbf{x}+\mathbf{h}$ belongs to $\mathcal{B}_r(\mathbf{x})$ and so, by the Mean Value Theorem (5.86), we can select a number $\theta$ with $0 < \theta < 1$ such that

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = \langle \nabla f(\mathbf{x}+\theta\mathbf{h}), \mathbf{h} \rangle.$$

Thus,

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - \langle \nabla f(\mathbf{x}), \mathbf{h} \rangle = \langle \nabla f(\mathbf{x}+\theta\mathbf{h}) - \nabla f(\mathbf{x}), \mathbf{h} \rangle,$$

so that, using the Cauchy-Schwarz Inequality, we obtain the estimate

$$|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - \langle \nabla f(\mathbf{x}), \mathbf{h} \rangle| \le \|\nabla f(\mathbf{x}+\theta\mathbf{h}) - \nabla f(\mathbf{x})\| \cdot \|\mathbf{h}\|.$$

Dividing this estimate by $\|\mathbf{h}\|$, we obtain

$$\frac{|f(\mathbf{x}+\mathbf{h}) - [f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h} \rangle]|}{\|\mathbf{h}\|} \le \|\nabla f(\mathbf{x}+\theta\mathbf{h}) - \nabla f(\mathbf{x})\|. \tag{5.129}$$

But the function $f : \mathcal{O} \to \mathbb{R}$ has been assumed to be continuously differentiable, so

$$\lim_{\mathbf{h}\to\mathbf{0}} \|\nabla f(\mathbf{x}+\theta\mathbf{h}) - \nabla f(\mathbf{x})\| = 0,$$

and thus (5.128) follows from the estimate (5.129).

For a continuously differentiable function $f : \mathcal{O} \to \mathbb{R}$ whose domain $\mathcal{O}$ is an open subset of the plane $\mathbb{R}^2$ and a point $(x_0, y_0)$ in $\mathcal{O}$, if we denote a general point in $\mathcal{O}$ by $(x, y)$ and set $\mathbf{h} = (x - x_0, y - y_0)$, it is clear that $\mathbf{h}$ approaches $\mathbf{0}$ if and only if $(x, y)$ approaches $(x_0, y_0)$ and that $\|\mathbf{h}\| = \sqrt{(x-x_0)^2 + (y-y_0)^2}$. Hence the approximation property (5.128) can be rewritten as

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - [f(x_0,y_0) + \partial f/\partial x\,(x_0,y_0)\,(x-x_0) + \partial f/\partial y\,(x_0,y_0)\,(y-y_0)]}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0. \tag{5.130}$$

This last formula has a geometric interpretation involving the existence of a tangent plane. To describe this, we state the following definition.

Definition: Let $\mathcal{O}$ be an open subset of the plane $\mathbb{R}^2$ and suppose that the function $f:\mathcal{O}\to\mathbb{R}$ is continuous at the point $(x_0,y_0)$ in $\mathcal{O}$. By the tangent plane to the graph of $f:\mathcal{O}\to\mathbb{R}$ at the point $(x_0,y_0,f(x_0,y_0))$, we mean the graph of a function $\psi:\mathbb{R}^2\to\mathbb{R}$ of the form

$$\psi(x,y)=a+b\,(x-x_0)+c\,(y-y_0)\qquad\text{for } (x,y) \text{ in } \mathbb{R}^2,$$

where $a,b$, and $c$ are real numbers, which has the property that

$$\lim_{(x,y)\to(x_0,y_0)}\frac{f(x,y)-\psi(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0. \tag{5.131}$$

A continuous function of two variables $f:\mathcal{O}\to\mathbb{R}$ can have directional derivatives in all directions at the point $(x_0,y_0)$ in $\mathcal{O}$ without having a tangent plane at the point $(x_0,y_0,f(x_0,y_0))$. Such examples occur because the definition of tangent plane requires that the limit (5.131) exist independently of the way in which the point $(x,y)$ approaches $(x_0,y_0)$. A case in point is Example 5.10.

However, for continuously differentiable functions, the approximation property (5.130) is exactly what is required in order to prove the following corollary. This is the same as (5.48).

Corollary: Suppose that $\mathcal{O}$ is an open subset of the plane $\mathbb{R}^2$ that contains point $(x_0,y_0)$ and that the function $f:\mathcal{O}\to\mathbb{R}$ is continuously differentiable. Then there is a tangent plane to the graph of the function $f:\mathcal{O}\to\mathbb{R}$ at the point $(x_0,y_0,f(x_0,y_0))$. This tangent plane is the graph of the function $\psi:\mathbb{R}^2\to\mathbb{R}$ defined for $(x,y)$ in $\mathbb{R}^2$ by

$$\psi(x,y)=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\,(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)\,(y-y_0)\,. \tag{5.132}$$

> Proof: For a general point $(x,y)$ in $\mathcal{O}$, set $\mathbf{h}=(x,y)-(x_0,y_0)$ and observe that the total differential of $f$ at $(x_0,y_0)$, from (5.38), is
>
> $$\langle\nabla f(x_0,y_0)\,,\mathbf{h}\rangle=\frac{\partial f}{\partial x}(x_0,y_0)\,(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)\,(y-y_0)\,.$$
>
> Since $f:\mathcal{O}\to\mathbb{R}$ is continuously differentiable, the First-Order Approximation Theorem (5.128) implies that
>
> $$\lim_{(x,y)\to(x_0,y_0)}\frac{f(x,y)-\psi(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0;$$
>
> that is, the graph of the function $\psi:\mathbb{R}^2\to\mathbb{R}$ is the tangent plane to the graph of the function $f:\mathcal{O}\to\mathbb{R}$ at the point $(x_0,y_0,f(x_0,y_0))$.

We can reason geometrically to see why the tangent plane described in the preceding corollary is necessarily described by equation (5.132). Indeed, suppose that $\mathcal{O}$ is an open subset of the plane $\mathbb{R}^2$ and consider the function $f:\mathcal{O}\to\mathbb{R}$. At the point $(x_0,y_0)$ in $\mathcal{O}$, we look for a plane that is tangent to the graph of $f:\mathcal{O}\to\mathbb{R}$ at the point $(x_0,y_0,f(x_0,y_0))$. If the function $f:\mathcal{O}\to\mathbb{R}$ has first-order partial derivatives at $(x_0,y_0)$, then from the definition of a partial derivative and the meaning, in the case of functions of a single real variable, of the derivative as the slope of the tangent line, it follows that the vectors

$$\mathbf{T}_1=\left(1,0,\frac{\partial f}{\partial x}(x_0,y_0)\right)\qquad\text{and}\qquad\mathbf{T}_2=\left(0,1,\frac{\partial f}{\partial y}(x_0,y_0)\right) \tag{5.133}$$

should be parallel to the proposed tangent plane. See Figure 43. Thus, the proposed tangent plane should have a cross-product This is the same as (5.32) and Figure 40.

$$\eta = \mathbf{T}_1 \times \mathbf{T}_2 = \left(-\partial f/\partial x\left(x_0, y_0\right), -\partial f/\partial y\left(x_0, y_0\right), 1\right) \tag{5.134}$$

as a normal vector. Recalling (4.8), the plane that passes through the point $(x_0, y_0, f(x_0, y_0))$ and is normal to $\eta$ consists of all points $(x, y, z)$ in $\mathbb{R}^3$ that satisfy the equation

$$\langle \eta, (x - x_0, y - y_0, z - f(x_0, y_0))\rangle = 0,$$

and it is clear that this means that the point $(x, y, z)$ in $\mathbb{R}^3$ lies on the graph of the function defined by equation (5.132).

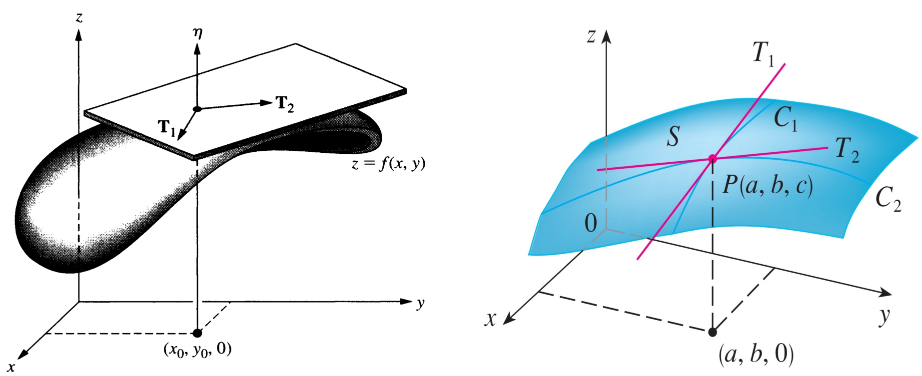



Figure 43: Left: From Fitzpatrick, p. 376: The tangent plane to the graph at the point $(x_0, y_0, z_0)$. Note in (5.133) that $\mathbf{T}_1$, viewed in the $xz$-plane, can be seen as a vector originating at the origin, with slope $(D_1 f)(x_0, y_0)/1$. Similar for $\mathbf{T}_2$ in the $yz$-plane.
Right: From Stewart, Multivariate Calculus, 7th ed., p. 927: The curve $C_1$ is the graph of the function $g(x) = f(x, b)$, so the slope of its tangent $T_1$ at $P$ is $g'(a) = (D_1 f)(a, b)$. The curve $C_2$ is the graph of the function $h(y) = f(a, y)$, so the slope of its tangent $T_2$ at $P$ is $h'(b) = (D_2 f)(a, b)$. The partial derivatives $(D_1 f)(a, b)$ and $(D_2 f)(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces $C_1$ and $C_2$ of $S$ in the planes $y = b$ and $x = a$.

The First-Order Approximation Theorem (5.128) is also useful from another, less geometric, perspective. It enables us to approximate rather complicated functions by simpler ones and to assert precisely the manner in which the functions are close to one another. Of course, the simplest type of function is a constant function. The next two simplest types of functions are linear functions and affine functions, which are defined as follows.

Definition: A function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be affine if it is defined by

$$g(\mathbf{u}) = c + \sum_{i=1}^{n} a_i u_i \quad \text{for } \mathbf{u} \text{ in } \mathbb{R}^n,$$

where $c$ and the $a_i$ are prescribed numbers. If $c = 0$, the function is called linear.

Corollary: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Then there is an affine function that is a first-order approximation of $f$ at the point $\mathbf{x}$, namely, the function $g : \mathbb{R}^n \to \mathbb{R}$ defined by

$$g(\mathbf{u}) = f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{u} - \mathbf{x}\rangle \quad \text{for } \mathbf{u} \text{ in } \mathbb{R}^n.$$

Proof: Observe that the function $g : \mathbb{R}^n \to \mathbb{R}$ is affine and that

$$g(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h} \rangle \quad \text{for } \mathbf{x}+\mathbf{h} \text{ in } \mathbb{R}^n.$$

The First-Order Approximation Theorem (5.128) asserts that the functions $f : \mathcal{O} \to \mathbb{R}$ and $g : \mathcal{O} \to \mathbb{R}$ are first-order approximations of one another at the point $\mathbf{x}$.

**Example 5.22** *For $(x, y) \in \mathbb{R}^2$, define $f(x, y) = \sin(x - y - y^2)$. The function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable. Computing partial derivatives at the point $(0, 0)$, we find that the affine function that is a first-order approximation of $f$ at the point $(0, 0)$ is defined by $\psi(x, y) = x - y$ for $(x, y) \in \mathbb{R}^2$. Computing partial derivatives at the point $(\pi, 0)$, we find that the affine function that is a first-order approximation of $f$ at the point $(\pi, 0)$ is given by $\psi(x, y) = \pi - x + y$ for $(x, y) \in \mathbb{R}^2$.* ■

Definition: Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix. The function $Q : \mathbb{R}^n \to \mathbb{R}$ defined by $Q(\mathbf{x}) \equiv \langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle$, $\mathbf{x} \in \mathbb{R}^n$, is called the quadratic function associated with the matrix $\mathbf{A}$.

We have seen this before, in (4.90). Observe that $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle = (\mathbf{A}\mathbf{x})'\mathbf{x} = \mathbf{x}\mathbf{A}'\mathbf{x}$, but from the symmetry property of the inner product (see the beginning of §4.5.1), $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{A}\mathbf{x} \rangle = \mathbf{x}'\mathbf{A}\mathbf{x}$, so, without loss of generality, matrix $\mathbf{A}$ can be taken to be symmetric. A non-symmetric matrix $\mathbf{A}$ can be replaced with $\mathbf{B} = (\mathbf{A}' + \mathbf{A})/2$.

Observe that, for $\mathbf{x} \in \mathbb{R}^n$, $Q(\mathbf{x}) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_j x_i$, so $Q(\mathbf{x})$ is a linear combination of $x_j x_i$'s; hence the name quadratic function.

Definition: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has second-order partial derivatives. The Hessian matrix of the function $f : \mathcal{O} \to \mathbb{R}$ at the point $\mathbf{x}$ in $\mathcal{O}$, denoted by $\nabla^2 f(\mathbf{x})$, is defined to be the $n \times n$ matrix that, for each pair of indices $i$ and $j$, with $1 \le i \le n$ and $1 \le j \le n$, has the $(i, j)$th entry defined by

$$\left(\nabla^2 f(\mathbf{x})\right)_{ij} \equiv \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}). \tag{5.135}$$

Observe that, for each index $i$ with $1 \le i \le n$, and $\mathbf{x} \in \mathcal{O}$,

$$\text{the } i\text{th row of } \nabla^2 f(\mathbf{x}) \text{ is the gradient of the function } \partial f/\partial x_i : \mathcal{O} \to \mathbb{R}. \tag{5.136}$$

To illustrate, for $n = 2$, the Hessian matrix of $f$ at $(x_0, y_0) \in \mathcal{O}$ is

$$\nabla^2 f(x_0, y_0) = \begin{bmatrix} \partial^2 f/\partial x \partial x\,(x_0, y_0) & \partial^2 f/\partial y \partial x\,(x_0, y_0) \\ \partial^2 f/\partial x \partial y\,(x_0, y_0) & \partial^2 f/\partial y \partial y\,(x_0, y_0) \end{bmatrix}.$$

Also observe that, in view of the equality of cross-partial derivatives, it follows that the Hessian matrix $\nabla^2 f(\mathbf{x})$ is symmetric; that is, the $(i, j)$th entry equals the $(j, i)$th entry, provided that the function $f : \mathcal{O} \to \mathbb{R}$ has continuous second-order partial derivatives.

Theorem: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has continuous second-order partial derivatives. Choose a positive number $r$ such that the open ball about $\mathbf{x}$, $\mathcal{B}_r(\mathbf{x})$, is contained in $\mathcal{O}$. Then if $\|\mathbf{h}\| < r$ and $|t| \leq 1$, it is useful to recall and review (5.85)

$$\frac{d}{dt}[f(\mathbf{x} + t\mathbf{h})] = \langle \nabla f(\mathbf{x} + t\mathbf{h}), \mathbf{h} \rangle \tag{5.137}$$

and

$$\frac{d^2}{dt^2}[f(\mathbf{x} + t\mathbf{h})] = \left\langle \nabla^2 f(\mathbf{x} + t\mathbf{h})\mathbf{h}, \mathbf{h} \right\rangle. \tag{5.138}$$

Let $\mathbf{H} = \nabla^2 f(\mathbf{x} + t\mathbf{h})$ as in (5.135). Then (5.138) is $\mathbf{h}'\mathbf{H}\mathbf{h}$.

Proof: Let $I$ be an open interval of real numbers that contains the points 0 and 1 and is such that the point $\mathbf{x} + t\mathbf{h}$ belongs to $\mathcal{O}$ if $t$ belongs to $I$. Define

$$\phi(t) = f(\mathbf{x} + t\mathbf{h}) \quad \text{for } t \text{ in } I.$$

The Directional Derivative Theorem (5.79) implies that, if $t$ is in $I$, then

$$\phi'(t) = \frac{d}{dt}[f(\mathbf{x} + t\mathbf{h})] = \langle \nabla f(\mathbf{x} + t\mathbf{h}), \mathbf{h} \rangle = \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x} + t\mathbf{h}).$$

However, for each index $i$ with $1 \leq i \leq n$, we can again apply the Directional Derivative Theorem to the partial derivative $\partial f / \partial x_i : \mathcal{O} \to \mathbb{R}$, and, hence, by differentiating each side of the preceding equality, we see that

$$\phi''(t) = \frac{d}{dt}\left[\phi'(t)\right] = \sum_{i=1}^{n} h_i \frac{d}{dt}\left[\frac{\partial f}{\partial x_i}(\mathbf{x} + t\mathbf{h})\right],$$

i.e., and recalling (5.136)

$$\begin{aligned} \phi''(t) &= \sum_{i=1}^{n} h_i \left\langle \nabla \left[\frac{\partial f}{\partial x_i}\right](\mathbf{x} + t\mathbf{h}), \mathbf{h} \right\rangle \\ &= \sum_{i=1}^{n} \left\langle \nabla \left[\frac{\partial f}{\partial x_i}\right](\mathbf{x} + t\mathbf{h}), \mathbf{h} \right\rangle h_i = \left\langle \nabla^2 f(\mathbf{x} + t\mathbf{h})\mathbf{h}, \mathbf{h} \right\rangle. \end{aligned}$$

Definition: The norm of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, denoted by $\|\mathbf{A}\|$, is defined by[35]

$$\|\mathbf{A}\| \equiv \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}^2}. \tag{5.139}$$

Observe that, if we define the point $\mathbf{A}_i$ in $\mathbb{R}^n$ to be the $i$th row of the $n \times n$ matrix $\mathbf{A}$, then the square of the norm of $\mathbf{A}$ can be written as

$$\|\mathbf{A}\|^2 = \|\mathbf{A}_1\|^2 + \|\mathbf{A}_2\|^2 + \cdots + \|\mathbf{A}_n\|^2.$$

[35] The matrix norm defined in (5.139) is one of several popular norms for matrices. An excellent discussion of matrix norms is given by Terrell, A Passage to Modern Analysis, 2019, §9.5.1.

The above definition of the norm of a matrix is introduced because, with this definition of the norm, we have the following useful variant of the Cauchy-Schwarz Inequality.

Theorem (A Generalized Cauchy-Schwarz Inequality): Let $\mathbf{A}$ be an $n \times n$ matrix and let $\mathbf{u}$ be a point in $\mathbb{R}^n$. Then

$$\|\mathbf{Au}\| \leq \|\mathbf{A}\| \cdot \|\mathbf{u}\|. \tag{5.140}$$

Proof: Squaring both sides of (5.140), this inequality holds if and only if

$$\|\mathbf{Au}\|^2 \leq \|\mathbf{A}\|^2 \|\mathbf{u}\|^2. \tag{5.141}$$

If for each index $i$ with $1 \leq i \leq n$ we let the point $\mathbf{A}_i$ in $\mathbb{R}^n$ be the $i$th row of $\mathbf{A}$, then

$$\mathbf{Au} = (\langle \mathbf{A}_1, \mathbf{u} \rangle, \dots, \langle \mathbf{A}_n, \mathbf{u} \rangle).$$

Thus, by the standard Cauchy-Schwarz Inequality,

$$\begin{aligned}
\|\mathbf{Au}\|^2 &= (\langle \mathbf{A}_1, \mathbf{u} \rangle)^2 + \cdots + (\langle \mathbf{A}_n, \mathbf{u} \rangle)^2 \\
&\leq \|\mathbf{A}_1\|^2 \|\mathbf{u}\|^2 + \cdots + \|\mathbf{A}_n\|^2 \|\mathbf{u}\|^2 \\
&= \left(\|\mathbf{A}_1\|^2 + \cdots + \|\mathbf{A}_n\|^2\right) \|\mathbf{u}\|^2 = \|\mathbf{A}\|^2 \|\mathbf{u}\|^2.
\end{aligned}$$

We have verified inequality (5.141) and, hence, also inequality (5.140).

Corollary: Let $\mathbf{A}$ be an $n \times n$ matrix, let $Q : \mathbb{R}^n \to \mathbb{R}$ be the quadratic function associated with $\mathbf{A}$, and let $\mathbf{u}$ be a point in $\mathbb{R}^n$. Then

$$|Q(\mathbf{u})| \leq \|\mathbf{A}\| \|\mathbf{u}\|^2. \tag{5.142}$$

Proof: By definition, $|Q(\mathbf{u})| = |\langle \mathbf{Au}, \mathbf{u} \rangle|$. Thus, if we first use the standard Cauchy-Schwarz Inequality and then the Generalized Cauchy-Schwarz Inequality, it follows that

$$|Q(\mathbf{u})| = |\langle \mathbf{Au}, \mathbf{u} \rangle| \leq \|\mathbf{Au}\| \cdot \|\mathbf{u}\| \leq \|\mathbf{A}\| \|\mathbf{u}\|^2.$$

Definition: An $n \times n$ matrix $\mathbf{A}$ is said to be positive definite provided that

$$\langle \mathbf{Au}, \mathbf{u} \rangle > 0 \quad \text{for all nonzero points } \mathbf{u} \text{ in } \mathbb{R}^n,$$

and is said to be negative definite provided that

$$\langle \mathbf{Au}, \mathbf{u} \rangle < 0 \quad \text{for all nonzero points } \mathbf{u} \text{ in } \mathbb{R}^n.$$

These are commonly expressed, in shorthand, as $\mathbf{A} > 0$ and $\mathbf{A} < 0$, respectively.

Proposition: Let $\mathbf{A}$ be an $n \times n$ positive definite matrix. Then there is a positive number $c$ such that, for all $\mathbf{u} \in \mathbb{R}^n$,

$$Q(\mathbf{u}) = \langle \mathbf{Au}, \mathbf{u} \rangle \geq c\|\mathbf{u}\|^2. \tag{5.143}$$

Proof: Since the quadratic function $Q : \mathbb{R}^n \to \mathbb{R}$ is the sum of products of continuous functions, namely, the component projection functions, it is continuous. On the other hand, from (5.4), the unit sphere $S = \{\mathbf{u} \text{ in } \mathbb{R}^n \mid \|\mathbf{u}\| = 1\}$ is a closed and bounded subset of $\mathbb{R}^n$. From (3.59), the unit sphere is therefore sequentially compact.

Thus, by Extreme Value Theorem (3.77), there is a point in $S$ that is a minimizer for the restriction of the quadratic function to $S$. Define $c$ to be the value of the quadratic function at this minimizer. Observe that $c$ is positive, since we have assumed that the matrix $\mathbf{A}$ is positive definite, and that

$$Q(\mathbf{u}) \geq c \quad \text{for all points } \mathbf{u} \text{ in } S. \tag{5.144}$$

Now, for all points $\mathbf{u}$ in $\mathbb{R}^n$ and all real numbers $\lambda$, $\mathbf{A}(\lambda\mathbf{u}) = \lambda\mathbf{A}\mathbf{u}$, so

$$Q(\lambda\mathbf{u}) = \lambda^2 Q(\mathbf{u}). \tag{5.145}$$

Moreover, note that, if $\mathbf{u}$ is any nonzero point in $\mathbb{R}^n$, then

$$Q(\mathbf{u}) = Q\left(\|\mathbf{u}\|\frac{\mathbf{u}}{\|\mathbf{u}\|}\right).$$

From equality (5.145), it follows that

$$Q(\mathbf{u}) = \|\mathbf{u}\|^2 Q\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right).$$

But $\mathbf{u}/\|\mathbf{u}\|$ is a point in $S$, so by inequality (5.144), $Q(\mathbf{u}) \geq c\|\mathbf{u}\|^2$. It is clear that this inequality also holds if $\mathbf{u} = \mathbf{0}$.

Definition: Let $A \subset \mathbb{R}^n$ and $\mathbf{x} \in A$. For function $f : A \to \mathbb{R}$:

i. The point $\mathbf{x}$ is called a local maximizer for the function $f : A \to \mathbb{R}$, provided that there is some positive number $r$ such that

$$f(\mathbf{x} + \mathbf{h}) \leq f(\mathbf{x}) \quad \text{if } \mathbf{x} + \mathbf{h} \text{ is in } A \text{ and } \|\mathbf{h}\| < r. \tag{5.146}$$

ii. The point $\mathbf{x}$ is called a local minimizer for the function $f : A \to \mathbb{R}$, provided that there is some positive number $r$ such that

$$f(\mathbf{x} + \mathbf{h}) \geq f(\mathbf{x}) \quad \text{if } \mathbf{x} + \mathbf{h} \text{ is in } A \text{ and } \|\mathbf{h}\| < r. \tag{5.147}$$

iii. The point $\mathbf{x}$ is called a local extreme point for the function $f : A \to \mathbb{R}$, provided that it is either a local minimizer or a local maximizer for $f : A \to \mathbb{R}$.

We immediately find the following necessary condition for a point to be a local extreme point for a function.

Proposition: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has first-order partial derivatives. If the point $\mathbf{x}$ is a local extreme point for the function $f : \mathcal{O} \to \mathbb{R}$, then

$$\nabla f(\mathbf{x}) = \mathbf{0}. \tag{5.148}$$

Proof: Since $\mathbf{x}$ is an interior point of $\mathcal{O}$, we can choose a positive number $r$ such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$. Fix an index $i$ with $1 \leq i \leq n$ and define the function $\phi : (-r, r) \to \mathbb{R}$ by $\phi(t) = f(\mathbf{x} + t\mathbf{e}_i)$, for $|t| < r$. Then the point 0 is an extreme point of the function $\phi : (-r, r) \to \mathbb{R}$, so recalling (2.99)

$$\phi'(0) = \frac{\partial f}{\partial x_i}(\mathbf{x}) = 0.$$

But this holds for each index $i$ with $1 \leq i \leq n$, which means that (5.148) holds.

Observe that, in order to search for local extreme points, we must first find the points $\mathbf{x}$ in $\mathcal{O}$ at which

$$\nabla f(\mathbf{x}) = \mathbf{0}. \tag{5.149}$$

However, equation (5.149) is a system of $n$ scalar equations in $n$ real unknowns. Unless the function $f : \mathcal{O} \to \mathbb{R}$ has a very simple form, it is not possible to find explicit solutions of (14.18). This should not be so surprising since in fact even for a differentiable function of a single variable $f : \mathbb{R} \to \mathbb{R}$, unless $f : \mathbb{R} \to \mathbb{R}$ is very simple, it is not possible to explicitly find all the numbers $x$ that are solutions of the equation $f'(x) = 0$.

Theorem: Let $I$ be an open interval of real numbers and suppose that the function $f : I \to \mathbb{R}$ has a second derivative. Then for each pair of points $x$ and $x + h$ in the interval $I$, there is a number $\theta$ with $0 < \theta < 1$ such that

$$f(x + h) = f(x) + f'(x)h + \frac{1}{2}f''(x + \theta h)h^2. \tag{5.150}$$

Proof: This is (5.125) for $k = 1$.

From (5.150) for functions of a single variable, and the derivative calculations for functions of several variables we obtained above, we obtain the following theorem.

Theorem: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has continuous second-order partial derivatives. For each pair of points $\mathbf{x}$ and $\mathbf{x} + \mathbf{h}$ in $\mathcal{O}$ with the property that the segment between these points also lies in $\mathcal{O}$, there is a number $\theta$ with $0 < \theta < 1$ such that

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h} \rangle + \frac{1}{2}\left\langle \nabla^2 f(\mathbf{x} + \theta\mathbf{h})\mathbf{h}, \mathbf{h} \right\rangle. \tag{5.151}$$

Proof: Choose $I$ to be an open interval of real numbers containing both 0 and 1 such that $\mathbf{x} + t\mathbf{h}$ belongs to $\mathcal{O}$ if $t$ is in $I$. Then define the function $\psi : I \to \mathbb{R}$ by

$$\psi(t) = f(\mathbf{x} + t\mathbf{h}) \quad \text{for } t \text{ in } I.$$

Recalling (5.138) and (5.140), the function $\psi : I \to \mathbb{R}$ has a second derivative and we have the following formulas for the first and second derivatives, for $t \in I$:

$$\psi'(t) = \langle \nabla f(\mathbf{x} + t\mathbf{h}), \mathbf{h} \rangle \quad \text{and} \quad \psi''(t) = \left\langle \nabla^2 f(\mathbf{x} + t\mathbf{h})\mathbf{h}, \mathbf{h} \right\rangle. \tag{5.152}$$

We now apply (5.150) to the function $\psi : I \to \mathbb{R}$ with $x = 0$ and $h = 1$ to choose a number $\theta$ with $0 < \theta < 1$ such that

$$\psi(1) = \psi(0) + \psi'(0) + \frac{1}{2}\psi''(\theta), \tag{5.153}$$

an equality that, after substituting the values of $\psi(1)$ and $\psi(0)$ and using the above formulas for $\psi'(0)$ and $\psi''(\theta)$, is seen to be precisely formula (5.151).

Theorem (The Second-Order Approximation Theorem): Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has continuous second-order partial derivatives. Then

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{x}+\mathbf{h}) - [f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h}\rangle + 1/2\,\langle \nabla^2 f(\mathbf{x})\mathbf{h}, \mathbf{h}\rangle]}{\|\mathbf{h}\|^2} = 0. \tag{5.154}$$

Proof: Since the point $\mathbf{x}$ is an interior point of $\mathcal{O}$, we can choose a positive number $r$ such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$. It is convenient to define

$$R(\mathbf{h}) = f(\mathbf{x}+\mathbf{h}) - \left[f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h}\rangle + \frac{1}{2}\left\langle \nabla^2 f(\mathbf{x})\mathbf{h}, \mathbf{h}\right\rangle\right] \quad \text{for } \|\mathbf{h}\| < r.$$

We must show that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{R(\mathbf{h})}{\|\mathbf{h}\|^2} = 0. \tag{5.155}$$

Fix the point $\mathbf{h}$ in $\mathbb{R}^n$ with $0 < \|\mathbf{h}\| < r$. Using (5.151), we can choose a number $\theta$ with $1 < \theta < 1$ such that

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h}\rangle + \frac{1}{2}\left\langle \nabla^2 f(\mathbf{x}+\theta\mathbf{h})\mathbf{h}, \mathbf{h}\right\rangle,$$

so that

$$\begin{aligned} R(\mathbf{h}) &= \frac{1}{2}\left\langle \nabla^2 f(\mathbf{x}+\theta\mathbf{h})\mathbf{h}, \mathbf{h}\right\rangle - \frac{1}{2}\left\langle \nabla^2 f(\mathbf{x}+\mathbf{h})\mathbf{h}, \mathbf{h}\right\rangle \\ &= \frac{1}{2}\left\langle \left[\nabla^2 f(\mathbf{x}+\theta\mathbf{h}) - \nabla^2 f(\mathbf{x}+\mathbf{h})\right]\mathbf{h}, \mathbf{h}\right\rangle. \end{aligned} \tag{5.156}$$

Let $\mathbf{A} = [\nabla^2 f(\mathbf{x}+\theta\mathbf{h}) - \nabla^2 f(\mathbf{x}+\mathbf{h})]$, so that (5.156) is $\mathbf{h}'\mathbf{A}\mathbf{h}/2$. Then the regular Cauchy-Schwarz implies $|\mathbf{h}'\mathbf{A}\mathbf{h}| \leq \|\mathbf{h}\| \cdot \|\mathbf{A}\mathbf{h}\|$, and the generalized one, (5.140), implies $\|\mathbf{A}\mathbf{h}\| \leq \|\mathbf{A}\| \cdot \|\mathbf{h}\|$. This yields (5.157).

We can use this formula and the Generalized Cauchy-Schwarz Inequality to obtain the estimate

$$|R(\mathbf{h})| \leq \frac{1}{2}\left\|\nabla^2 f(\mathbf{x}+\theta\mathbf{h}) - \nabla^2 f(\mathbf{x})\right\| \|\mathbf{h}\|^2. \tag{5.157}$$

Dividing this estimate by $\|\mathbf{h}\|^2$, we obtain

$$\frac{|R(\mathbf{h})|}{\|\mathbf{h}\|^2} \leq \frac{1}{2}\left\|\nabla^2 f(\mathbf{x}+\theta\mathbf{h}) - \nabla^2 f(\mathbf{x})\right\|. \tag{5.158}$$

But the function $f : \mathcal{O} \to \mathbb{R}$ has been assumed to have continuous second-order partial derivatives, so, from linearity of limits (5.15) and continuity of the norm (5.21),

$$\lim_{\mathbf{h}\to\mathbf{0}} \left\|\nabla^2 f(\mathbf{x}+\theta\mathbf{h}) - \nabla^2 f(\mathbf{x})\right\| = \mathbf{0}.$$

Hence, from the Squeeze Theorem (2.9), (5.155) follows from the estimate (5.158).

For the following, it is useful to recall the univariate results (2.105) and (2.106).

Theorem (The Second-Derivative Test): Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has continuous second-order partial derivatives. Assume that $\nabla f(\mathbf{x}) = \mathbf{0}$.

i. If the Hessian matrix $\nabla^2 f(\mathbf{x})$ is positive definite, then the point $\mathbf{x}$ is a strict local minimizer of the function $f : \mathcal{O} \to \mathbb{R}$.
ii. If the Hessian matrix $\nabla^2 f(\mathbf{x})$ is negative definite, then the point $\mathbf{x}$ is a strict local maximizer of the function $f : \mathcal{O} \to \mathbb{R}$.

In short,

$$\textbf{If } \nabla^2 f(\mathbf{x}) > 0\textbf{, then } \mathbf{x} \textbf{ is a strict local minimizer of } f. \tag{5.159}$$

$$\textbf{If } \nabla^2 f(\mathbf{x}) < 0\textbf{, then } \mathbf{x} \textbf{ is a strict local maximizer of } f. \tag{5.160}$$

Proof: We need only consider case (i) since case (ii) follows from (i) if we replace $f$ with $-f$. So suppose that the Hessian matrix $\nabla^2 f(\mathbf{x})$ is positive definite. Since the point $\mathbf{x}$ is an interior point of $\mathcal{O}$, we can choose a positive number $r$ such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$. The strategy of the proof is to write the difference $f(\mathbf{x}+\mathbf{h})-f(\mathbf{x})$ as

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = Q(\mathbf{h}) + R(\mathbf{h}), \tag{5.161}$$

where $Q : \mathbb{R}^n \to \mathbb{R}$ is a positive definite quadratic function and

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{R(\mathbf{h})}{\|\mathbf{h}\|^2} = 0. \tag{5.162}$$

Indeed, if we define for $\|\mathbf{h}\| < r$

$$R(\mathbf{h}) = f(\mathbf{x}+\mathbf{h}) - \left[ f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h} \rangle + \frac{1}{2} \left\langle \nabla^2 f(\mathbf{x})\mathbf{h}, \mathbf{h} \right\rangle \right], \tag{5.163}$$

then the Second-Order Approximation Theorem asserts that (5.162) holds. Moreover, if we define $Q : \mathbb{R}^n \to \mathbb{R}$ to be the quadratic function associated with one-half the Hessian matrix $\nabla^2 f(\mathbf{x})$, then this quadratic function is positive definite. Finally, since $\nabla f(\mathbf{x}) = \mathbf{0}$, we can rewrite (5.163) to obtain (5.161).

Since the quadratic function $Q : \mathbb{R}^n \to \mathbb{R}$ is positive definite, we can use (5.143) to choose a positive number $c$ such that, $\forall \mathbf{h} \in \mathbb{R}^n$, $Q(\mathbf{h}) \geq c\|\mathbf{h}\|^2$. On the other hand, using (5.162), it follows that we can choose a positive number $\delta$ less than $r$ such that, for $0 < \|\mathbf{h}\| < \delta$,

$$\frac{|R(\mathbf{h})|}{\|\mathbf{h}\|^2} < \frac{c}{2}, \quad \text{i.e.,} \quad -\frac{c}{2}\|\mathbf{h}\|^2 < R(\mathbf{h}) < \frac{c}{2}\|\mathbf{h}\|^2. \tag{5.164}$$

Combining these two estimates, it follows from (5.161) that, if $0 < \|\mathbf{h}\| < \delta$,

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = Q(\mathbf{h}) + R(\mathbf{h}) \geq c\|\mathbf{h}\|^2 + R(\mathbf{h}) > \frac{c}{2}\|\mathbf{h}\|^2, \tag{5.165}$$

so, from (5.147), the point $\mathbf{x}$ is a strict local minimizer of the function $f : \mathcal{O} \to \mathbb{R}$.

## 5.9 Approximating Nonlinear Mappings By Linear Mappings

Subsections 5.9.1 and 5.9.2 come from parts of Fitzpatrick, Ch. 15.

### 5.9.1 Derivative Matrix and Differential

Definition: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and consider a mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ represented in component functions as $\mathbf{F} = (F_1, \ldots, F_m)$.

i. The mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ is said to have first-order partial derivatives at the point $\mathbf{x}$ in $\mathcal{O}$ provided that for each index $i$ such that $1 \leq i \leq m$, the component function $F_i : \mathcal{O} \to \mathbb{R}$ has first-order partial derivatives at $\mathbf{x}$.

ii. Moreover, the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ is said to have first-order partial derivatives provided that it has first partial derivatives at every point in $\mathcal{O}$.

iii. Finally, the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ is said to be continuously differentiable provided that each component function is continuously differentiable.

Proposition: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ is continuously differentiable. Then the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ is continuous.

> Proof: By definition, each of the component functions of the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ is continuously differentiable. It follows from Theorem (Fitzpatrick, 13.20) (page 330) that each component function is continuous. Consequently, from the Componentwise Continuity Criterion (5.20), we conclude that the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ is itself continuous.

Definition: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ has first-order partial derivatives at the point $\mathbf{x}$ in $\mathcal{O}$. The **derivative matrix** of $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ at the point $\mathbf{x}$ is defined to be the $m \times n$ matrix $\mathbf{DF}(\mathbf{x})$, which, for each index $i$ such that $1 \leq i \leq m$, has an $i$th row equal to $\nabla F_i(\mathbf{x})$. Thus, the $(i, j)$th entry of this derivative matrix is given by the formula

$$(\mathbf{DF}(\mathbf{x}))_{ij} \equiv \frac{\partial F_i}{\partial x_j}(\mathbf{x}). \tag{5.166}$$

Theorem (The Mean Value Theorem for General Mappings): Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ is continuously differentiable. Suppose that the points $\mathbf{x}$ and $\mathbf{x} + \mathbf{h}$ are in $\mathcal{O}$ and that the segment joining these points also lies in $\mathcal{O}$. Then there are numbers $\theta_1, \theta_2, \ldots, \theta_m$ in the open interval $(0, 1)$ such that, for $1 \leq i \leq m$,

$$F_i(\mathbf{x} + \mathbf{h}) - F_i(\mathbf{x}) = \langle \nabla F_i \left(\mathbf{x} + \theta_i \mathbf{h}\right), \mathbf{h} \rangle \,; \tag{5.167}$$

that is, now necessarily with $\mathbf{x}, \mathbf{h}$ $n \times 1$ column vectors, and $\mathbf{F}(\mathbf{x})$ an $m \times 1$ column vector,

$$\mathbf{F}(\mathbf{x} + \mathbf{h}) - \mathbf{F}(\mathbf{x}) = \mathbf{A}\mathbf{h}, \tag{5.168}$$

where $\mathbf{A}$ is the $m \times n$ matrix whose $i$th row is $\nabla F_i \left(\mathbf{x} + \theta_i \mathbf{h}\right)$.

> Proof: Just apply the Mean Value Theorem for real-valued functions (5.86) to each of the continuously differentiable component functions and we obtain formula (5.167). Formula (5.168) is simply a rewriting of (5.167).

Recall the First-Order Approximation Theorem for scalar-valued functions, which asserts that, if $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ and the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable, then, at each point $\mathbf{x}$ in $\mathcal{O}$, fixing a typo in Fitzpatrick; and as in (5.128),

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{x} + \mathbf{h}) - [f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h} \rangle]}{\|\mathbf{h}\|} = 0. \tag{5.169}$$

The following is an extension of this result to general mappings.

Theorem (First-Order Approximation Theorem for Mappings): Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ is continuously differentiable. Then

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{F}(\mathbf{x}+\mathbf{h}) - [\mathbf{F}(\mathbf{x}) + \mathbf{DF}(\mathbf{x})\mathbf{h}]\|}{\|\mathbf{h}\|} = 0. \tag{5.170}$$

Proof: Since $\mathcal{O}$ is open, we can choose a positive number $r$ such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$. For a point $\mathbf{h}$ in $\mathbb{R}^n$ such that $\|\mathbf{h}\| < r$, define

$$\mathbf{R}(\mathbf{h}) = \mathbf{F}(\mathbf{x}+\mathbf{h}) - [\mathbf{F}(\mathbf{x}) + \mathbf{DF}(\mathbf{x})\mathbf{h}].$$

We must show that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{R}(\mathbf{h})\|}{\|\mathbf{h}\|} = 0. \tag{5.171}$$

But if we represent the mappings $\mathbf{F}$ and $\mathbf{R}$ as $\mathbf{F} = (F_1, \ldots, F_m)$ and $\mathbf{R} = (R_1, \ldots, R_m)$, then it is clear that, for each index $i$ such that $1 \le i \le m$, and for $\|\mathbf{h}\| < r$,

$$R_i(\mathbf{h}) = F_i(\mathbf{x}+\mathbf{h}) - [F_i(\mathbf{x}) + \langle \nabla F_i(\mathbf{x}), \mathbf{h}\rangle].$$

Since the function $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ is continuously differentiable, the First-Order Approximation Theorem for real-valued functions (5.128), also (5.169), implies that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{R_i(\mathbf{h})}{\|\mathbf{h}\|} = 0.$$

Since, for $0 < \|\mathbf{h}\| < r$, taking limits and using result (2.43) or (2.44),

$$\frac{\|\mathbf{R}(\mathbf{h})\|}{\|\mathbf{h}\|} = \left( \sum_{i=1}^{m} \left[ \frac{R_i(\mathbf{h})}{\|\mathbf{h}\|} \right]^2 \right)^{1/2},$$

it follows that (5.171) holds.

For a function $f : I \to \mathbb{R}$, where $I$ is an open interval, assume at the point $x \in I$ there is a number $a$ such that Recall (5.124), and also the Lang excerpt starting on page 316.

$$\lim_{h\to 0} \frac{f(x+h) - [f(x) + ah]}{h} = 0. \tag{5.172}$$

If $h \neq 0$ and $x + h$ is in $I$,

$$\frac{f(x+h) - [f(x) + ah]}{h} = \frac{f(x+h) - f(x)}{h} - a.$$

It follows that $f$ is differentiable at $x$ and that $f'(x) = a$. This property generalizes to mappings as follows. Note how (5.173) is a direct generalization of (5.172).

Theorem: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and consider a mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$. Suppose that $\mathbf{A}$ is an $m \times n$ matrix with the property that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{F}(\mathbf{x}+\mathbf{h}) - [\mathbf{F}(\mathbf{x}) + \mathbf{A}\mathbf{h}]\|}{\|\mathbf{h}\|} = 0. \tag{5.173}$$

Then the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ has first-order partial derivatives at the point $\mathbf{x}$ and

$$\mathbf{A} = \mathbf{DF}(\mathbf{x}). \tag{5.174}$$

Proof: Represent the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ in component functions as $\mathbf{F} = (F_1, \ldots, F_m)$ and set $a_{ij} = (\mathbf{A})_{ij}$. We must show that, for each pair of indices $i$ and $j$ such that $1 \le i \le m$ and $1 \le j \le n$,

$$a_{ij} = \frac{\partial F_i}{\partial x_j}(\mathbf{x}).$$

For each index $i$ such that $1 \le i \le m$, define $\mathbf{A}_i$ to be the $i$th row of the matrix $\mathbf{A}$. Since $\mathcal{O}$ is open, we can choose a positive number $r$ such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$. Now observe that if $1 \le i \le m$ and $\|\mathbf{h}\| < r$, then with $p_i$ the $i$th component projection function (5.5),

$$F_i(\mathbf{x} + \mathbf{h}) - [F_i(\mathbf{x}) + \langle \mathbf{A}_i, \mathbf{h} \rangle] = p_i(\mathbf{F}(\mathbf{x} + \mathbf{h}) - [\mathbf{F}(\mathbf{x}) + \mathbf{A}\mathbf{h}]),$$

so that

$$|F_i(\mathbf{x} + \mathbf{h}) - [F_i(\mathbf{x}) + \langle \mathbf{A}_i, \mathbf{h} \rangle]| \le \|\mathbf{F}(\mathbf{x} + \mathbf{h}) - [\mathbf{F}(\mathbf{x}) + \mathbf{A}\mathbf{h}]\|.$$

From (5.173) it follows that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{F_i(\mathbf{x} + \mathbf{h}) - [F_i(\mathbf{x}) + \langle \mathbf{A}_i, \mathbf{h} \rangle]}{\|\mathbf{h}\|} = 0.$$

In particular, for an index $j$ such that $1 \le j \le n$,

$$\lim_{t \to 0} \frac{F_i\left(\mathbf{x} + t\mathbf{e}_j\right) - [F_i(\mathbf{x}) + \langle \mathbf{A}_i, t\mathbf{e}_j \rangle]}{\|t\mathbf{e}_j\|} = 0. \tag{5.175}$$

However, $\|t\mathbf{e}_j\| = |t|$, so (5.175) is equivalent to recall (5.24) or (5.28); and (5.166)

$$\lim_{t \to 0} \frac{F_i\left(\mathbf{x} + t\mathbf{e}_j\right) - F_i(\mathbf{x})}{t} = \left\langle \mathbf{A}_i, \mathbf{e}_j \right\rangle,$$

thus proving that $\mathbf{F} : \mathcal{O} \to \mathbb{R}^n$ has first partial derivatives at $\mathbf{x}$ and

$$a_{ij} = \langle \mathbf{A}_i, \mathbf{e}_j \rangle = \frac{\partial F_i}{\partial x_j}(\mathbf{x}), \quad \text{for } 1 \le i \le m,\ \ 1 \le j \le n.$$

The above theorem implies that, for a continuously differentiable mapping, the derivative matrix is the only matrix having the first-order approximation property (5.170). That it needs to be continuously differentiable requires justification. I assume it is from (5.48) and (5.79).

Definition: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ has first-order partial derivatives at the point $\mathbf{x}$. The linear mapping $\mathbf{dF}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$ defined by $\mathbf{dF}(\mathbf{x})(\mathbf{h}) \equiv \mathbf{DF}(\mathbf{x})\mathbf{h}$, for all $\mathbf{h} \in \mathbb{R}^n$ is called the **differential of the mapping** $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ at the point $\mathbf{x}$. Compare to the total differential (5.38).

### 5.9.2 The Chain Rule

From the Chain Rule (2.71) for real-valued functions of a single variable, it follows that, if $\mathcal{O}$ and $\mathcal{U}$ are open sets of real numbers and the functions $f : \mathcal{O} \to \mathbb{R}$ and $g : \mathcal{U} \to \mathbb{R}$ are continuously differentiable, with $f(\mathcal{O})$ contained in $\mathcal{U}$, then the composite function $g \circ f : \mathcal{O} \to \mathbb{R}$ is also differentiable: For each point $x$ in $\mathcal{O}$, $(g \circ f)'(x) = g'(f(x))f'(x)$, and $g \circ f$ is also continuously differentiable, from (2.39).

The Chain Rule carries over to compositions of general continuously differentiable mappings in which the derivative matrix replaces the derivative and matrix multiplication replaces scalar multiplication. The general Chain Rule follows from the following special case of the composition of a mapping with a real-valued function.

In order to clearly state the Chain Rule, it is helpful to use the following notation: For an open subset $\mathcal{U}$ of $\mathbb{R}^m$ and a function $g : \mathcal{U} \to \mathbb{R}$ that has first-order partial derivatives, at each point $\mathbf{p}$ in $\mathcal{U}$, and for each index $i$ such that $1 \leq i \leq m$, we define as in (5.24)

$$D_i g(\mathbf{p}) \equiv \lim_{t \to 0} \frac{g(\mathbf{p} + t\mathbf{e}_i) - g(\mathbf{p})}{t}. \tag{5.176}$$

This notation has the advantage that the partial derivative with respect to the $i$th component is denoted by a symbol independent of the notation being used for the points in the domain. Moreover, for each $\mathbf{p} \in \mathcal{U}$, as in (5.25), and is an $m$-length row vector, i.e., $1 \times m$,

$$\nabla g(\mathbf{p}) = (D_1 g(\mathbf{p}), \ldots, D_m g(\mathbf{p})). \tag{5.177}$$

Theorem (The Chain Rule): Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ is continuously differentiable. Suppose also that $\mathcal{U}$ is an open subset of $\mathbb{R}^m$ and that the function $g : \mathcal{U} \to \mathbb{R}$ is continuously differentiable. Finally, suppose that $\mathbf{F}(\mathcal{O})$ is contained in $\mathcal{U}$. Then the composition $g \circ \mathbf{F} : \mathcal{O} \to \mathbb{R}$ is also continuously differentiable. Moreover, for each point $\mathbf{x}$ in $\mathcal{O}$ and each index $i$ such that $1 \leq i \leq n$,

$$\frac{\partial}{\partial x_i}(g \circ \mathbf{F})(\mathbf{x}) = \sum_{j=1}^{m} D_j g(\mathbf{F}(\mathbf{x})) \frac{\partial F_j}{\partial x_i}(\mathbf{x}); \tag{5.178}$$

that is, with respective matrix sizes $1 \times n$; $1 \times m$; and $m \times n$,

$$\nabla(g \circ \mathbf{F})(\mathbf{x}) = \nabla g(\mathbf{F}(\mathbf{x}))\mathbf{DF}(\mathbf{x}). \tag{5.179}$$

> Proof: Let $\mathbf{x}$ be a point in $\mathcal{O}$. Since $\mathcal{O}$ is open, we can select a positive number $r$ such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$. Moreover, since the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ is continuous and $\mathcal{U}$ is an open subset of $\mathbb{R}^m$, we can also suppose that the segment joining the points $\mathbf{F}(\mathbf{x})$ and $\mathbf{F}(\mathbf{x} + \mathbf{h})$ lies in $\mathcal{U}$ if $\|\mathbf{h}\| < r$. For each $\mathbf{h}$ in $\mathbb{R}^n$ such that $\|\mathbf{h}\| < r$, define
>
> $$\mathbf{R}(\mathbf{h}) = \mathbf{F}(\mathbf{x} + \mathbf{h}) - \mathbf{F}(\mathbf{x}) - \mathbf{DF}(\mathbf{x})\mathbf{h}.$$
>
> According to the First-Order Approximation Theorem for Mappings (5.170),
>
> $$\lim_{\mathbf{h} \to \mathbf{0}} \frac{\|\mathbf{R}(\mathbf{h})\|}{\|\mathbf{h}\|} = 0, \tag{5.180}$$
>
> and, by the definition of $\mathbf{R}(\mathbf{h})$, if $\|\mathbf{h}\| < r$,
>
> $$\mathbf{F}(\mathbf{x} + \mathbf{h}) - \mathbf{F}(\mathbf{x}) = \mathbf{DF}(\mathbf{x})\mathbf{h} + \mathbf{R}(\mathbf{h}). \tag{5.181}$$

Now for each $\mathbf{h}$ in $\mathbb{R}^n$ such that $\|\mathbf{h}\| < r$, we can apply the MVT (5.86) to the function $g : \mathcal{U} \to \mathbb{R}$ on the segment joining the points $\mathbf{F}(\mathbf{x})$ and $\mathbf{F}(\mathbf{x}+\mathbf{h})$ in order to select a point on this segment, which we label $\mathbf{v}(\mathbf{h})$, at which

$$g(\mathbf{F}(\mathbf{x}+\mathbf{h})) - g(\mathbf{F}(\mathbf{x})) = \langle \nabla g(\mathbf{v}(\mathbf{h})), \mathbf{F}(\mathbf{x}+\mathbf{h}) - \mathbf{F}(\mathbf{x})\rangle.$$

Substituting (5.181) and using properties (i), (ii), and (iii) of (4.1) gives

$$(g \circ \mathbf{F})(\mathbf{x}+\mathbf{h}) - (g \circ \mathbf{F})(\mathbf{x}) = \langle \nabla g(\mathbf{v}(\mathbf{h})), \mathbf{DF}(\mathbf{x})\mathbf{h}\rangle + \langle \nabla g(\mathbf{v}(\mathbf{h})), \mathbf{R}(\mathbf{h})\rangle. \tag{5.182}$$

Observe that the continuity of $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ implies that

$$\lim_{\mathbf{h}\to\mathbf{0}} \mathbf{v}(\mathbf{h}) = \mathbf{F}(\mathbf{x}). \tag{5.183}$$

We now verify (5.178). Fix an index $i$, with $1 \le i \le n$. For a number $t$ such that $0 < |t| < r$, if we define $\mathbf{h} = t\mathbf{e}_i$, then from (5.182) we obtain

$$\frac{(g \circ \mathbf{F})\left(\mathbf{x}+t\mathbf{e}_i\right) - (g \circ \mathbf{F})(\mathbf{x})}{t} = \left\langle \nabla g\left(\mathbf{v}\left(t\mathbf{e}_i\right)\right), \mathbf{DF}(\mathbf{x})\mathbf{e}_i\right\rangle + \left\langle \nabla g\left(\mathbf{v}\left(t\mathbf{e}_i\right)\right), \frac{\mathbf{R}\left(t\mathbf{e}_i\right)}{t}\right\rangle.$$

From this equality, recalling (5.24), and by using (5.180) and (5.183), it follows that

$$\frac{\partial}{\partial x_i}(g \circ \mathbf{F})(\mathbf{x}) = \left\langle \nabla g(\mathbf{F}(\mathbf{x})), \mathbf{DF}(\mathbf{x})\mathbf{e}_i\right\rangle. \tag{5.184}$$

But (note $\mathbf{DF}(\mathbf{x})$ is $m \times n$; $\mathbf{e}_i$ is $n \times 1$, so the rhs is an $m \times 1$ column vector)

$$\mathbf{DF}(\mathbf{x})\mathbf{e}_i = \left(\frac{\partial F_1}{\partial x_i}(\mathbf{x}), \ldots, \frac{\partial F_m}{\partial x_i}(\mathbf{x})\right),$$

so the scalar equation (5.184) is exactly equation (5.178). In particular, this shows that the function $g \circ \mathbf{F} : \mathcal{O} \to \mathbb{R}$ has first-order partial derivatives, and then, because of the continuity with respect to $\mathbf{x}$ of the right-hand side of formula (5.178), that $g \circ \mathbf{F} : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. To conclude the proof, simply observe that (5.179) is a rewriting of (5.178) in matrix notation.

**Example 5.23** *Suppose that the functions $\psi : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable. Suppose also that $\mathcal{O}$ is an open subset of the plane $\mathbb{R}^2$ and that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Finally, suppose that $(\psi(x,y), \varphi(x,y))$ is in $\mathcal{O}$ for all $(x,y)$ in $\mathbb{R}^2$. Then*

$$\begin{aligned}\frac{\partial}{\partial x}(f(\psi(x,y), \varphi(x,y))) =& D_1 f(\psi(x,y), \varphi(x,y))\frac{\partial \psi}{\partial x}(x,y) \\ &+ D_2 f(\psi(x,y), \varphi(x,y))\frac{\partial \varphi}{\partial x}(x,y)\end{aligned}$$

*and*

$$\begin{aligned}\frac{\partial}{\partial y}(f(\psi(x,y), \varphi(x,y))) =& D_1 f(\psi(x,y), \varphi(x,y))\frac{\partial \psi}{\partial y}(x,y) \\ &+ D_2 f(\psi(x,y), \varphi(x,y))\frac{\partial \varphi}{\partial y}(x,y). \qquad \blacksquare\end{aligned}$$

In books in which there are calculations involving partial derivatives, the reader will find a large variety of notation. For example, the second of the two derivative formulas in the previous example is often abbreviated as

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \psi}\frac{\partial \psi}{\partial y} + \frac{\partial f}{\partial \varphi}\frac{\partial \varphi}{\partial y}. \tag{5.185}$$

As another common instance of terse but useful notational devices, we note that if the function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable and the function $g : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$g(r, \theta) = f(r\cos\theta, r\sin\theta) \quad \text{ for } (r, \theta) \text{ in } \mathbb{R}^2,$$

then according to the Chain Rule, for each point $(r, \theta)$ in $\mathbb{R}^2$,

$$\frac{\partial g}{\partial r}(r, \theta) = D_1 f(r\cos\theta, r\sin\theta)\cos\theta + D_2 f(r\cos\theta, r\sin\theta)\sin\theta.$$

The last formula is frequently abbreviated as

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta. \tag{5.186}$$

One must carefully interpret this formula in order to understand that it signifies the same thing as its predecessor. Formulas such as (5.185) and (5.186) are useful in compressing long equations. But such formulas are not precise because there is no indication of where the derivatives are to be evaluated, and there is ambiguity about what the variables are. When we analyze functions of two or three variables, especially when computing higher derivatives, it is notationally useful to denote

$$D_1 g(\mathbf{p}) \text{ by } \frac{\partial g}{\partial x}(\mathbf{p}), \quad D_2 g(\mathbf{p}) \text{ by } \frac{\partial g}{\partial y}(\mathbf{p}), \quad \text{ and } \quad D_3 g(\mathbf{p}) \text{ by } \frac{\partial g}{\partial z}(\mathbf{p}),$$

even when $x, y$, and $z$ have not been explicitly introduced as notation for the component variables. In the following example, we use this notational convention.

We make some notes in preparation for the next example: Function $u : \mathbb{R}^2 \to \mathbb{R}$ is not specified, but we are told it is harmonic. We label its inputs as $(x, y) \in \mathbb{R}^2$. Let function $\mathbf{H} : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\mathbf{H}(w, z) = (H_1, H_2) = (w^2 - z^2, 2wz)$, and define the composite function $v : \mathbb{R}^2 \to \mathbb{R}$ as $v = (u \circ \mathbf{H})$. Then $v(x, y) = (u \circ \mathbf{H})(x, y) = u(x^2 - y^2, 2xy)$ and, with $\mathbf{x} = (x, y)$, (5.179) is

$$\begin{aligned}
\left[\begin{array}{cc} \frac{\partial v}{\partial x}(\mathbf{x}) & \frac{\partial v}{\partial y}(\mathbf{x}) \end{array}\right] &= \nabla(u \circ \mathbf{H})(\mathbf{x}) = \nabla u(\mathbf{H}(\mathbf{x}))\, \mathbf{DH}(\mathbf{x}) \\
&= \left[\begin{array}{cc} (D_1 u)(\mathbf{H}(\mathbf{x})) & (D_2 u)(\mathbf{H}(\mathbf{x})) \end{array}\right] \left[\begin{array}{cc} \frac{\partial H_1}{\partial x}(\mathbf{x}) & \frac{\partial H_1}{\partial y}(\mathbf{x}) \\ \frac{\partial H_2}{\partial x}(\mathbf{x}) & \frac{\partial H_2}{\partial y}(\mathbf{x}) \end{array}\right] \\
&= \left[\begin{array}{cc} (D_1 u)(x^2 - y^2, 2xy) & (D_2 u)(x^2 - y^2, 2xy) \end{array}\right] \left[\begin{array}{cc} 2x & 2y \\ 2y & 2x \end{array}\right],
\end{aligned}$$

and, writing out the two rows separately (equivalently as in (5.178)),

$$\begin{aligned}
\frac{\partial v}{\partial x}(\mathbf{x}) &= \frac{\partial}{\partial x}(u \circ \mathbf{H})(\mathbf{x}) = (D_1 u)(\mathbf{H}(\mathbf{x}))\frac{\partial H_1}{\partial x}(\mathbf{x}) + (D_2 u)(\mathbf{H}(\mathbf{x}))\frac{\partial H_2}{\partial x}(\mathbf{x}), \\
\frac{\partial v}{\partial y}(\mathbf{x}) &= \frac{\partial}{\partial y}(u \circ \mathbf{H})(\mathbf{x}) = (D_1 u)(\mathbf{H}(\mathbf{x}))\frac{\partial H_1}{\partial y}(\mathbf{x}) + (D_2 u)(\mathbf{H}(\mathbf{x}))\frac{\partial H_2}{\partial y}(\mathbf{x}).
\end{aligned}$$

Using now the informal notation,

$$\begin{aligned}
\frac{\partial v}{\partial x}(x,y) &= \frac{\partial u}{\partial x}\left(x^2-y^2,2xy\right)2x+\frac{\partial u}{\partial y}\left(x^2-y^2,2xy\right)2y,\\
\frac{\partial v}{\partial y}(x,y) &= \frac{\partial u}{\partial x}\left(x^2-y^2,2xy\right)2y+\frac{\partial u}{\partial y}\left(x^2-y^2,2xy\right)2x.
\end{aligned}$$

The following example also involves computing second derivatives as in (5.135).

**Example 5.24** *(Fitzpatrick, p. 417) A function $u:\mathbb{R}^2\to\mathbb{R}$ is said to be harmonic provided it has continuous second-order partial derivatives that satisfy the identity*

$$\frac{\partial^2 u}{\partial x^2}(x,y)+\frac{\partial^2 u}{\partial y^2}(x,y)=0 \quad \textit{for all } (x,y) \textit{ in } \mathbb{R}^2.$$

*Suppose that the function $u:\mathbb{R}^2\to\mathbb{R}$ is harmonic. Define*

$$v(x,y)=u\left(x^2-y^2,2xy\right) \quad \textit{for all } (x,y) \textit{ in } \mathbb{R}^2.$$

*Then it turns out that the function $v:\mathbb{R}^2\to\mathbb{R}$ is also harmonic. To verify this, we must show that*

$$\frac{\partial^2 v}{\partial x^2}(x,y)+\frac{\partial^2 v}{\partial y^2}(x,y)=0 \quad \textit{for all } (x,y) \textit{ in } \mathbb{R}^2.$$

*However, for $(x,y)$ in $\mathbb{R}^2$,*

$$\frac{\partial v}{\partial x}(x,y)=\frac{\partial u}{\partial x}\left(x^2-y^2,2xy\right)2x+\frac{\partial u}{\partial y}\left(x^2-y^2,2xy\right)2y,$$

*So*

$$\begin{aligned}
\frac{\partial^2 v}{\partial x^2}(x,y)=&\frac{\partial^2 u}{\partial x^2}\left(x^2-y^2,2xy\right)4x^2+\frac{\partial u}{\partial x}\left(x^2-y^2,2xy\right)2\\
&+\frac{\partial^2 u}{\partial x\partial y}\left(x^2-y^2,2xy\right)8xy+\frac{\partial^2 u}{\partial y^2}\left(x^2-y^2,2xy\right)4y^2.
\end{aligned}$$

*We carry out a similar computation for $\partial^2 v/\partial y^2(x,y)$, and since*

$$\frac{\partial^2 u}{\partial x^2}\left(x^2-y^2,2xy\right)+\frac{\partial^2 u}{\partial y^2}\left(x^2-y^2,2xy\right)=0 \quad \textit{for all } (x,y) \textit{ in } \mathbb{R}^2,$$

*a calculation shows that*

$$\frac{\partial^2 v}{\partial x^2}(x,y)+\frac{\partial^2 v}{\partial y^2}(x,y)=0 \quad \textit{for } (x,y) \textit{ in } \mathbb{R}^2. \qquad \blacksquare$$

The special case of the Chain Rule that we have just proved leads to the proof of the general case.

Theorem (The Chain Rule for General Mappings): Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the mapping $\mathbf{F}:\mathcal{O}\to\mathbb{R}^m$ is continuously differentiable. Suppose also that $\mathcal{U}$ is an open subset of $\mathbb{R}^m$ and that the mapping $\mathbf{G}:\mathcal{U}\to\mathbb{R}^k$ is continuously differentiable. Finally, suppose that $\mathbf{F}(\mathcal{O})$ is contained in $\mathcal{U}$. Then the composite mapping $\mathbf{G}\circ\mathbf{F}:\mathcal{O}\to\mathbb{R}^k$ is also continuously differentiable. Moreover, for each point $\mathbf{x}$ in $\mathcal{O}$,
with respective matrix sizes $k\times n$, $k\times m$, and $m\times n$,

$$\mathbf{D}(\mathbf{G}\circ\mathbf{F})(\mathbf{x})=\mathbf{DG}(\mathbf{F}(\mathbf{x}))\cdot\mathbf{DF}(\mathbf{x}). \tag{5.187}$$

Proof: Represent the mapping $\mathbf{G}$ in component functions by $\mathbf{G} = (G_1, \ldots, G_k)$. Then observe that the composition $\mathbf{G} \circ \mathbf{F} : \mathcal{O} \to \mathbb{R}^k$ is represented in component functions by $\mathbf{G} \circ \mathbf{F} = (G_1 \circ \mathbf{F}, G_2 \circ \mathbf{F}, \ldots, G_k \circ \mathbf{F})$. For an index $j$ such that $1 \leq j \leq k$, and that the component function $G_j : \mathcal{U} \to \mathbb{R}$ is continuously differentiable, recall the Componentwise Continuity Criterion (5.20) it follows from the Chain Rule (5.178) and (5.179) that, for all points $\mathbf{x}$ in $\mathcal{O}$,

$$\nabla (G_j \circ \mathbf{F})(\mathbf{x}) = \nabla G_j(\mathbf{F}(\mathbf{x}))\mathbf{DF}(\mathbf{x}).$$

This formula is an assertion of the equality of the $j$th rows of each of the matrices in formula (5.187) for $1 \leq j \leq k$. Thus, the matrix formula (5.187) holds. Therefore, the composition $\mathbf{G} \circ \mathbf{F} : \mathcal{O} \to \mathbb{R}^k$ has first-order partial derivatives at each point, and from the continuity of the entries on the right-hand side of (5.187), again from (5.20) we conclude that the composition is continuously differentiable.

### 5.9.3 Directional Derivatives

In (5.78), we defined the directional derivative of the function $f : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}$ in the direction $\mathbf{p}$ at the point $\mathbf{x}$. Further, in (5.79) and (5.85), we expressed $(D_{\mathbf{p}} f)(\mathbf{x})$ as $(\nabla f)(\mathbf{x}) \cdot \mathbf{p}$, this being a matrix product of $1 \times n$ and $n \times 1$ vectors.

This extends in a natural way to the case of $\mathbf{F} : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}^m$ continuously differentiable, with derivative matrix $\mathbf{DF}(\mathbf{x})$ from (5.166). In particular, for $\mathbf{p}, \mathbf{x} \in \mathcal{O}$, $\mathbf{D_p F}(\mathbf{x})$ is the $m \times 1$ vector $\mathbf{DF}(\mathbf{x}) \cdot \mathbf{p}$.

**Example 5.25** *(Dineen, Multivariate Calculus and Geometry, 3rd ed., p. 8) Let $\mathbf{F} : \mathbb{R}^4 \to \mathbb{R}^3$ be defined by*

$$\mathbf{F}(x, y, z, w) = \left(x^2 y, xyz, x^2 + y^2 + zw^2\right).$$

*Then $\mathbf{F} = (f_1, f_2, f_3)$, where $f_1(x, y, z, w) = x^2 y$, $f_2(x, y, z, w) = xyz$ and $f_3(x, y, z, w) = x^2 + y^2 + zw^2$. Moreover, $\nabla f_1(x, y, z, w) = (2xy, x^2, 0, 0)$, $\nabla f_2(x, y, z, w) = (yz, xz, xy, 0)$ and $\nabla f_3(x, y, z, w) = (2x, 2y, w^2, 2zw)$. Hence*

$$\mathbf{DF}(x, y, z, w) = \begin{pmatrix} 2xy & x^2 & 0 & 0 \\ yz & xz & xy & 0 \\ 2x & 2y & w^2 & 2zw \end{pmatrix}.$$

*If $\mathbf{x} = (1, 2, -1, -2)'$ and $\mathbf{p} = (0, 1, 2, -2)'$ then*

$$\mathbf{D_p F}(\mathbf{x}) = \mathbf{DF}(\mathbf{x}) \cdot \mathbf{p} = \begin{pmatrix} 4 & 1 & 0 & 0 \\ -2 & -1 & 2 & 0 \\ 2 & 4 & 4 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}. \quad \blacksquare$$

# 6 Multivariate Integration

> Mathematics is not a deductive science – that's a cliche. When you try to prove a theorem, you don't just list the hypotheses, and then start to reason. What you do is trial and error, experimentation, guesswork. (Paul R. Halmos)

## 6.1 Definitions, Existence, and Properties

This subsection is based on Fitzpatrick, §18.1. The quote from Pugh at the beginning of §5.2 pertains precisely to this material, i.e., "The multivariable case in which $f : \mathbb{R}^n \to \mathbb{R}$ offers no new ideas, only new notation", compared to the univariate case in §2.5.1. Still, it is worth spelling out, and also serves as a refresher of the univariate material.

Recall from §2.5.1 that, if $I = [a, b]$, $a < b$, is a closed bounded interval of real numbers, $m$ is a positive integer, and $P = \{x_0, \ldots, x_m\}$ are $m + 1$ real numbers such that

$$a = x_0 < x_1 < \cdots < x_i < \cdots < x_m = b, \tag{6.1}$$

then $P$ is called a partition of $[a, b]$, and the intervals $[x_{i-1}, x_i]$, for $i$ an index between 1 and $m$, are called intervals in the partition $P$. We define the length of the interval $I = [a, b]$ to be $b - a$. Let $n$ be a positive integer and for each index $i$ between 1 and $n$ let $I_i = [a_i, b_i]$ be a closed bounded interval of real numbers. The Cartesian product of these intervals,

$$\mathbf{I} = I_1 \times \cdots \times I_i \times \cdots \times I_n = \{\mathbf{x} = (x_1, \ldots, x_n) \text{ in } \mathbb{R}^n \mid x_i \text{ in } I_i \text{ for } 1 \leq i \leq n\}, \tag{6.2}$$

is called a **generalized rectangle**. It is convenient to refer to the interval $I_i$ as being the $i$th edge of $\mathbf{I}$. We define the volume of $\mathbf{I}$, denoted by vol $\mathbf{I}$, to be the product of the lengths of the $n$ edges; that is,

$$\operatorname{vol} \mathbf{I} \equiv \prod_{i=1}^{n} [b_i - a_i].$$

In the case where $n = 1$, the volume is simply the length; in the case where $n = 2$, the volume is called the area.

Definition: Given a generalized rectangle $\mathbf{I} = I_1 \times \cdots \times I_i \times \cdots \times I_n$, for each index $i$ between 1 and $n$, let $P_i$ be a partition of the $i$th edge $I_i$. The collection of generalized rectangles of the form

$$\mathbf{J} = J_1 \times \cdots \times J_i \times \cdots \times J_n,$$

where each $J_i$ is an interval in the partition $P_i$, is called a partition of $\mathbf{I}$ and is denoted by

$$\mathbf{P} \equiv (P_1, \ldots, P_n).$$

Consider the rectangle $[a, b] \times [c, d]$ in the plane $\mathbb{R}^2$. Let $P_1 = \{x_0, \ldots, x_m\}$ and $P_2 = \{y_0, \ldots, y_\ell\}$ be partitions of $[a, b]$ and $[c, d]$, respectively, and define $\mathbf{P} = (P_1, P_2)$. Then

$$\begin{aligned}\sum_{\mathbf{J} \text{ in } \mathbf{P}} \operatorname{vol} \mathbf{J} &= \sum_{j=1}^{\ell} \sum_{i=1}^{m} [x_i - x_{i-1}]\,[y_j - y_{j-1}] = \sum_{j=1}^{\ell} \left\{ \sum_{i=1}^{m} [x_i - x_{i-1}] \right\} [y_j - y_{j-1}] \\ &= \sum_{j=1}^{\ell} \{[b - a]\}\,[y_j - y_{j-1}] = [b - a] \sum_{j=1}^{\ell} [y_j - y_{j-1}] = [b - a][d - c] = \operatorname{vol} \mathbf{I}.\end{aligned}$$

An induction argument shows that the above formula also holds in general: For each natural number $n$, if $\mathbf{P}$ is a partition of the generalized rectangle $\mathbf{I}$ in $\mathbb{R}^n$, then

$$\operatorname{vol}\mathbf{I} = \sum_{\mathbf{J} \text{ in } \mathbf{P}} \operatorname{vol}\mathbf{J}. \tag{6.3}$$

Let $f : \mathbf{I} \to \mathbb{R}$ is a bounded function whose domain $\mathbf{I}$ is a generalized rectangle and let $\mathbf{P}$ be a partition of $\mathbf{I}$. For $\mathbf{J}$ a generalized rectangle in the partition $\mathbf{P}$, we define

$$m(f, \mathbf{J}) \equiv \inf\{f(\mathbf{x}) \mid \mathbf{x} \text{ in } \mathbf{J}\} \quad \text{and} \quad M(f, \mathbf{J}) \equiv \sup\{f(\mathbf{x}) \mid \mathbf{x} \text{ in } \mathbf{J}\}.$$

Remark: Note that $\mathbf{I}$ is closed and bounded. If $f$ is continuous, then, for $n = 1$, from (2.55), it is uniformly continuous. From the EVT (5.22), the inf and sup can be replaced with min and max.

We then define the lower and upper Darboux sums for the function $f : \mathbf{I} \to \mathbb{R}$ with respect to the partition $\mathbf{P}$, by

$$L(f, \mathbf{P}) \equiv \sum_{\mathbf{J} \text{ in } \mathbf{P}} m(f, \mathbf{J}) \operatorname{vol}\mathbf{J}, \quad \text{and} \quad U(f, \mathbf{P}) \equiv \sum_{\mathbf{J} \text{ in } \mathbf{P}} M(f, \mathbf{J}) \operatorname{vol}\mathbf{J}.$$

Lemma: Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on a generalized rectangle $\mathbf{I}$. Suppose that the two numbers $m$ and $M$ have the property that $\forall \mathbf{x} \in \mathbf{I}$, $m \le f(\mathbf{x}) \le M$. Then, for any partition $\mathbf{P}$ of $\mathbf{I}$,

$$m \operatorname{vol}\mathbf{I} \le L(f, \mathbf{P}) \le U(f, \mathbf{P}) \le M \operatorname{vol}\mathbf{I}. \tag{6.4}$$

> Proof: Let $\mathbf{P}$ be a partition of $\mathbf{I}$. For a generalized rectangle $\mathbf{J}$ in $\mathbf{I}$, it is clear that
>
> $$m \le \inf\{f(\mathbf{x}) \mid \mathbf{x} \text{ in } \mathbf{J}\} = m(f, \mathbf{J}) \le M(f, \mathbf{J}) = \sup\{f(\mathbf{x}) \mid \mathbf{x} \text{ in } \mathbf{J}\} \le M,$$
>
> so
>
> $$m \operatorname{vol}\mathbf{J} \le m(f, \mathbf{J}) \operatorname{vol}\mathbf{J} \le M(f, \mathbf{J}) \operatorname{vol}\mathbf{J} \le M \operatorname{vol}\mathbf{J}.$$
>
> Summing over all the generalized rectangles $\mathbf{J}$ in the partition $\mathbf{P}$ and using the sum of volumes formula (6.3), we conclude that the inequality (6.4) holds.

Given a partition $\mathbf{P} = (P_1, \ldots, P_n)$ of a generalized rectangle $\mathbf{I}$, another partition $\mathbf{P}^* = (P_1^*, \ldots, P_n^*)$ of $\mathbf{I}$ is said to be a refinement of $\mathbf{P}$ provided that, for each index $i$ between 1 and $n$, $P_i^*$ is a refinement of $P_i$. Recall (2.156). Observe that if $\mathbf{P}^*$ is a refinement of $\mathbf{P}$, then (i) each generalized rectangle $\mathbf{J}$ in $\mathbf{P}^*$ is contained in exactly one generalized rectangle in $\mathbf{P}$, and (ii) given a generalized rectangle $\mathbf{J}$ in $\mathbf{P}$, the collection of generalized rectangles in $\mathbf{P}^*$ contained in $\mathbf{J}$ induces a partition of $\mathbf{J}$ that we denote by $\mathbf{P}^*(\mathbf{J})$. The following distribution formulas for the lower and upper Darboux sums follow from these two properties:

$$L\left(f, \mathbf{P}^*\right) = \sum_{\mathbf{J} \text{ in } \mathbf{P}} L\left(f, \mathbf{P}^*(\mathbf{J})\right) \quad \text{and} \quad U\left(f, \mathbf{P}^*\right) = \sum_{\mathbf{J} \text{ in } \mathbf{P}} U\left(f, \mathbf{P}^*(\mathbf{J})\right). \tag{6.5}$$

Lemma (The Refinement Lemma): Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on a generalized rectangle $\mathbf{I}$. Let $\mathbf{P}$ be a partition of $\mathbf{I}$ and let $\mathbf{P}^*$ be a refinement of $\mathbf{P}$. Then

$$L(f, \mathbf{P}) \le L\left(f, \mathbf{P}^*\right) \le U\left(f, \mathbf{P}^*\right) \le U(f, \mathbf{P}). \tag{6.6}$$

Proof: Let $\mathbf{J}$ be a generalized rectangle in $\mathbf{P}$ and denote by $\mathbf{P}^*(\mathbf{J})$ the partition of $\mathbf{J}$ induced by $\mathbf{P}^*$. From (6.4), with $\mathbf{J}$ playing the role of $\mathbf{I}$, it follows that

$$m(f,\mathbf{J})\text{ vol }\mathbf{J}\le L\left(f,\mathbf{P}^*(\mathbf{J})\right)\le U\left(f,\mathbf{P}^*(\mathbf{J})\right)\le M(f,\mathbf{J})\text{ vol }\mathbf{J}.$$

If we sum these inequalities over all generalized rectangles $\mathbf{J}$ in $\mathbf{P}$ and use the distribution formulas (6.5), we arrive at the inequality (6.6).

For two partitions $P$ and $P'$ of a closed bounded interval of real numbers $I$, by taking the partition consisting of all points that are partition points in at least one of the two partitions, we obtain a partition that is a common refinement of the two given partitions, meaning that it is a refinement of both $P$ and $P'$. Similarly, suppose that $\mathbf{P}$ and $\mathbf{P}'$ are two partitions of a generalized rectangle $\mathbf{I}$ in $\mathbb{R}^n$ represented as $\mathbf{P}=(P_1,\ldots,P_n)$ and $\mathbf{P}'=(P_1',\ldots,P_n')$. For each index $i$ between 1 and $n$, choose $P_i''$ to be a common refinement of $P_i$ and $P_i'$ and define $\mathbf{P}''=(P_1'',\ldots,P_n'')$. Then $\mathbf{P}''$ is a partition of $\mathbf{I}$ that is a common refinement of the partitions $\mathbf{P}$ and $\mathbf{P}'$. The existence of common refinements is what is necessary to establish the following proposition.

Proposition: Let $f:\mathbf{I}\to\mathbb{R}$ be a bounded function on a generalized rectangle $\mathbf{I}$. For any two partitions $\mathbf{P}_1$ and $\mathbf{P}_2$ of $\mathbf{I}$,

$$L\left(f,\mathbf{P}_1\right)\le U\left(f,\mathbf{P}_2\right). \tag{6.7}$$

Proof: Choose $\mathbf{P}$ to be a common refinement of the two partitions $\mathbf{P}_1$ and $\mathbf{P}_2$. By the Refinement Lemma,

$$L\left(f,\mathbf{P}_1\right)\le L(f,\mathbf{P})\le U(f,\mathbf{P})\le U\left(f,\mathbf{P}_2\right).$$

Definition: Let $f:\mathbf{I}\to\mathbb{R}$ be a bounded function on a generalized rectangle $\mathbf{I}$. We define the lower and upper integrals of $f$ on $\mathbf{I}$, by

$$\underline{\int_{\mathbf{I}}} f\equiv\sup\{L(f,\mathbf{P})\mid\mathbf{P}\text{ a partition of the generalized rectangle }\mathbf{I}\} \tag{6.8}$$

and

$$\overline{\int_{\mathbf{I}}} f\equiv\inf\{U(f,\mathbf{P})\mid\mathbf{P}\text{ a partition of the generalized rectangle }\mathbf{I}\}. \tag{6.9}$$

Lemma: Let $f:\mathbf{I}\to\mathbb{R}$ be a bounded function on a generalized rectangle $\mathbf{I}$. Then

$$\underline{\int_{\mathbf{I}}} f\le\overline{\int_{\mathbf{I}}} f.$$

Proof: Let $\mathbf{P}$ be a partition of $\mathbf{I}$. Proposition (6.7) asserts that $U(f,\mathbf{P})$ is an upper bound for the collection of all lower Darboux sums for $f$. Therefore, by the definition of supremum,

$$\underline{\int_{\mathbf{I}}} f\le U(f,\mathbf{P}).$$

But this inequality asserts that $\underline{\int_{\mathbf{I}}} f$ is a lower bound for the collection of upper Darboux sums for $f$. Thus, by the definition of infimum,

$$\underline{\int_{\mathbf{I}}} f\le\overline{\int_{\mathbf{I}}} f.$$

Definition: Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on a generalized rectangle $\mathbf{I}$. Then we say that $f : \mathbf{I} \to \mathbb{R}$ is integrable, or $f$ is integrable on $\mathbf{I}$, provided that

$$\underline{\int_{\mathbf{I}}} f = \overline{\int_{\mathbf{I}}} f. \tag{6.10}$$

When this is so, the integral of the function $f : \mathbf{I} \to \mathbb{R}$, denoted by $\int_{\mathbf{I}} f$, is defined by

$$\int_{\mathbf{I}} f \equiv \underline{\int_{\mathbf{I}}} f = \overline{\int_{\mathbf{I}}} f, \quad \text{and we write} \quad f \in \mathcal{R}[\mathbf{I}]. \tag{6.11}$$

Definition: Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on a generalized rectangle. For each natural number $k$, let $\mathbf{P}_k$ be a partition of $\mathbf{I}$. The sequence of partitions $\{\mathbf{P}_k\}$ is said to be an Archimedean sequence of partitions for the function $f : \mathbf{I} \to \mathbb{R}$ provided that

$$\lim_{k \to \infty} \left[U\left(f, \mathbf{P}_k\right) - L\left(f, \mathbf{P}_k\right)\right] = 0.$$

Theorem (The Archimedes-Riemann Theorem): Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on the generalized rectangle $\mathbf{I}$. Then $f$ is integrable on $\mathbf{I}$ if and only if there is an Archimedean sequence of partitions for $f : \mathbf{I} \to \mathbb{R}$. Moreover, for any such Archimedean sequence of partitions $\{\mathbf{P}_k\}$,

$$\lim_{k \to \infty} L\left(f, \mathbf{P}_k\right) = \int_{\mathbf{I}} f \quad \text{and} \quad \lim_{k \to \infty} U\left(f, \mathbf{P}_k\right) = \int_{\mathbf{I}} f. \tag{6.12}$$

The following theorem links this result with the univariate result in (2.161).

Theorem: Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on the generalized rectangle $\mathbf{I}$. Then the following two assertions are equivalent:

i. There is an Archimedean sequence of partitions for $f : \mathbf{I} \to \mathbb{R}$.

ii. For each $\epsilon > 0$ there is a partition $\mathbf{P}$ of $\mathbf{I}$ such that

$$U(f, \mathbf{P}) - L(f, \mathbf{P}) < \epsilon.$$

> Proof: First we suppose that (i) holds. Let $\{\mathbf{P}_k\}$ be an Archimedean sequence of partitions for $f : \mathbf{I} \to \mathbb{R}$. To verify criterion (ii) we let $\epsilon$ be any positive number. By the definition of convergent sequence we can choose an index $k$ such that $U\left(f, \mathbf{P}_k\right) - L\left(f, \mathbf{P}_k\right) < \epsilon$. Thus, setting $\mathbf{P} = \mathbf{P}_k$, we have $U(f, \mathbf{P}) - L(f, \mathbf{P}) < \epsilon$. Thus, criterion (ii) holds.
>
> Now suppose that criterion (ii) holds. Let $k$ be a natural number. Then, setting $\epsilon = 1/k$, according to (ii) there is a partition $\mathbf{P}$ such that $U(f, \mathbf{P}) - L(f, \mathbf{P}) < 1/k$. Choose such a partition and label it $\mathbf{P}_k$. This defines a sequence of partitions $\{\mathbf{P}_k\}$ of the generalized interval $\mathbf{I}$ that is Archimedean since
>
> $$0 \leq \lim_{k \to \infty} \left[U\left(f, \mathbf{P}_k\right) - L\left(f, \mathbf{P}_k\right)\right] \leq \lim_{k \to \infty} 1/k = 0.$$

Here is the generalization of the *domain additivity*, or *additivity over partitions* from (2.171), which we state without proof: See, e.g., Fitzpatrick, p. 479.

Theorem (Additivity over Partitions): Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on the generalized rectangle $\mathbf{I}$. Let $\mathbf{P}$ be a partition of $\mathbf{I}$. Then $f \in \mathcal{R}[\mathbf{I}]$ if and only if for each generalized rectangle $\mathbf{J}$ in $\mathbf{P}$, the restriction of $f$ to $\mathbf{J}$, $f : \mathbf{J} \to \mathbb{R}$, is integrable: In this case,

$$\int_{\mathbf{I}} f = \sum_{\mathbf{J} \text{ in } \mathbf{P}} \int_{\mathbf{J}} f.$$

We also have the following results, which are clear analogs of their univariate counterparts.

Theorem (Monotonicity of the Integral): Suppose that the functions $f : \mathbf{I} \to \mathbb{R}$ and $g : \mathbf{I} \to \mathbb{R}$ are integrable, where $\mathbf{I}$ is a generalized rectangle in $\mathbb{R}^n$, and also suppose that $\forall \mathbf{x} \in \mathbf{I}$, $f(\mathbf{x}) \le g(\mathbf{x})$. Then $\int_{\mathbf{I}} f \le \int_{\mathbf{I}} g$.

Theorem (Linearity of the Integral): Suppose that the functions $f : \mathbf{I} \to \mathbb{R}$ and $g : \mathbf{I} \to \mathbb{R}$ are integrable, where $\mathbf{I}$ is a generalized rectangle in $\mathbb{R}^n$. Then for any two numbers $\alpha$ and $\beta$, the function $\alpha f + \beta g : \mathbf{I} \to \mathbb{R}$ also is integrable and

$$\int_{\mathbf{I}} [\alpha f + \beta g] = \alpha \int_{\mathbf{I}} f + \beta \int_{\mathbf{I}} g.$$

As hoped and expected, we have the generalization of (2.163), which we did prove.

Theorem: Let $f : \mathbf{I} \to \mathbb{R}$ be a continuous function on a generalized rectangle $\mathbf{I}$. Then $f$ is integrable on $\mathbf{I}$. That is,

$$f \in \mathcal{C}^0 \implies f \in \mathcal{R}[\mathbf{I}]. \tag{6.13}$$

A proof in the general $n$ case can be found in, e.g., Fitzpatrick, p. 484; and, somewhat more advanced, Terrell, A Passage to Modern Analysis, 2019, Prop. 12.5.2, p. 374.

For vector-valued functions $f : B \subset \mathbb{R}^n \to \mathbb{R}^m$, there is a natural extension of Riemann integration: It is just the elementwise integration of each component function. From Terrell, p. 364, we have:

Definition: Let $\mathbf{F} : B \to \mathbf{R}^m$ be a function bounded on the closed interval $B$ in $\mathbf{R}^n$, and let us write $\mathbf{F} = (f_1, \dots, f_m)$ where the $f_j$ are the real valued component functions. We say that $\mathbf{F}$ is Riemann integrable on $B$ if and only if each component function $f_j : B \to \mathbf{R}, 1 \le j \le m$, is Riemann integrable on $B$. Then the vector

$$\int_B \mathbf{F} = \left( \int_B f_1, \dots, \int_B f_m \right)$$

is called the Riemann integral of $\mathbf{F}$ on $B$.

We close this subsection by mentioning a characterization of Riemann integrability that had to wait until the 20th century to be discovered (by Lebesgue). Some first-course analysis books cover the (long, without the use of measure theory) proof in the $n = 1$ case, such as the enjoyable presentation in Stoll (2021). We state that result in (2.164). The general $n$ case is proved in Terrell, 2019, §12.5. The theorem refers to Lebesgue measure zero, the definition of which is not complicated and does not require a deep dive into measure theory. We discuss this in the subsequent subsection §6.2.

Theorem: Let $B$ be a closed interval in $\mathbf{R}^n$. A bounded function $f : B \to \mathbf{R}$ is Riemann integrable on $B$ if and only if the set of points where $f$ is discontinuous has Lebesgue measure zero.

## 6.2 Bounded Sets, Jordan Measure, Volume Zero, and Lebesgue Measure Zero

A proper, rigorous development of the multivariate Riemann integral requires the concepts of Jordan measure (or Jordan content), volume, and volume zero. In this subsection, we only define these quantities and draw a contrast to the concept of Lebesgue measure zero.

### 6.2.1 Introduction and Useful Results

The Lebesgue integral has desirable features not possessed by the Riemann integral, and is the main reason that some analysis books cover the univariate Riemann integral, but then, for the multivariate case, skip the development of the Riemann integral, and go straight to the Lebesgue integral. The latter is an early 20th century development, and considered one of the most important advances in analysis. To understand it requires first learning what is called measure theory, this being a topic for a subsequent course. Below, we indicate two advantages of the notion of Lebesgue measure zero, as compared to that of Jordan volume zero, thus scratching the surface of why the Lebesgue integral is considered superior to the Riemann integral.

As a slight counterbalance, the formulation of the Lebesgue integral does not give rise to a method for numerically computing it, whereas the construction of the Riemann integral does. All algorithms for numeric integration (e.g., Simpson's rule, though there are far more sophisticated methods, and these are conveniently built in to numeric software, such as Matlab) are based on the Riemann formulation. The key result is that, if a function is Riemann integrable, then it is Lebesgue integrable, but not vice-versa. The Lebesgue integral formulation is of great use for theoretical reasons, but for computation, it is required that the function is also Riemann integrable.

We collect some useful definitions that we will require below.

Definition: An **open interval** in $\mathbb{R}^n$, $n \geq 2$, is a Cartesian product of $n$ real intervals,

$$B = (a_1, b_1) \times \cdots \times (a_n, b_n), \tag{6.14}$$

where $a_i < b_i$ for $1 \leq i \leq n$. A **closed interval** in $\mathbb{R}^n$, $n \geq 2$, has the form

$$B = [a_1, b_1] \times \cdots \times [a_n, b_n], \tag{6.15}$$

where $a_i \leq b_i$ for $1 \leq i \leq n$. (Note that if $a_i < b_i$, $1 \leq i \leq n$, then the interior of a closed interval is the open interval having the same endpoints for each interval factor.) The volume of either of these types of intervals, described by the Cartesian product of real intervals, is defined to be $\nu(B) = \prod_{i=1}^{n} (b_i - a_i)$. In particular, note how the volumes of (6.14) and (6.15) are defined to be equal.

Definition: The volume of a union of finitely many intervals, any two of which intersect (if at all) only along boundary segments, is defined to be the sum (finite) of the volumes of the intervals.

This definition leads to (and is a special case of) result (6.20) below. Some authors refer to two intervals that intersect only along boundary segments as "nonoverlapping" (in the sense that, while they have points in common, in $\mathbb{R}^n$, this is a set of measure zero; recall (1.5) and also see below. Indeed, so does Terrell, on page 519, in his chapter on measure theory:

Two intervals in $\mathbb{R}^n$ (whether open, closed, or otherwise) are nonoverlapping if their interiors are disjoint, that is, they intersect only in some boundary points, if at all. Thus the intersection of the two intervals equals the intersection of their boundaries. Similarly, the intervals in an arbitrary collection of intervals are called nonoverlapping if any two of them are nonoverlapping.

The following presentation comes from Terrell, §12.2, §12.3, §12.4, and §12.5.

### 6.2.2 Bounded Sets, Jordan Measure, Volume Zero

The first task is to generalize the Riemann integral (6.11) to integration over other bounded sets. Let $S \subset \mathbb{R}^n$ be a bounded set, and $f : S \to \mathbb{R}$ a bounded function. We may extend $f$ to all of $\mathbb{R}^n$ by defining

$$f_S(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \text{if } \mathbf{x} \in S, \\ 0, & \text{if } \mathbf{x} \notin S. \end{cases} \tag{6.16}$$

This is called the extension of $f$ by zero. Let $B$ be a closed interval in $\mathbb{R}^n$ that contains the bounded set $S$. We want to say that $f$ is integrable on $S$ if $f_S$ is integrable on $B$, that is, if the integral $\int_B f_S$ exists. However, we have to show that the existence of the integral, and its value, is independent of the enclosing interval $B$.

Lemma: Let $S$ be a bounded subset of $\mathbb{R}^n$ and $f : S \to \mathbb{R}$ a bounded function such that $\int_B f_S$ exists for some closed interval $B$ containing $S$. Then

$$\int_B f_S = \int_{B'} f_S \tag{6.17}$$

for any other closed interval $B'$ in $\mathbb{R}^n$ containing $S$.

We omit the proof, which can be found in Terrell, p. 366. The Lemma justifies the following definition.

Definition: If $S \subset \mathbb{R}^n$ is a bounded set and $f : S \to \mathbb{R}$ is a bounded function for which $\int_B f_S$ exists for some closed interval $B$ containing $S$, then $f$ is integrable on $S$ and

$$\int_S f = \int_B f_S \tag{6.18}$$

is the integral of $f$ on $S$. Thus the existence of $\int_S f$, and its value, are independent of the enclosing interval $B$.

Definition: If $A \subset \mathbb{R}^n$ is a bounded set, the **characteristic function** of $A$ is the mapping $\chi_A : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\chi_A(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in A, \\ 0, & \text{if } \mathbf{x} \notin A. \end{cases} \tag{6.19}$$

Definition: The set $A$ is **Jordan measurable**, or $A$ **has volume**, if $\chi_A$ is integrable on $A$, that is, $\int_A \chi_A$ exists.

Definition: The **volume** of $A$, denoted $\nu(A)$, is defined by

$$\nu(A) = \int_A \chi_A.$$

Definition: For open interval $S$ and its closure $\bar{S}$,

$$S = (a_1, b_1) \times \cdots \times (a_n, b_n), \qquad \bar{S} = [a_1, b_1] \times \cdots \times [a_n, b_n],$$

their volumes equal $\prod_{i=1}^{n} (b_i - a_i)$ by axiom. We define, for consistency,

$$\int_S \chi_S = \int_{\bar{S}} \chi_{\bar{S}} = \prod_{i=1}^{n} (b_i - a_i).$$

As important special cases: For a subset $A$ of the two-dimensional plane, the volume is the area of the region, and this area is numerically equal to the (three-dimensional) volume of the solid lying between the graph of $\chi_A$ and the region $A$ in the plane. For an interval $A = [a, b]$ of real numbers, the volume is the length of the interval, and this length is numerically the same as the area of the region between the graph of $\chi_{[a,b]}$ and the interval $A = [a, b]$ on the real line.

Definition: The volume of $A$, when it exists, is also called the **Jordan measure**, or **Jordan content**, of $A$.

Definition: A set $A$ with volume such that $\nu(A) = 0$ is said to have **volume zero**. The concept of volume zero is also called **Jordan measure zero** or **Jordan content zero**.

The following results on Jordan measure (from Terrell, p. 379) parallel fundamental results for Lebesgue measure (which we do not state here):

Theorem: Let $S_1$ and $S_2$ be subsets of $\mathbb{R}^n$ that have volume. Then:

1. $S_1 \cup S_2$ and $S_1 \cap S_2$ have volume, and

$$\nu(S_1 \cup S_2) = \nu(S_1) + \nu(S_2) - \nu(S_1 \cap S_2).$$

2. If $\operatorname{Int} S_1 \cap \operatorname{Int} S_2$ is the empty set, then

$$\nu(S_1 \cup S_2) = \nu(S_1) + \nu(S_2). \tag{6.20}$$

3. If $S_1 \subseteq S_2$, then $S_2 - S_1 = S_2 \cap S_1^c$ has volume and

$$\nu(S_2 \cap S_1^c) = \nu(S_2) - \nu(S_1).$$

4. If $S_1 \subseteq S_2$, then

$$\nu(S_1) \leq \nu(S_2).$$

Proposition: From the definition of integrability of $\chi_A$, a set $A$ has volume zero if and only if, for every $\epsilon > 0$, there is a finite collection of closed intervals $C_1, \ldots, C_N$ such that

$$A \subseteq \bigcup_{i=1}^{N} C_i \quad \text{and} \quad \sum_{i=1}^{N} \nu(C_i) < \epsilon. \tag{6.21}$$

Author Terrell assigns this as an exercise (12.3.2), with the hint (and note he refers to boxes in $\mathbb{R}^n$ as intervals): For each implication, think of the intervals $C_i$ as intervals of a partition $P$, involved in defining an upper sum $U(\chi_A, P)$ for that partition.

> ($\Rightarrow$) We have $\nu(A) = \int_A \chi_A = 0$, i.e., by the definition of volume, $\chi_A$ is integrable. From (6.11), this means that $\chi_A$ is bounded, which is trivially true; but also that $A$ is either a generalized rectangle, which from (6.1) and (6.2) means it is bounded; or, from definition (6.16), a more general bounded set. From (6.18), we can take $A$ to be its closure, $\bar{A}$, which is necessarily closed and bounded. Note it is the smallest closed set containing $A$. Being bounded, there exists a generalized rectangle $\mathbf{I}$ that covers $A$, and from (6.18), $\int_A \chi_A = \int_{\mathbf{I}} \chi'_A$, where $\chi'_A$ is the extension of function $\chi_A$ defined in (6.16).
>
> Let $\mathbf{P}_1 = \mathbf{I}$; and let $\mathbf{P}_k$ be a sequence of partitions of $A$ with $\mathbf{P}_k \subset \mathbf{P}_{k+1}$, i.e., the latter is a refinement of the former. From (6.12), $\lim_{k\to\infty} U(\chi_A, \mathbf{P}_k) = \int_A \chi_A = 0$, and we know the rhs integral exists by assumption, while for any $k \in \mathbb{N}$, the lhs exists. From the definition of limit, for each $j \in \mathbb{N}$, $\exists K_j \in \mathbb{N}$ such that, for $k \geq K_j$, $U(\chi_A, \mathbf{P}_k) < 1/j$. For $j > 1/\epsilon$, the finite partition $\mathbf{P}_m$, $m = K_j$, satisfies $U(\chi_A, \mathbf{P}_m) < \epsilon$. The value $N$ in (6.21) is the number of generalized rectangles in $\mathbf{P}_m$.
>
> ($\Leftarrow$) We are given that, $\forall \epsilon > 0$, $\exists \{C_i\}_{i=1}^N$, $C_i$ closed, such that (6.21) holds. We require use of (6.18) to handle $A \subseteq \cup_{i=1}^N C_i$; and, if we assume the $C_i$ are nonoverlapping, (6.20) to justify the equality $\nu\left(\cup_{i=1}^N C_i\right) = \sum_{i=1}^N \nu(C_i)$, so $\nu\left(\cup_{i=1}^N C_i\right) < \epsilon$. We can take the $\{C_i\}$ to be nonoverlapping, and let $\mathbf{P}_k = \{C_i\}$. Then, by the definition of $\chi_A$ in (6.19) (namely that it is constant on $A$ with value 1, and zero otherwise), we have $0 \leq L(\chi_A, \mathbf{P}_k) \leq U(\chi_A, \mathbf{P}_k) < \epsilon$. Thus, from (6.12) and the Squeeze Theorem (2.9), $\int_A \chi_A$ exists and equals zero, i.e., $A$ has volume zero.

Having now secured the solutions manual to Terrell's book, we can compare my above attempt to his proof.

> Proof (Terrell): Suppose $A$ has volume zero, that is, $\nu(A) = \int_A \chi_A = 0$, and $B$ is a closed interval containing $A$. Then for every $\epsilon > 0$, there is a partition $P$ of $B$ such that $U(\chi_A, P) < \epsilon$. Let the listing $S_1, \ldots, S_N$ be an enumeration of the intervals of the partition $P$. Then $A \subseteq \bigcup_{i=1}^N S_i$ and $\sum_{i=1}^N \nu(S_i) = U(\chi_A, P) < \epsilon$.
>
> Suppose that for every $\epsilon > 0$ there is a finite collection of closed intervals $S_1, \ldots, S_N$ such that $A \subseteq \bigcup_{i=1}^N S_i$ and $\sum_{i=1}^N \nu(S_i) < \epsilon$. Let $B$ be any closed interval that contains the union of the $S_i$. The intervals $S_i$ may intersect or not, but in any case, there is a partition $P$ of $B$ that includes all the lattice points defined by the endpoints of the $S_i, 1 \leq i \leq N$. Then we have $U(\chi_A, P) \leq \sum_{i=1}^N \nu(S_i) < \epsilon$. Since $L(\chi_A, P) = 0$, this shows, by the Riemann criterion, that $\chi_A$ is integrable and $\nu(A) = \int_A \chi_A = 0$.

One of the weaknesses of the volume concept is that it does not apply to unbounded sets. Another weakness is that a countable union of sets having volume is not necessarily a set having volume, even in some cases where we think it probably should be. As an example, on the real line, the open set $\bigcup_{k=1}^\infty \left(k - 1/2^k, k + 1/2^k\right)$ has what we call finite total length, given by

$$\sum_{k=1}^\infty \frac{2}{2^k} = \sum_{k=1}^\infty \frac{1}{2^{k-1}} = 2,$$

but it does not have volume since it is an unbounded set.

Now consider the next example: Any single point, that is, a singleton set $\{\mathbf{x}\}$, has volume zero, since it can be covered by a single closed interval of arbitrarily small volume. On the real line, consider the rational numbers in $[0,1]$, that is, $S = \mathbf{Q} \cap [0,1]$. Then $S$ is bounded, and it is the union of countably many (singleton) sets of volume zero, but $S$ does not have volume, much less volume zero, since $\chi_S$ is not integrable. There are similar examples in the plane and in higher dimensions. For example, the rational points (the points with rational coordinates) in the unit square in the plane, $\mathbf{Q} \times \mathbf{Q} \cap [0,1] \times [0,1]$, is a countable union of sets, each with volume zero, but it does not have volume, since its characteristic function is not integrable.

We have seen that there are open sets that do not have volume. Since open sets play a fundamental role in analysis, this must be seen as a weakness in the theory of Jordan measure we are discussing, and the weakness is tied to the Riemann integral concept through the above definition of volume, which relies on the existence of the Riemann integral. The central issue that prevents some bounded sets $S$ from having volume is that the boundary $\partial S$ may be too complicated to allow integrability of the characteristic function $\chi_S$. This issue about the boundary $\partial S$ is discussed subsequently.

### 6.2.3 Lebesgue Measure Zero

We defined Lebesgue measure zero in the univariate case in (1.5). Its extension to $\mathbb{R}^n$ is very natural.

Definition: Let $S \subset \mathbb{R}^n$, bounded or unbounded. We say that $S$ has $n$-dimensional Lebesgue measure zero (or simply measure zero) if, for every $\epsilon > 0$, there is a sequence of open intervals, $J_i$, in $\mathbb{R}^n$ such that $S \subseteq \bigcup_i J_i$ and

$$\sum_i \nu(J_i) < \epsilon.$$

The concepts of measure zero and volume zero depend on the dimension, and one can write $m_n(S) = 0$ and $\nu_n(S) = 0$ to indicate $n$-dimensional Lebesgue measure zero and $n$-dimensional volume zero, respectively, if needed.

**Example 6.1** *Let $S$ be the set of rational numbers in the unit interval, $S = \mathbf{Q} \cap [0,1]$. Then $S$ has Lebesgue measure zero. We enumerate these rationals by the listing $\{r_1, r_2, r_3, \ldots\}$, and then cover the numbers individually by open intervals whose lengths sum to less than a given $\epsilon > 0$. For example, cover $r_1$ by an open interval of length $\epsilon/2$, $r_2$ by an open interval of length $\epsilon/2^2$; in general, cover $r_k$ by an open interval of length $\epsilon/2^k$. Then the countable collection of these open intervals covers $S$ and has total length less than $\sum_{k=1}^{\infty} \epsilon/2^k = \epsilon$. Therefore $S$ has Lebesgue measure zero.* ■

**Example 6.2** *The set $S = \{(x,0) : 0 \le x \le 1\}$ has 2-dimensional Lebesgue measure zero. To verify this, observe that $S$ can be covered by the single closed interval $[0,1] \times [0,\delta]$ for any $\delta > 0$. Since this interval has volume $\delta$, we conclude that $S$ has volume zero, and hence $S$ has measure zero. Alternatively, given $0 < \epsilon < 1$, $S$ can be covered by a single open interval, for example,*

$$R = \left\{(x,y) : -\frac{\epsilon}{4} < x < 1 + \frac{\epsilon}{4}, -\frac{\epsilon}{4} < y < \frac{\epsilon}{4}\right\}$$

*which has volume $\nu(R) = (1+\epsilon/2)(\epsilon/2) < \epsilon$. Therefore $S$ has measure zero.* ■

Theorem: If $S$ has $n$-dimensional volume zero, then it has $n$-dimensional Lebesgue measure zero.

> Proof: If $S$ has volume zero, then, from (6.21), for any $\epsilon > 0$, $S$ can be covered by a finite collection of closed intervals $I_i, 1 \le i \le N$, such that $\sum_{i=1}^{N} \nu(I_i) < \epsilon/2$. For each $i$, we can cover $I_i$ with an open interval $J_i$ of volume $\nu(I_i) + \epsilon/2^{i+1}$, and $\sum_{i=1}^{N} \nu(J_i) = \sum_{i=1}^{N} \nu(I_i) + \sum_{i=1}^{N} \epsilon/2^{i+1} < \epsilon/2 + \epsilon/2 = \epsilon$. Since countable means finite or countably infinite, $S$ has Lebesgue measure zero.

On the other hand, there are sets having Lebesgue measure zero that do not have volume, as we see in the next example.

**Example 6.3** *Let $S$ be the set of points in the unit square $[0,1] \times [0,1]$ having rational coordinates, that is,*

$$S = \mathbf{Q} \times \mathbf{Q} \cap [0,1] \times [0,1].$$

*Then $S$ has Lebesgue measure zero, since $S$ is countable. However, $S$ does not have volume, because the characteristic function of $S$ is not integrable.* ■

We have seen that volume zero implies Lebesgue measure zero; however, the converse does not generally hold. An exception is described in the next proposition.

Proposition: A compact set in $\mathbb{R}^n$ that has Lebesgue measure zero also has volume zero.

> Proof: Suppose $A \subset \mathbb{R}^n$ is compact (that is, closed and bounded) and has Lebesgue measure zero. Let $\epsilon > 0$. Since $A$ has Lebesgue measure zero, there is a sequence of open intervals, $J_i$, in $\mathbb{R}^n$ such that $A \subseteq \bigcup_i J_i$ and $\sum_i \nu(J_i) < \epsilon$. Since $A$ is compact, there is a finite subcover $\{J_{i_1}, J_{i_2}, \ldots, J_{i_M}\}$ of $A$. By taking the closure of each of these $M$ open intervals, we have the collection $\left\{\bar{J}_{i_1}, \bar{J}_{i_2}, \ldots, \bar{J}_{i_M}\right\}$ of closed intervals, which covers $A$, and
>
> $$\sum_{j=1}^{M} \nu\left(\bar{J}_{i_j}\right) \le \sum_{i} \nu(J_i) < \epsilon.$$
>
> This argument holds for every $\epsilon > 0$, and therefore, again from (6.21), $A$ has volume zero.

Observe that, if $J_1 \times \cdots \times J_n$ is an interval in $\mathbb{R}^n$, then its boundary is given by

$$\bigcup_{k=1}^{n} J_1 \times \cdots \times J_{k-1} \times (\partial J_k) \times J_{k+1} \times \cdots \times J_n.$$

It is not difficult to see that the boundary of an interval in $\mathbb{R}^n$ has volume zero, and thus the boundary has $n$-dimensional Lebesgue measure zero. On the other hand, it is not difficult to directly see that the boundary of an interval in $\mathbb{R}^n$ has Lebesgue measure zero, and that the boundary is compact (since it is closed and bounded); hence, the boundary of an interval has volume zero by the proposition.

It follows directly from the definition that every subset of a set of measure zero has measure zero. In particular, since the empty set is a subset of every set, it is covered by a single interval of arbitrarily small volume, and hence has Lebesgue measure zero. By a similar argument, any singleton set $\{\mathbf{x}\}$ has measure zero, and thus every finite set in $\mathbb{R}^n$ has Lebesgue measure zero.

**Example 6.4** *Let us show that the graph of a continuous function $f : [a, b] \to \mathbf{R}$ has 2-dimensional Lebesgue measure zero. It suffices to show that the graph has 2-dimensional volume zero. Let $G = \{(x, f(x)) : x \in [a, b]\}$ be the graph.*

*From (2.55) or (2.61), $f$ is uniformly continuous on $[a, b]$. This means that, for every $\epsilon > 0$, there is a $\delta > 0$ such that $|x_1 - x_2| < \delta$ implies $|f(x_1) - f(x_2)| < \epsilon/(b-a)$. Thus, for every $\epsilon > 0$, we can find a finite cover of $G$ by closed rectangles having height $\epsilon/(b-a)$ and nonoverlapping interiors. Thus, $G$ is covered by finitely many closed intervals in $\mathbf{R}^2$ whose total volume is less than or equal to $\epsilon$, so $\nu(G) = 0$.* ■

Remark: Despite the result of this last example, a continuous image of a set with $n$-dimensional volume zero need not have $n$-dimensional volume zero. This fact is demonstrated by the existence of so-called space-filling curves.

**Example 6.5** *It is not true that the boundary of every set $E \subseteq \mathbb{R}$ has measure zero. For example, the set of rationals $\mathbb{Q}$ has measure zero, but its boundary is $\partial\mathbb{Q} = \mathbb{R}$, whose measure is infinite.* ■

An advantage of the concept of measure zero over that of volume zero is that the union of a countable infinity of sets, each having measure zero, is also a set of measure zero:

Theorem: A countably infinite union of sets of $n$-dimensional Lebesgue measure zero is a set of $n$-dimensional Lebesgue measure zero.

The proof of this result invokes a result involving a monotone increasing sequence and a subsequence thereof, namely:

> If $(b_k)$ is an increasing sequence and if some subsequence $(b_{n_k})$ of $(b_k)$ converges and $\lim_{k\to\infty} b_{n_k} = b$, then $(b_k)$ itself converges to the same limit, $\lim_{k\to\infty} b_k = b$. See, e.g., Terrell, p. 42 for proof.

Proof: Let $\{E_k\}$ be a countable collection of subsets $E_k \subset \mathbb{R}^n$, each having Lebesgue measure zero. Given $\epsilon > 0$, there exists a doubly indexed collection of open intervals $\left\{J_j^k\right\}$ in $\mathbb{R}^n$ such that, for each $k$,

$$\bigcup_j J_j^k \supset E_k \quad \text{and} \quad \sum_j \nu\left(J_j^k\right) < \frac{\epsilon}{2^{k+1}}.$$

Thus, $\bigcup_{k,j} J_j^k$ covers $\bigcup_k E_k$. The problem now is to arrange this doubly indexed collection of intervals into a sequence and then sum the volumes according to that definite sequence.

We arrange the volumes $\nu\left(J_j^k\right)$ of these intervals in a matrix with $\nu\left(J_1^1\right)$ as the upper left entry and, using $k$ as the row index, $j$ as the column index, we can list the volumes, by tracing the diagonals of slope one in our matrix, starting from the upper left entry, in the order

$$\nu\left(J_1^1\right), \quad \nu\left(J_1^2\right), \nu\left(J_2^1\right), \quad \nu\left(J_1^3\right), \nu\left(J_2^2\right), \nu\left(J_3^1\right), \quad \nu\left(J_1^4\right), \nu\left(J_2^3\right), \nu\left(J_3^2\right), \nu\left(J_4^1\right), \tag{6.22}$$

and so on. Let $\sigma_n$ denote the $n$th partial sum based on this ordering of the volumes; thus, $\sigma_1 = \nu\left(J_1^1\right)$, $\sigma_2 = \nu\left(J_1^1\right) + \nu\left(J_1^2\right)$, $\sigma_3 = \nu\left(J_1^1\right) + \nu\left(J_1^2\right) + \nu\left(J_2^1\right)$, etc.. With the diagonals of slope one in our matrix in view, we can form the triangular partial sums

$$s_n = \sum_{k+j \le n} \nu\left(J_j^k\right)$$

and notice that we have

$$s_1 = \sigma_1, \quad s_2 = \sigma_3, \quad s_3 = \sigma_6, \quad \ldots, \quad s_n = \sigma_{(n(n+1))/2}, \quad \ldots.$$

Thus $(\sigma_n)$ is an increasing sequence with subsequence $(s_n)$. By (6.22), we may write

$$s_n = \sigma_{(n(n+1))/2} = \sum_{k=1}^{n} \left(\nu\left(J_1^k\right) + \nu\left(J_2^{k-1}\right) + \cdots + \nu\left(J_k^1\right)\right).$$

Since $\nu\left(J_j^k\right) \geq 0$ for all $k, j \in \mathbf{N}$, we have

$$\begin{aligned} s_n = \sigma_{(n(n+1))/2} &= \sum_{k=1}^{n} \left(\nu\left(J_1^k\right) + \nu\left(J_2^{k-1}\right) + \cdots + \nu\left(J_k^1\right)\right) \leq \sum_{k=1}^{n}\left(\sum_{j=1}^{n} \nu\left(J_j^k\right)\right) \\ &\leq \sum_{k=1}^{n}\left(\sum_{j=1}^{\infty} \nu\left(J_j^k\right)\right) < \sum_{k=1}^{n} \frac{\epsilon}{2^{k+1}} < \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Therefore the sequence $(s_n) = \left(\sigma_{(n(n+1))/2}\right)$ is increasing and bounded above by $\epsilon$, and it converges to a limit $s < \epsilon$. Since it is a subsequence of the increasing sequence $(\sigma_n)$, we conclude from the above theorem that $(\sigma_n)$ itself converges to the same limit, hence $\lim_{n\to\infty} \sigma_n = s < \epsilon$. We conclude that the sum of the volumes of the intervals $J_j^k$, listed in (6.22), is less than $\epsilon$. Since $\epsilon$ is arbitrary, this shows that $\bigcup_k E_k$ has Lebesgue measure zero.

**Example 6.6** *The real numbers of the form $a + b\sqrt{2}, a, b \in \mathbb{Q}$, are all irrational. The set of such numbers is a countable union of countable sets, and hence has measure zero.* ■

We noted that the real interval [0,1], considered as a subset of the plane, has 2-dimensional Lebesgue measure zero. Let us show that the entire real line, considered as a subset of the plane, has measure zero.

**Example 6.7** *The entire real line, considered as a subset of the plane, has measure zero. The proof depends on showing that increasingly larger chunks of the embedded line can be covered by smaller and smaller 2-dimensional interval volumes. Given $0 < \epsilon < 1$, we must find open intervals $J_j$ in $\mathbf{R}^2$ such that $\sum_j \nu(J_j) < \epsilon$. For example, we may choose*

$$J_j = \left(-\frac{j}{2}, \frac{j}{2}\right) \times \left(-\frac{\epsilon}{2j\left(2^j\right)}, \frac{\epsilon}{2j\left(2^j\right)}\right).$$

*The 2-dimensional volume of $J_j$ is $\nu(J_j) = \epsilon/2^j$, and thus $\sum_j \nu(J_j) < \epsilon$. Moreover, the entire real line, considered as a subset of the plane (i.e., the set $\{(x, 0) : x \in \mathbf{R}\}$), is contained in the union of the $J_j$. Thus, the embedded real line has measure zero in $\mathbb{R}^2$.* ■

We state some further fundamental results, without proof, from Terrell, §12.5. The first generalizes the $n = 1$ case stated in (2.164) (where other references for the proof can be found):

Proposition: Let $B$ be a closed interval in $\mathbb{R}^n$. If $f : B \to \mathbb{R}$ is a bounded function and the set $D$ of discontinuities of $f$ has Lebesgue measure zero, then $f \in \mathcal{R}[B]$.

Proposition: Let $B$ be a closed interval in $\mathbb{R}^n$. If $f : B \to \mathbb{R}$ is Riemann integrable on $B$, then the set $D$ of discontinuities of $f$ has Lebesgue measure zero.

These two previous propositions imply:

Theorem: Let $B$ be a closed interval in $\mathbb{R}^n$, and $f : B \to \mathbb{R}$ bounded. $f \in \mathcal{R}[B]$ if and only if the set of points where $f$ is discontinuous has Lebesgue measure zero.

Corollary: Let $S$ be a bounded set in $\mathbb{R}^n$ and $B$ any closed interval containing $S$. A bounded function $f : S \to \mathbb{R}$ is integrable on $S$ if and only if the set of discontinuities of $f_S$, the extension of $f$ by zero to $B$, has Lebesgue measure zero;

Corollary: A bounded set $S$ in $\mathbb{R}^n$ has volume if and only if $\partial S$ has Lebesgue measure zero.

Definition: If a property or statement involving points of $B$ holds for all points except on a subset of $B$ of Lebesgue measure zero, we say that the property holds almost everywhere (a.e.) in $B$.

Corollary: Let $S$ be a bounded set that has volume. A bounded function $f : S \to \mathbb{R}$ is integrable on $S$ if and only if $f$ is continuous a.e. in the interior of $S$.

**Example 6.8** *(Heil, Measure Theory for Scientists and Engineers, p. 97)*
*Define $f : [0, \infty) \to [0, \infty]$ by*

$$f(t) = \begin{cases} 1/t, & \text{if } t > 0, \\ \infty, & \text{if } t = 0. \end{cases}$$

*This function takes finite values at all but a single point. Hence the set*

$$Z = \{f = \pm\infty\} = \{t \in [0, \infty) : f(t) = \pm\infty\}$$

*where $f$ is not finite has measure zero, so we say that*

$$f(t) \text{ is finite for almost every } t \in [0, \infty),$$

*or simply that $f$ is finite a.e..*

*Every bounded function is certainly finite a.e., but the function $f$ is an example of function that is finite almost everywhere but not bounded.*

*If $f : E \to [-\infty, \infty]$ is a generic extended real-valued function, then $f$ is bounded implies $f$ is finite a.e.; but $f$ is finite a.e. does not imply $f$ is bounded.* ■

**Example 6.9** *(Heil, Measure Theory for Scientists and Engineers, p. 166, Problem 3.2.19)*
*The* Heaviside function *is defined as $H = \chi_{[0,\infty)}$. Being piecewise constant, $H$ is clearly continuous at every point $t \neq 0$. We wish to prove that there is no continuous function $g$ such that $H = g$ a.e..*

*Case 1. Suppose that $g(0) = H(0) = 1$. Since $g$ is continuous at 0, there exists a $\delta > 0$ such that $|g(x) - g(0)| < 1/2$ for all $|x| < \delta$. Since $g(0) = 1$, it follows that $1/2 < g(x) < 3/2$ for all $|x| < \delta$. Since $H(x) = 0$ for all $x < 0$, it follows that $g(x) \neq H(x)$ for all $x \in (-\delta, 0)$. This is a set of positive measure, so $g$ does not equal $H$ almost everywhere.*

*Case 2. Suppose that $g(0) \neq H(0) = 1$. Let $\varepsilon = |1 - g(0)|/2$. Then since $g$ is continuous at 0, there is a $\delta > 0$ such that $|g(x) - g(0)| < \varepsilon$ for all $|x| < \delta$. In particular, for $0 < x < \delta$ we have ($H(x) = 1$ and)*

$$2\varepsilon = |g(0) - 1| \leq |g(0) - g(x)| + |g(x) - H(x)| < \varepsilon + |g(x) - H(x)|.$$

*Therefore $|g(x) - H(x)| > \varepsilon$ for all $x \in (0, \delta)$. This is a set of positive measure, so $g$ does not equal $H$ almost everywhere.* ■

## 6.3 Exchange of Derivative and Integral

Throughout this subsection, let $a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $a_1 < b_1$, $a_2 < b_2$, and let $D := [a_1, b_1] \times [a_2, b_2]$ be a closed rectangle in $\mathbb{R}^2$. To get to the result we want, we first prove some basic results. We will also need these in the next subsection on Fubini's theorem.

Theorem: If $f : D \to \mathbb{R}$ is continuous, then so are

$$\phi_1(x) := \int_{a_1}^{b_1} f(t,x)\,dt \quad \text{and} \quad \phi_2(t) := \int_{a_2}^{b_2} f(t,x)\,dx. \tag{6.23}$$

We prove a more general statement just below. For now, recall the FTC (ii, a) in (2.178):

For $f \in \mathcal{R}[a,b]$, $F(x) = \int_a^x f$, and $x \in I = [a,b]$, we have $F \in \mathcal{C}^0[a,b]$.

Recall from (2.163) that continuity implies integrability, giving an important special case of FTC (ii, a). Observe how result (6.23) generalizes this.

The second statement in (6.23) is a special case of the following result, also required below.

Theorem: If $f : D \to \mathbb{R}$ is continuous,

$$\forall (t,x) \in D, \quad \psi(t,x) := \int_{a_2}^{x} f(t,u)\,du \quad \textbf{is continuous}. \tag{6.24}$$

Proof: Recalling (5.11), we need to show that, $\forall (t,x) \in D$ and any given $\epsilon > 0$,

$$\exists \delta \text{ such that } (t_0, x_0) \in \mathcal{B}_\delta((t,x)) \cap D \implies |\psi(t,x) - \psi(t_0,x_0)| < \epsilon. \tag{6.25}$$

Using domain additivity (2.171), and defining $A$ and $B$ as the indicated two integrals,

$$\begin{aligned}\psi(t,x) - \psi(t_0,x_0) &= \int_{a_2}^{x} f(t,u)\,du - \int_{a_2}^{x_0} f(t_0,u)\,du \\ &= \int_{a_2}^{x_0} [f(t,u) - f(t_0,u)]\,du + \int_{x_0}^{x} f(t,u)\,du \\ &=: A + B.\end{aligned} \tag{6.26}$$

As $D$ is closed and bounded and $f$ is continuous on $D$, from (2.58), $f$ is bounded by some number, say $K$; and, from (2.55), is uniformly continuous on $D$.

Let $\epsilon > 0$. To bound the (absolute value of the) second integral in (6.26), choose $x_0$ such that $|x - x_0| < \epsilon/K$.

For the first integral, as $f$ is uniformly continuous, there exists $\delta_1$ such that, whenever $|t - t_0| < \delta_1$, $|f(t,u) - f(t_0,u)| < \epsilon$.

Let $\delta = \min(\epsilon/K, \delta_1)$. Then for $|x - x_0| < \delta$ and $|t - t_0| < \delta$, (6.26) is such that, from the triangle inequality,

$$\psi(t,x) - \psi(t_0,x_0) \le |\psi(t,x) - \psi(t_0,x_0)| \le |A| + |B| < \epsilon\,|x_0 - a_2| + \epsilon.$$

Observe that this is equivalent to (6.25), thus proving $\psi$ is continuous.

Our goal is to know the conditions under which differentiation and integration can be exchanged. Let $f : D \to \mathbb{R}$ and $D_2 f$ be continuous on $D$, where, from (5.23),

$$D_2 f(t, x) = \lim_{h \to 0} \frac{f(t, x+h) - f(t, x)}{h}.$$

Theorem: Function $g(x) := \int_{a_1}^{b_1} f(t, x)\, dt$ is differentiable, and

$$g'(x) = \int_{a_1}^{b_1} D_2 f(t, x)\, dt. \tag{6.27}$$

Proof: As $D_2 f$ is continuous on $D$, (2.163) implies that $\int_{a_1}^{b_1} D_2 f(t, x)\, dt$ exists, so if (6.27) is true, then $g$ is differentiable. To show (6.27), as in Lang (1997, p. 276), write

$$\frac{g(x+h) - g(x)}{h} - \int_{a_1}^{b_1} D_2 f(t, x)\, dt = \int_{a_1}^{b_1} \left[ \frac{f(t, x+h) - f(t, x)}{h} - D_2 f(t, x) \right] dt.$$

By the MVT (2.94), for each $t$ there exists a number $c_{t,h}$ between $x$ and $x + h$ such that

$$\frac{f(t, x+h) - f(t, x)}{h} = D_2 f(t, c_{t,h}).$$

From (2.61), $D_2 f$ is uniformly continuous on the closed, bounded interval $D$, so

$$\left| \frac{f(t, x+h) - f(t, x)}{h} - D_2 f(t, x) \right| = |D_2 f(t, c_{t,h}) - D_2 f(t, x)| < \frac{\epsilon}{b_1 - a_1},$$

where $\epsilon > 0$, whenever $h$ is sufficiently small.

NOTE: A sufficient condition for result (6.27) to be true when $b_1 = \infty$ is that $f$ and $D_2 f$ are absolutely convergent. That is, for $D := [a_1, \infty) \times [a_2, b_2]$, $a_2 < b_2$, (6.27) holds if there are nonnegative functions $\phi(t)$ and $\psi(t)$ such that $|f(t, x)| \le \phi(t)$ and $|D_2 f(t, x)| \le \psi(t)$ for all $t, x \in D$, and $\int_{a_1}^{\infty} \phi$ and $\int_{a_1}^{\infty} \psi$ converge. See, e.g., Lang (1997, p. 337) for proof.

**Example 6.10** *To calculate the derivative at zero of the function* $f(t) = \int_{-1}^{1} \sin(ts) e^{s+t} ds$, *differentiate under the integral sign, giving*

$$f'(t) = \int_{-1}^{1} \left( s \cos(ts) e^{s+t} + \sin(ts) e^{s+t} \right) ds,$$

*so that*

$$f'(0) = \int_{-1}^{1} \left( s \cos(0s) e^{s+0} + \sin(0s) e^{s+0} \right) ds = \int_{-1}^{1} (s e^s + 0)\, ds = (s-1) e^s \Big|_{-1}^{1} = 2e^{-1}.$$

*This method is quite straightforward (at least for $t = 0$) and obviates the need for a direct calculation of the complicated integral expression of $f$.* ∎

## 6.4 Fubini's Theorem

We concentrate on stating and proving the two dimensional case, from which the full generalization becomes plausible.

Theorem: Let $a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $a_1 < b_1$, $a_2 < b_2$, and let $D := [a_1, b_1] \times [a_2, b_2]$. Let $f : D \to \mathbb{R}$ be a continuous function. From (6.13), $f$ is Riemann integrable on the set $D$, and we can use (a special case of the more general, given below) *Fubini's theorem*, due to Guido Fubini (1879-1943), to calculate its integral:

$$\int_D f(\mathbf{x})\, d\mathbf{x} = \int_{a_2}^{b_2} \left[ \int_{a_1}^{b_1} f(x_1, x_2) dx_1 \right] dx_2 = \int_{a_1}^{b_1} \left[ \int_{a_2}^{b_2} f(x_1, x_2) dx_2 \right] dx_1. \qquad (6.28)$$

Observe that (6.28) is a set of nested *univariate* Riemann integrals. This can be extended in an obvious way to the $n$-dimensional case with $\mathbf{x} = (x_1, \ldots, x_n)$. Fubini's theorem holds whenever $f$ is Riemann integrable; in particular, when $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is continuous and $A$ is closed and bounded.

Proof: As in Lang (1997, p. 277), we wish to show that

$$\int_{a_2}^{b_2} \left[ \int_{a_1}^{b_1} f(t, x)\, dt \right] dx = \int_{a_1}^{b_1} \left[ \int_{a_2}^{b_2} f(t, x)\, dx \right] dt.$$

Let $\psi(t, x) = \int_{a_2}^{x} f(t, u)\, du$, so that $D_2 \psi(t, x) = f(t, x)$ from the FTC (2.179), and $\psi$ is continuous from (6.24). We can now apply (6.27) to $\psi$ and $D_2 \psi = f$. Let $g(x) = \int_{a_1}^{b_1} \psi(t, x)\, dt$. Then

$$g'(x) = \int_{a_1}^{b_1} D_2 \psi(t, x)\, dt = \int_{a_1}^{b_1} f(t, x)\, dt,$$

and, from the FTC (2.176),

$$g(b_2) - g(a_2) = \int_{a_2}^{b_2} g'(x)\, dx = \int_{a_2}^{b_2} \left[ \int_{a_1}^{b_1} f(t, x)\, dt \right] dx.$$

On the other hand, from FTC (i) (2.176), $\int_{a_2}^{b_2} f(t, u)\, du = \psi(t, b_2) - \psi(t, a_2)$, so that

$$g(b_2) - g(a_2) = \int_{a_1}^{b_1} \psi(t, b_2)\, dt - \int_{a_1}^{b_1} \psi(t, a_2)\, dt = \int_{a_1}^{b_1} \left[ \int_{a_2}^{b_2} f(t, u)\, du \right] dt,$$

and the theorem is proved.

The above proof, from Lang's book, is the most elegant, easy, and short I have seen. The reader can compare the other approach, as taken in Fitzpatrick, §19.1, and Terrell, §12.7. All three books provide further extensions and examples, and are worth reading. We look at two important extensions below.

**Example 6.11** *(Shimamoto, Example 5.3) Consider the iterated integral* $\int_0^2 \left( \int_{x^2}^4 x^3 e^{y^3} dy \right) dx$. *We wish to sketch the domain of integration $D$ in the $xy$-plane; and then evaluate the integral.*

*The domain of integration can be reconstructed from the endpoints of the integrals. The outermost integral says that $x$ goes from $x = 0$ to $x = 2$. Geometrically, we are integrating* *areas of cross-sections that are perpendicular to the $x$-axis. Then the inner integral says that, for each $x, y$ goes from $y = x^2$ to $y = 4$. The left panel of Figure 44 exhibits the relevant input. Hence $D$ is described by the conditions:*

$$D = \left\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2, x^2 \leq y \leq 4\right\}.$$

*This is the region in the first quadrant bounded by the parabola $y = x^2$, the line $y = 4$, and the $y$-axis.*

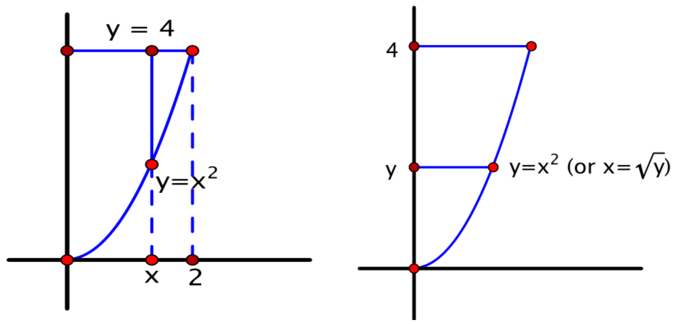



Figure 44: From Shimamoto, Multivariate Calculus, page 126.

*To evaluate the integral as presented, we would antidifferentiate first with respect to $y$, treating $x$ as constant:*

$$\int_0^2 \left(\int_{x^2}^4 x^3 e^{y^3}\, dy\right) dx = \int_0^2 x^3 \left(\int_{x^2}^4 e^{y^3}\, dy\right) dx.$$

*The innermost antiderivative looks hard. So, having nothing better to do, we try switching the order of antidifferentiation. Using cross-sections perpendicular to the $y$-axis, we see from the description of $D$ that $y$ goes from $y = 0$ to $y = 4$, and, for each $y, x$ goes from $x = 0$ to $x = \sqrt{y}$. The thinking behind the switched order is illustrated in the right panel of Figure 44. Therefore,*

$$\begin{aligned}
\int_0^2 \left(\int_{x^2}^4 x^3 e^{y^3}\, dy\right) dx &= \int_0^4 \left(\int_0^{\sqrt{y}} x^3 e^{y^3}\, dx\right) dy = \int_0^4 \left(\frac{1}{4} x^4 e^{y^3}\Big|_{x=0}^{x=\sqrt{y}}\right) dy \\
&= \int_0^4 \left(\frac{1}{4} y^2 e^{y^3} - 0\right) dy \quad \left(\text{let } u = y^3, du = 3y^2 dy\right) \\
&= \frac{1}{4} \cdot \frac{1}{3} e^{y^3}\Big|_0^4 = \frac{1}{12}\left(e^{64} - 1\right). \qquad \blacksquare
\end{aligned}$$

**Example 6.12** *The above proof of Fubini's theorem used the interchange of derivative and integral result (6.27). It is instructive to go the other way, proving (6.27) with the use of Fubini's theorem. With $D = [a_1, b_1] \times [a_2, b_2]$ and $f : D \to \mathbb{R}$ and $D_2 f$ continuous functions, we wish to show that, for $t \in (a_2, b_2)$,*

$$\frac{d}{dt} \int_{a_1}^{b_1} f(x_1, t)\, dx_1 = \int_{a_1}^{b_1} D_2 f(x_1, t)\, dx_1. \tag{6.29}$$

*Define $h : [a_2, b_2] \to \mathbb{R}$ as $h(x_2) := \int_{a_1}^{b_1} D_2 f(x_1, x_2)\, dx_1$, so that $h(t)$ is the rhs of (6.29), and, from FTC (ii, b) (2.179),*

$$\frac{d}{dt}\int_{a_2}^{t} h(x_2)\,dx_2 = h(t). \tag{6.30}$$

*As $D_2 f$ is continuous, it follows from (6.23) that h is continuous on $[a_2, b_2]$. Choosing an arbitrary t with $a_2 < t < b_2$ and integrating $h(x_2)$ over the interval $[a_2, t]$, we obtain*

$$\int_{a_2}^{t} h(x_2)\,dx_2 = \int_{a_2}^{t}\left[\int_{a_1}^{b_1}\frac{\partial f(x_1,x_2)}{\partial x_2}dx_1\right]dx_2. \tag{6.31}$$

*The order of integration in (6.31) can be reversed by Fubini's theorem, so that, using the FTC (i) in (2.176),*

$$\begin{aligned}
\int_{a_2}^{t} h(x_2)\,dx_2 &\overset{Fubini}{=} \int_{a_1}^{b_1}\left[\int_{a_2}^{t}\frac{\partial f(x_1,x_2)}{\partial x_2}dx_2\right]dx_1 \\
&\overset{FTC}{=} \int_{a_1}^{b_1}\left[f(x_1,t) - f(x_1,a_2)\right]dx_1 \\
&= \int_{a_1}^{b_1} f(x_1,t)dx_1 - \int_{a_1}^{b_1} f(x_1,a_2)dx_1. \qquad (6.32)
\end{aligned}$$

*From the FTC (ii, b) in (2.179) and differentiating both sides of (6.32) with respect to t, we obtain*

$$\frac{d}{dt}\int_{a_2}^{t} h(x_2)\,dx_2 = h(t) = \int_{a_1}^{b_1} D_2 f(x_1,t)\,dx_1 \overset{(6.32)}{=} \frac{d}{dt}\int_{a_1}^{b_1} f(x_1,t)\,dx_1.$$

*But the lhs of this is (6.30); the rhs is the lhs of (6.29); and the rhs of (6.29) is $h(t)$, thus showing the result.* ■

**Example 6.13** *Let $f : \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto y^2 e^{2x}$. From (6.13), the fact that f is continuous implies that it is Riemann integrable on bounded rectangles. Let $a_1 = a_2 = 0$ and $b_1 = b_2 = 1$. If $D = [a_1, b_1] \times [a_2, b_2]$ we have*

$$\begin{aligned}
\int_D f &= \int_0^1\int_0^1 y^2 e^{2x}\,dx\,dy = \int_0^1\left[y^2\frac{1}{2}e^{2x}\right]_{x=0}^{x=1}dy = \int_0^1 y^2\frac{1}{2}e^2 - y^2\frac{1}{2}1\,dy \\
&= \left[\frac{1}{3}y^3\left(\frac{1}{2}e^2 - \frac{1}{2}\right)\right]_{y=0}^{y=1} = \frac{1}{6}(e^2-1).
\end{aligned}$$

*The same result can be easily derived when interchanging the order of integration. However, in this example, the calculations can be simplified by* factorizing *the integrated function:*

$$\begin{aligned}
\int_D f &= \int_0^1\int_0^1 y^2 e^{2x}\,dx\,dy = \int_0^1 y^2\int_0^1 e^{2x}\,dx\,dy = \int_0^1 e^{2x}\,dx\int_0^1 y^2\,dy \\
&= \left[\frac{1}{2}e^{2x}\right]_0^1\left[\frac{1}{3}y^3\right]_0^1 = \frac{1}{2}(e^2-1)\frac{1}{3}.
\end{aligned}$$

*Usually, a factorization will not be possible.* ■

We state now the generalization of (6.28) to higher dimensions.

Theorem (Fubini): Suppose that the function $f : \mathbf{I} \to \mathbb{R}$ is integrable, where $\mathbf{I} = \mathbf{I_x} \times \mathbf{I_y}$ is a generalized rectangle in $\mathbb{R}^{n+k}$. For each point $\mathbf{x}$ in $\mathbf{I_x}$, define the function $F_\mathbf{x} : \mathbf{I_y} \to \mathbb{R}$ by

$$F_\mathbf{x}(\mathbf{y}) = f(\mathbf{x}, \mathbf{y}) \quad \text{for } \mathbf{y} \text{ in } \mathbf{I_y};$$

suppose that the function $F_\mathbf{x} : \mathbf{I_y} \to \mathbb{R}$ is integrable, and define

$$A(\mathbf{x}) = \int_{\mathbf{I_y}} f(\mathbf{x}, \mathbf{y}) d\mathbf{y}.$$

Then the function $A : \mathbf{I_x} \to \mathbb{R}$ is integrable, and

$$\int_\mathbf{I} f = \int_{\mathbf{I_x}} A(\mathbf{x}) d\mathbf{x} = \int_{\mathbf{I_x}} \left[ \int_{\mathbf{I_y}} f(\mathbf{x}, \mathbf{y}) d\mathbf{y} \right] d\mathbf{x}. \tag{6.33}$$

We are often interested in integration on more general regions, especially on sets that are unbounded, such as $\mathbb{R}^2$ or regions of the form $D_\infty := (-\infty, b_1] \times (-\infty, b_2]$. Recall the discussion of improper integrals in §2.5.3.

In order to define an integral on $D_\infty$, let $D_1 \subset D_2 \subset \ldots$ be a sequence of closed and bounded rectangles with $\bigcup_{k \in \mathbb{N}} D_k = D_\infty$. Then we define

$$\int_{D_\infty} f := \lim_{k \to \infty} \int_{D_k} f,$$

whenever the rhs exists. Fubini's theorem still applies in these more general cases; in particular,

$$\int_{D_\infty} f = \int_{-\infty}^{b_2} \left[ \int_{-\infty}^{b_1} f(x_1, x_2)\, dx_1 \right] dx_2 = \int_{-\infty}^{b_1} \left[ \int_{-\infty}^{b_2} f(x_1, x_2)\, dx_2 \right] dx_1.$$

See, for example, Lang (1997, §13.3) for details.

**Example 6.14** *As an example of a double integral for which the two corresponding iterated integrals are not equal, let $f(x, y) = (2 - xy)xy \exp(-xy)$. Then*

$$\int_0^1 \int_0^\infty f(x, y)\, dy\, dx = 0 \quad \textit{and} \quad \int_0^\infty \int_0^1 f(x, y)\, dx\, dy = 1.$$

*This is from Cornfield (1969, p. 630), who refers to notes from Courant in 1936.* ■

We now show another important extension of the baseline Fubini result.

Theorem: For continuous functions $h : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ with the property that $h(x) \le g(x)$ for all points $x$ in $[a, b]$, define

$$D = \{(x, y) \mid a \le x \le b,\ \ h(x) \le y \le g(x)\}.$$

Suppose that the function $f : D \to \mathbb{R}$ is continuous and bounded. Then

$$\int_D f = \int_a^b \left[ \int_{h(x)}^{g(x)} f(x, y) dy \right] dx. \tag{6.34}$$

Proofs can be found in, e.g., Fitzpatrick, p. 500; and Terrell, p. 385.

The idea in (6.34) can be extended to any number of dimensions. For the triple integral case, we have from Terrell, p. 386:

Theorem: Let $\alpha(x)$ and $\beta(x)$ be continuous functions for $a \le x \le b$, with $\alpha(x) \le \beta(x)$, and let $\gamma(x,y)$ and $\delta(x,y)$ be continuous functions for $a \le x \le b$, $\alpha(x) \le y \le \beta(x)$, with $\gamma(x,y) \le \delta(x,y)$. Let

$$D = \left\{(x,y,z) \in \mathbf{R}^3 : a \le x \le b,\ \ \alpha(x) \le y \le \beta(x),\ \ \gamma(x,y) \le z \le \delta(x,y)\right\}.$$

If $f : D \to \mathbf{R}$ is continuous, then $\int_D f$ exists and

$$\int_D f = \int_a^b \left( \int_{\alpha(x)}^{\beta(x)} \left( \int_{\gamma(x,y)}^{\delta(x,y)} f(x,y,z)dz \right) dy \right) dx.$$

**Example 6.15** *For $D = \{(x,y) \mid x^2+y^2 \le 1, y \ge 0\}$, define $f : D \to \mathbb{R}$ to be the constant function with value 1. Then*

$$D = \left\{(x,y) \mid -1 \le x \le 1, 0 \le y \le \sqrt{1-x^2}\right\},$$

*and*

$$\int_D f = \int_{-1}^1 \left[\int_0^{\sqrt{1-x^2}} dy\right] dx = \int_{-1}^1 \left[\sqrt{1-x^2}\right] dx = \frac{\pi}{2},$$

*where this integral is resolved in Example 2.33.* ■

Generalizing (6.33) and (6.34), we get (Fitzpatrick, p. 503):

Theorem: For a Jordan domain $K$ in $\mathbb{R}^n$, let $h : K \to \mathbb{R}$ and $g : K \to \mathbb{R}$ be continuous bounded functions with the property that, $\forall \mathbf{x} \in K$, $h(\mathbf{x}) \le g(\mathbf{x})$. Define

$$D = \left\{(\mathbf{x}, y) \text{ in } \mathbb{R}^{n+1} \mid \mathbf{x} \text{ in } K,\ \ h(\mathbf{x}) \le y \le g(\mathbf{x})\right\}.$$

Suppose that the function $f : D \to \mathbb{R}$ is continuous and bounded. Then

$$\int_D f = \int_K \left[\int_{h(\mathbf{x})}^{g(\mathbf{x})} f(\mathbf{x},y)dy\right] d\mathbf{x}.$$

## 6.5 Leibniz' Rule

As above, let $a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $a_1 < b_1$, $a_2 < b_2$, and let $D := [a_1, b_1] \times [a_2, b_2]$. Assume functions $f : D \to \mathbb{R}$ and $D_1 f$ are continuous. Also let $\lambda$ and $\theta$ be differentiable functions defined on $[a_2, b_2]$ such that $\lambda(x), \theta(x) \in [a_1, b_1]$ for all $x \in [a_2, b_2]$ and define the function $A : [a_2, b_2] \to \mathbb{R}$ by

$$A(x) := \int_{\lambda(x)}^{\theta(x)} f(x, y)\, dy. \tag{6.35}$$

We wish to determine if this function is differentiable and, if so, derive an expression for its derivative. First, define the function $H : [a_1, b_1] \times [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$, depending on the three variables $a, b$ and $x$, by

$$H(a, b, x) := \int_a^b f(x, y)\, dy.$$

Note that $A(x) = H(\lambda(x), \theta(x), x)$ for every $x \in [a_2, b_2]$. From the FTC (2.179), $H$ is differentiable for any $a, b \in (a_2, b_2)$ and $x \in (a_1, b_1)$, with (also recall Example 2.26)

$$\begin{aligned} \frac{\partial H}{\partial b}(a, b, x) &= \frac{\partial}{\partial b} \int_a^b f(x, y)\, dy = f(x, b), \\ \frac{\partial H}{\partial a}(a, b, x) &= \frac{\partial}{\partial a} \int_a^b f(x, y)\, dy = -\frac{\partial}{\partial a} \int_b^a f(x, y)\, dy = -f(x, a), \end{aligned}$$

and from (6.27), as $D_1 f$ was assumed continuous,

$$\frac{\partial H}{\partial x}(a, b, x) = \frac{\partial}{\partial x} \int_a^b f(x, y)\, dy = \int_a^b \frac{\partial f(x, y)}{\partial x}\, dy.$$

From the chain rule (5.102), it follows that, for $a_1 < x < b_1$, $A$ is differentiable and

$$\begin{aligned} A'(x) &= \frac{\partial H}{\partial \lambda} \frac{d\lambda}{dx} + \frac{\partial H}{\partial \theta} \frac{d\theta}{dx} + \frac{\partial H}{\partial x} \frac{dx}{dx} \\ &= -f(x, \lambda(x))\, \lambda'(x) + f(x, \theta(x))\, \theta'(x) + \int_{\lambda(x)}^{\theta(x)} \frac{\partial f(x, y)}{\partial x}\, dy. \end{aligned} \tag{6.36}$$

Formula (6.36) is sometimes called "Leibniz' rule for differentiating an integral".

**Example 6.16** *Consider the function $f(t) := \int_0^t e^{st}\, ds$. There are two possible ways to calculate its derivative at $t = 1$. Firstly, let us integrate in step one and then differentiate afterwards. For $t > 0$,*

$$f(t) = \left[\frac{1}{t} e^{st}\right]_0^t = \frac{1}{t}\left(e^{t^2} - 1\right),$$

*and*

$$f'(t) = \frac{-1}{t^2}\left(e^{t^2} - 1\right) + \frac{1}{t} 2t e^{t^2} = \frac{-1}{t^2}\left(e^{t^2} - 1\right) + 2e^{t^2},$$

*so that $f'(1) = -(e - 1) + 2e = 1 + e$.*

*Secondly, we can differentiate first, using Leibniz' rule, and then integrate in a second step. For $t > 0$, with $d(e^{st})/dt = s e^{st}$ continuous,*

$$f'(t) = \int_0^t s e^{st} ds + 1 \cdot e^{t^2},$$

*hence*

$$f'(1)=\int_0^1 se^s ds+e^1=[(s-1)e^s]_0^1+e=0-(-1)+e=1+e.$$

*In this case, Leibniz' rule saves us a bit of work.* ■

**Example 6.17** *(*https://brilliant.org/wiki/differentiate-through-the-integral/*).* *We wish to compute the definite integral*

$$\int_0^1 \frac{t^3-1}{\ln t}dt.$$

*Recall from Example 2.18 that, for $f(x)=t^x$ and $t>0$, $f'(x)=t^x=t^x\ln t$. Define*

$$g(x)=\int_0^1 \frac{t^x-1}{\ln t}dt,$$

*and we wish to evaluate $g(3)$. Observe that the given integral has been recast as member of a family of definite integrals $g(x)$ indexed by the variable $x$. With*

$$\frac{\partial}{\partial x}\frac{t^x-1}{\ln t}=\frac{1}{\ln t}\frac{d}{dx}\left(t^x-1\right)=t^x,$$

*we have, by Leibniz' rule, or, in this case, just from (6.27),*

$$g'(x)=\int_0^1 \frac{\partial}{\partial x}\frac{t^x-1}{\ln t}dt=\int_0^1 t^x dt=\left.\frac{t^{x+1}}{x+1}\right|_0^1=\frac{1}{x+1}.$$

*Recalling Example 2.37, it follows that $g(x)=\ln|x+1|+C$ for some constant $C$. To determine $C$, note that $g(0)=0$, so $0=g(0)=\ln 1+C=C$. Hence, $g(x)=\ln|x+1|$ for all $x$ such that the integral exists. In particular, $g(3)=\ln 4=2\ln 2$.* ■

Although Leibniz' rule (6.36) follows directly from the multivariate chain rule, the expression itself does not appear to hold much intuition, and one might wonder how the formula could have been postulated without knowledge of the chain rule. It turns out that, with the right geometric representation, the formula is essentially obvious! This pleasant result is due to Frantz (2001), on which the following is based. Firstly, it suffices to set the lower limit $\lambda(x)$ in (6.35) to zero because, from (2.171),

$$A(x)=\int_{\lambda(x)}^{\theta(x)} f(x,y)\,dy=\int_0^{\theta(x)} f(x,y)\,dy-\int_0^{\lambda(x)} f(x,y)\,dy.$$

The first graphic[36] in Figure 45 shows a cross-section, or "slice" (or *lamina*) of $A$ at a particular value of $x$, with $y=\theta(x)$ lying in the $xy$-plane. The second graphic also shows the lamina at $x+\triangle x$, with area $A(x+\triangle x)$, so that the change in height of the lamina, for any $y$, is $f(x+\triangle x,y)-f(x,y)\approx D_1 f(x,y)\triangle x =: f_x(x,y)\triangle x$. Similarly, the width of the lamina increases by approximately $\theta'(x)\triangle x$.

Figure 46 isolates this lamina for clarity, and defines regions $A_1$ and $A_2$. The change in the lamina's area, $\triangle A$, is then $A_1+A_2$, plus the upper-right corner, which, compared to

[36]I am very grateful to Marc Frantz, the author of Frantz (2001), for constructing and providing me with the three graphs shown in the figures.

the size of $A_1$ and $A_2$, can be ignored (it becomes negligible much faster than $A_1$ and $A_2$ as $\triangle x \to 0$). Thus,

$$A_1 \approx \int_0^{\theta(x)} f_x(x,y)\,dy\,\triangle x \quad \text{and} \quad A_2 \approx f(x,\theta(x))\,\theta'(x)\,\triangle x,$$

i.e., dividing by $\triangle x$ gives

$$\frac{\triangle A}{\triangle x} \approx \int_0^{\theta(x)} f_x(x,y)\,dy + f(x,\theta(x))\theta'(x),$$

which is indeed $A'(x)$ in (6.36) with $\lambda(x) = 0$.

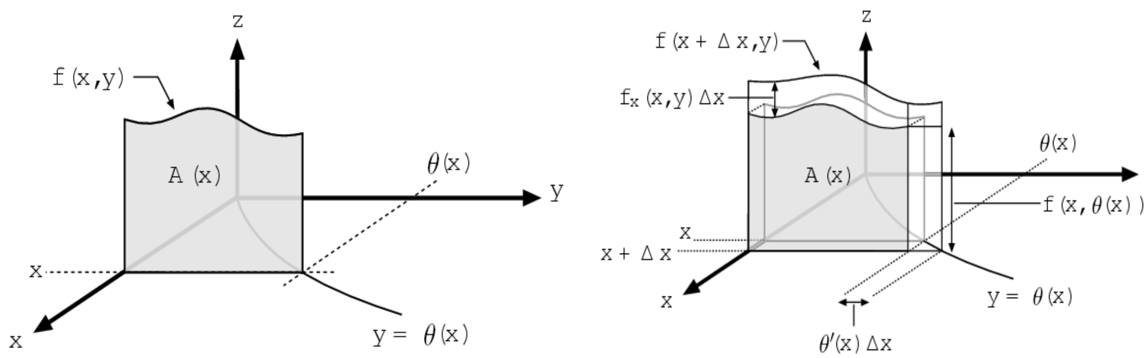



Figure 45: Geometric motivation for Leibniz' rule. Based on plots in M. Frantz, Visualizing Leibniz's Rule, *Mathematics Magazine*, 2001, 74(2):143–144.

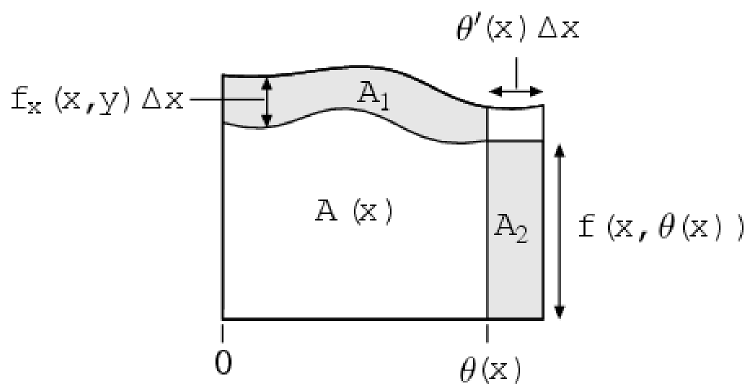



Figure 46: Magnified view of the relevant part of the second panel in Figure 45.

## 6.6 Integral Transformations, Polar and Spherical Coordinates

Theorem: Let $f : D \to \mathbb{R}$ be a continuous function, where domain $D$ is an open subset of $\mathbb{R}^n$ with typical element $\mathbf{x} = (x_1, \ldots, x_n)$. Let $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_n(\mathbf{x}) : D \to \mathbb{R}^n$ be a differentiable bijection with nonvanishing Jacobian

$$\mathbf{J} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}, \quad \text{where} \quad \mathbf{y} = \mathbf{g}(\mathbf{x}).$$

Then, for $S \subset D$,

$$\int_S f(\mathbf{x})\, d\mathbf{x} = \int_{\mathbf{g}(S)} f\left(\mathbf{g}^{-1}(\mathbf{y})\right) |\det \mathbf{J}|\, d\mathbf{y}, \tag{6.37}$$

where $\mathbf{g}(S) = \{\mathbf{y} : \mathbf{y} = \mathbf{g}(\mathbf{x}), \mathbf{x} \in S\}$.

This is referred to as the *multivariate change of variable formula.* The is a well-known and fundamental result in analysis, the rigorous proof of which, however, is somewhat involved.

**Example 6.18** *This example is a special case of the next example, and is shown to help illustrate the idea in just two dimensions. Let* $I = \int_T \exp(x_1 + x_2)\, dx_1 dx_2$, *where, for* $a_1, b_1, a_2, b_2 \in \mathbb{R}$ *and* $T = \{a_1 \le x_1 \le b_1,\ a_2 \le x_1 + x_2 \le b_2\}$. *For* $\mathbf{X} = (x_1, x_2)$, *Let* $\mathbf{Y} = (y_1, y_2) = F(\mathbf{X})$, *where* $\mathbf{F} = (f_1(\mathbf{X}), f_2(\mathbf{X}))$, *with* $y_1 = f_1(\mathbf{X}) = x_1$ *and* $y_2 = f_2(\mathbf{X}) = x_1 + x_2$. *Denote the bijective inverse function as* $\mathbf{X} = G(\mathbf{Y}) = F^{-1}(\mathbf{Y}) = (g_1(\mathbf{Y}), g_2(\mathbf{Y}))$, *where* $g_1(\mathbf{Y}) = y_1$ *and* $g_2(\mathbf{Y}) = y_2 - y_1$. *Then*

$$\mathbf{J} = \begin{bmatrix} \partial x_1/\partial y_1 & \partial x_1/\partial y_2 \\ \partial x_2/\partial y_1 & \partial x_2/\partial y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad \det(\mathbf{J}) = 1,$$

*and the range of* $\mathbf{Y}$ *is* $S = F(T) = [a_1, b_1] \times [a_2, b_2]$. *Thus,*

$$I = \int_S \exp(y_2)\, dy_1 dy_2 = \int_{a_1}^{b_1} dy_1 \int_{a_2}^{b_2} \exp(y_2)\, dy_2 = (b_1 - a_1)\left(e^{b_2} - e^{a_2}\right). \quad \blacksquare$$

**Example 6.19** *(Trench, 2013, Example 7.3.8) Evaluate*

$$I = \int_T e^{x_1 + x_2 + \cdots + x_n} d(x_1, x_2, \ldots, x_n),$$

*where* $T$ *is the region defined by*

$$a_i \le x_1 + x_2 + \cdots + x_i \le b_i, \quad 1 \le i \le n.$$

*Solution: We define the new variables* $y_1, y_2, \ldots, y_n$ *by* $\mathbf{Y} = \mathbf{F}(\mathbf{X})$, *where*

$$f_i(\mathbf{X}) = x_1 + x_2 + \cdots + x_i, \quad 1 \le i \le n.$$

*If* $\mathbf{G}(\mathbf{Y}) = \mathbf{F}^{-1}(\mathbf{Y})$, *then* $T = \mathbf{G}(S)$, *where*

$$S = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n],$$

*and* $\mathbf{J} = 1$. *Hence,*

$$I = \int_S e^{y_n} d\left(y_1, y_2, \ldots, y_n\right) \tag{6.38}$$

$$= \int_{a_1}^{b_1} dy_1 \int_{a_2}^{b_2} dy_2 \cdots \int_{a_{n-1}}^{b_{n-1}} dy_{n-1} \int_{a_n}^{b_n} e^{y_n} dy_n \tag{6.39}$$

$$= \left(b_1 - a_1\right)\left(b_2 - a_2\right) \cdots \left(b_{n-1} - a_{n-1}\right)\left(e^{b_n} - e^{a_n}\right). \quad \blacksquare \tag{6.40}$$

Consider the special case of (6.37) using polar coordinates, i.e., $x = c_1(r, \theta) = r\cos\theta$ and $y = c_2(r, \theta) = r\sin\theta$. Then

$$\det \mathbf{J} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r, \tag{6.41}$$

which is positive, so that (6.37) implies

$$\iint f(x, y)\, dxdy = \iint f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta. \tag{6.42}$$

**Example 6.20** *This is a particularly useful transformation when $f(x, y)$ depends only on the distance measure $r^2 = x^2 + y^2$ and the range of integration is a circle centered around $(0, 0)$. For example, if $f(x, y) = (k + x^2 + y^2)^p$ for constants $k$ and $p \geq 0$, and $S$ is such a circle with radius $a$, then (with $t = r^2$, $r = t^{1/2}$, $dr = \frac{1}{2}t^{-1/2}dt$),*

$$I = \iint_S f(x, y)\, dx\, dy = \int_0^{2\pi} \int_0^a \left(k + r^2\right)^p r\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi} \int_0^{a^2} (k + t)^p t^{1/2} t^{-1/2}\, dt\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} d\theta \cdot \int_0^{a^2} (k + t)^p\, dt = \pi \int_0^{a^2} (k + t)^p\, dt = \frac{\pi}{p+1}\left(\left(k + a^2\right)^{p+1} - k^{p+1}\right),$$

*having used the substitution $s = k + t$ in the last equality. For $k = p = 0$, $I$ reduces to $\pi a^2$, the area of a circle of radius $a$.* ■

The following example is very important, as it relates to the Gaussian distribution.

**Example 6.21** *To compute $I_1 = \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}t^2\right) dt = 2\int_0^{\infty} \exp\left(-\frac{1}{2}t^2\right) dt$, let $x^2 = t^2/2$ so that*

$$I_1 = 2\sqrt{2}\int_0^{\infty} \exp\left(-x^2\right) dx = \sqrt{2}\int_{-\infty}^{\infty} \exp\left(-x^2\right) dx =: \sqrt{2} I_2.$$

*Then, observe that*

$$I_2^2 = \int_{-\infty}^{\infty} \exp\left(-x^2\right) dx \int_{-\infty}^{\infty} \exp\left(-y^2\right) dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy,$$

*or, transforming to polar coordinates,*

$$I_2^2 = \int_0^{2\pi}\int_0^{\infty} \exp\left(-r^2\right) r\, dr\, d\theta = 2\pi \lim_{t\to\infty}\left(-\frac{1}{2}e^{-r^2}\Big|_0^t\right) = \pi,$$

*so that $I_2 = \sqrt{\pi}$ and $I_1 = \sqrt{2\pi}$.* ■

**Example 6.22** *(https://brilliant.org/wiki/differentiate-through-the-integral/). We wish to compute $\int_0^\infty e^{-x^2/2}dx$; recall (2.226). Define a function*

$$g(t) = \left(\int_0^t e^{-x^2/2}dx\right)^2.$$

*Our goal is to compute $g(\infty)$ and then take its square root. Differentiating with respect to $t$ gives*

$$g'(t) = 2\cdot\left(\int_0^t e^{-x^2/2}dx\right)\cdot\left(\frac{d}{dt}\int_0^t e^{-x^2/2}dx\right) = 2e^{-t^2/2}\int_0^t e^{-x^2/2}dx = 2\int_0^t e^{-\left(t^2+x^2\right)/2}dx.$$

*Make the change of variables $u = x/t$, so that the integral transforms to*

$$g'(t) = 2\int_0^1 te^{-\left(1+u^2\right)t^2/2}du.$$

*Now, the integrand has a closed-form antiderivative with respect to $t$ :*

$$g'(t) = -2\int_0^1 \frac{\partial}{\partial t}\frac{e^{-\left(1+u^2\right)t^2/2}}{1+u^2}du = -2\frac{d}{dt}\int_0^1 \frac{e^{-\left(1+u^2\right)t^2/2}}{1+u^2}du.$$

*Set*

$$h(t) = \int_0^1 \frac{e^{-\left(1+x^2\right)t^2/2}}{1+x^2}dx$$

*Then by the above calculation, $g'(t) = -2h'(t)$, so $g(t) = -2h(t) + C$. To determine $C$, take $t \to 0$ in the equation; since $g(0) = 0$ and*

$$h(0) = \int_0^1 \frac{1}{1+x^2}dx = \tan^{-1}x\Big|_0^1 = \frac{\pi}{4},$$

*it follows that $0 = -\pi/2 + C \implies C = \pi/2$. Finally, taking $t \to \infty$, we conclude $g(\infty) = -2h(\infty) + \pi/2 = \pi/2$. Thus, $\int_0^\infty e^{-x^2/2}dx = \sqrt{\frac{\pi}{2}}$, which of course agrees with the result from Example 6.21.* ■

We now turn to a useful change of variables formula in three dimensions. For each point $\mathbf{u} = (x, y, z)$ in $\mathbb{R}^3$ that does not lie on the $z$-axis, we define $\rho = \sqrt{x^2 + y^2 + z^2}$. It is not difficult to see that there are unique numbers $\theta$ in the interval $[0, 2\pi)$ and $\phi$ in the interval $(0, \pi)$ such that

$$\mathbf{u} = (x, y, z) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi).$$

The triple of numbers $(\rho, \phi, \theta)$ is called a choice of spherical coordinates for the point $\mathbf{u}$. Define $\mathcal{O}$ to be the open subset of $\mathbb{R}^3$ consisting of points $(\rho, \phi, \theta)$ with $\rho > 0$, $0 < \phi < \pi$, and $0 < \theta < 2\pi$ and then define $\Psi : \mathcal{O} \to \mathbb{R}^3$ by

$$\Psi(\rho, \phi, \theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \quad \text{for } (\rho, \phi, \theta) \text{ in } \mathcal{O}.$$

See Figure 47. It is clear that the mapping $\Psi : \mathcal{O} \to \mathbb{R}^3$ is both continuously differentiable and one-to-one. Also, at each point $(\rho, \phi, \theta)$ in $\mathcal{O}$, the derivative matrix is given by

$$\mathbf{D}\Psi(\rho, \phi, \theta) = \begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{pmatrix}.$$

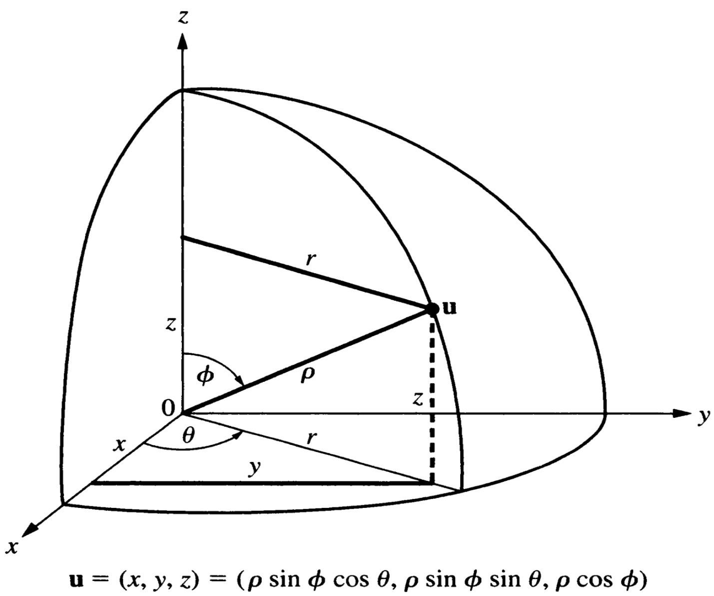



Figure 47: Spherical coordinates. From Fitzpatrick, p. 508.

A brief computation yields $\det \mathbf{D}\Psi(\rho, \phi, \theta) = \rho^2 \sin\phi \neq 0$. Thus, the derivative matrix $\mathbf{D}\Psi(\rho, \phi, \theta)$ is invertible, so $\Psi : \mathcal{O} \to \mathbb{R}^3$ is a smooth change of variables. For $0 < \rho_1 < \rho_2$, $0 < \phi_1 < \phi_2 < \pi$, and $0 < \theta_1 < \theta_2 < 2\pi$, define $K = [\rho_1, \rho_2] \times [\phi_1, \phi_2] \times [\theta_1, \theta_2]$.

Suppose that the function $f : \Psi(K) \to \mathbb{R}$ is continuous. Then by the integral transformation formula (6.37) and Fubini's Theorem,

$$\begin{aligned}
&\int_{\Psi(K)} f(x, y, z)\, dx\, dy\, dz \\
&\quad = \int_K \left[f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\right]\, d\rho\, d\phi\, d\theta \\
&\quad = \int_{\theta_1}^{\theta_2} \left[\int_{\phi_1}^{\phi_2} \left\{\int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi d\rho\right\} d\phi\right] d\theta. \qquad (6.43)
\end{aligned}$$

**Example 6.23** *For $a > 0$, we find the volume of the ball in $\mathbb{R}^3$ of radius $a$,*

$$B_a = \left\{(x, y, z) \mid x^2 + y^2 + z^2 \leq a^2\right\}.$$

*Indeed, by formula (6.43),*

$$\begin{aligned}
\operatorname{vol} B_a &= \int_{B_a} 1\, dx\, dy\, dz \\
&= \int_0^{2\pi} \left[\int_0^{\pi} \left\{\int_0^a \rho^2 \sin\phi d\rho\right\} d\phi\right] d\theta = [4/3]\pi a^3.
\end{aligned}$$

*As Fitzpatrick, p. 509 states, Archimedes discovered the formula for the volume of a ball. He was so proud of this accomplishment that he had the formula inscribed on his tomb.* ■

## 6.7 Multivariate Transformations for Random Variables

The Jacobian transformation is the key result required to compute the distribution of a set of random variables that are suitable functions of another set of random variables. The result is as follows:

Theorem: Let $\mathbf{X} = (X_1, \ldots, X_n)$ be an $n$-dimensional continuous random variable and let function $\mathbf{g} = (g_1(\mathbf{x}), \ldots, g_n(\mathbf{x}))$ be a continuous bijection that maps $\mathcal{S}_{\mathbf{X}} \subset \mathbb{R}^n$, the support of $\mathbf{X}$, onto $\mathcal{S}_{\mathbf{Y}} \subset \mathbb{R}^n$. Then the probability density function (pdf) of $\mathbf{Y} = (Y_1, \ldots, Y_n) = \mathbf{g}(\mathbf{X})$ is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(\mathbf{x})\,|\det \mathbf{J}|\,, \tag{6.44}$$

where $\mathbf{x} = \mathbf{g}^{-1}(\mathbf{y}) = \left(g_1^{-1}(\mathbf{y}), \ldots, g_n^{-1}(\mathbf{y})\right)$ and

$$\mathbf{J} = \begin{pmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} & \cdots & \dfrac{\partial x_1}{\partial y_n} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} & & \dfrac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial x_n}{\partial y_1} & \dfrac{\partial x_n}{\partial y_2} & \cdots & \dfrac{\partial x_n}{\partial y_n} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial g_1^{-1}(\mathbf{y})}{\partial y_1} & \dfrac{\partial g_1^{-1}(\mathbf{y})}{\partial y_2} & \cdots & \dfrac{\partial g_1^{-1}(\mathbf{y})}{\partial y_n} \\ \dfrac{\partial g_2^{-1}(\mathbf{y})}{\partial y_1} & \dfrac{\partial g_2^{-1}(\mathbf{y})}{\partial y_2} & & \dfrac{\partial g_2^{-1}(\mathbf{y})}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial g_n^{-1}(\mathbf{y})}{\partial y_1} & \dfrac{\partial g_n^{-1}(\mathbf{y})}{\partial y_2} & \cdots & \dfrac{\partial g_n^{-1}(\mathbf{y})}{\partial y_n} \end{pmatrix} \tag{6.45}$$

is the Jacobian of $\mathbf{g}$.

Notice that (6.44) reduces to the simple equation relevant for the univariate case.

> Outline of proof: Let $h : \mathbb{R}^n \to \mathbb{R}$ be a bounded, measurable function so that
>
> $$\mathbb{E}[h(\mathbf{Y})] = \mathbb{E}[h(\mathbf{g}(\mathbf{X}))] = \int_{\mathcal{S}_{\mathbf{X}}} h(\mathbf{g}(\mathbf{x})) f_{\mathbf{X}}(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_{\mathcal{S}_{\mathbf{Y}}} h(\mathbf{y}) f_{\mathbf{X}}\left(\mathbf{g}^{-1}(\mathbf{y})\right) |\det \mathbf{J}| d\mathbf{y}.$$
>
> In particular, let $h = \mathbb{I}_B(\mathbf{y})$ for a Borel set $B \in \mathcal{B}^n$, so that
>
> $$\mathbb{E}[h(\mathbf{Y})] = \Pr(\mathbf{Y} \in B) = \int_{\mathcal{S}_{\mathbf{Y}}} \mathbb{I}_B(\mathbf{y}) f_{\mathbf{X}}\left(\mathbf{g}^{-1}(\mathbf{y})\right) |\det \mathbf{J}| d\mathbf{y}.$$
>
> As this holds for all $B \in \mathcal{B}^n$, $f_{\mathbf{X}}\left(\mathbf{g}^{-1}(\mathbf{y})\right) |\det \mathbf{J}|$ is a probability density function for $\mathbf{Y}$.

The following examples are from Paolella, Fundamental Probability, Ch. 9).

**Example 6.24** *Consider calculating the joint distribution of $S = X + Y$ and $D = X - Y$ and their marginals for $X, Y \overset{\text{iid}}{\sim} \text{Exp}(\lambda)$. Adding and subtracting the two equations yields $X = (S + D)/2$ and $Y = (S - D)/2$. From these,*

$$\mathbf{J} = \begin{bmatrix} \partial x/\partial s & \partial x/\partial d \\ \partial y/\partial s & \partial y/\partial d \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}, \quad \det \mathbf{J} = -\frac{1}{2},$$

*so that*

$$f_{S,D}(s, d) = |\det \mathbf{J}|\, f_{X,Y}(x, y) = \frac{\lambda^2}{2} \exp\{-\lambda s\}\, \mathbb{I}_{(0,\infty)}(s + d)\, \mathbb{I}_{(0,\infty)}(s - d)\,.$$

*The constraints imply $s > d$ and $s > -d$; if $d < 0$ ($d > 0$) then $s > -d$ ($s > d$) is relevant. Thus,*

$$\begin{aligned} f_D(d) &= \mathbb{I}_{(-\infty,0)}(d) \int_{-d}^{\infty} f_{S,D}(s,d)\, ds + \mathbb{I}_{(0,\infty)}(d) \int_{d}^{\infty} f_{S,D}(s,d)\, ds \\ &= \mathbb{I}_{(-\infty,0)}(d) \frac{\lambda}{2} e^{\lambda d} + \mathbb{I}_{(0,\infty)}(d) \frac{\lambda}{2} e^{-\lambda d} = \frac{\lambda}{2} e^{-\lambda |d|}, \end{aligned}$$

*i.e., $D \sim \mathrm{Lap}(0, \lambda)$. Next, $s + d > 0$ and $s - d > 0$ imply that $-s < d < s$, giving*

$$f_S(s) = \int_{-s}^{s} \frac{\lambda^2}{2} \exp\{-\lambda s\}\, dd = \frac{\lambda^2}{2} \exp\{-\lambda s\} \int_{-s}^{s} dd = s\lambda^2 e^{-\lambda s} \mathbb{I}_{(0,\infty)}(s),$$

*which follows because $S > 0$ from its definition. Thus, $S \sim \mathrm{Gam}(2, \lambda)$, which agrees with the more general result regarding sums of iid exponentials.* ■

**Example 6.25** *Let $Z_i \overset{\text{iid}}{\sim} \mathrm{N}(0,1)$, $i = 1, 2$. The inverse transformation and Jacobian of $S = Z_1 + Z_2$ and $D = Z_1 - Z_2$ was derived in Example 6.24, so that*

$$\begin{aligned} f_{S,D}(s,d) &= \frac{1}{2} f_{Z_1,Z_2}(z_1, z_2) \\ &= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} z_1^2\right\} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} z_2^2\right\} \\ &= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} \left(\frac{s+d}{2}\right)^2\right\} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} \left(\frac{s-d}{2}\right)^2\right\} \\ &= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} \left[\frac{1}{2} d^2 + \frac{1}{2} s^2\right]\right\} \\ &= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} \left(\frac{d}{\sqrt{2}}\right)^2\right\} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} \left(\frac{s}{\sqrt{2}}\right)^2\right\}, \end{aligned}$$

*i.e., $S$ and $D$ are independent, with $S \sim \mathrm{N}(0, 2)$ and $D \sim \mathrm{N}(0, 2)$.* ■

**Example 6.26** *Let $X_i \overset{i.i.d.}{\sim} \mathrm{N}(0, 1)$. Interest centers on the distribution of*

$$Y_1 = g_1(\mathbf{X}) = \sum_{i=1}^{n} X_i^2.$$

*To derive it, let $Y_i = g_i(\mathbf{X}) = X_i, i = 2, 3, \ldots, n$, perform the multivariate transformation to get $f_{\mathbf{Y}}$, and then integrate out $Y_2, \ldots, Y_n$ to get the distribution of $Y_1$. The following steps can be used.*

1. *First do the $n = 2$ case, for which the fact that*

$$\int \frac{1}{\sqrt{y_1 - y_2^2}} dy_2 = c + \arcsin\left(\frac{y_2}{\sqrt{y_1}}\right)$$

*can be helpful.*

*Let* $n = 2$, *so that* $X_2 = g_2^{-1}(\mathbf{Y}) = Y_2$ *and* $X_1 = g_1^{-1}(\mathbf{Y}) = \pm\sqrt{Y_1 - Y_2^2}$. *Splitting this into two regions,*

$$f_{\mathbf{Y}}(\mathbf{y}) = |\det \mathbf{J}_1| \, f_{\mathbf{X}}(x_1, x_2) \, \mathbb{I}_{(-\infty,0)}(x_1) + |\det \mathbf{J}_2| \, f_{\mathbf{X}}(x_1, x_2) \, \mathbb{I}_{(0,-\infty)}(x_1)$$

*where*

$$\mathbf{J}_1 = \begin{bmatrix} \partial x_1/\partial y_1 & \partial x_1/\partial y_2 \\ \partial x_2/\partial y_1 & \partial x_2/\partial y_2 \end{bmatrix} = \begin{bmatrix} -\left(y_1 - y_2^2\right)^{-1/2}/2 & y_2\left(y_1 - y_2^2\right)^{-1/2} \\ 0 & 1 \end{bmatrix}$$

*and similarly for* $\mathbf{J}_2$, *and, in both cases,* $|\mathbf{J}_i| = \left(y_1 - y_2^2\right)^{-1/2}/2$. *Thus,*

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{2}\left(y_1 - y_2^2\right)^{-1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(y_1 - y_2^2\right)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_2^2} + \text{ same}$$

*or*

$$f_{\mathbf{Y}}(\mathbf{y}) = \left(y_1 - y_2^2\right)^{-1/2} \frac{1}{2\pi} e^{-\frac{1}{2}y_1} \mathbb{I}_{\left(y_2^2,\infty\right)}(y_1),$$

*where the indicator function follows from*

$$Y_1 = X_1^2 + X_2^2 \quad \text{and} \quad 0 \le X_2^2 \le X_1^2 + X_2^2$$

*or* $0 \le Y_2^2 \le Y_1$. *This also implies* $-\sqrt{Y_1} \le Y_2 \le \sqrt{Y_1}$, *from which we have*

$$f_{Y_1}(y_1) = \int_{-\sqrt{y_1}}^{\sqrt{y_1}} f_{\mathbf{Y}}(\mathbf{y}) dy_2 = \frac{1}{2\pi} e^{-\frac{1}{2}y_1} \int_{-\sqrt{y_1}}^{\sqrt{y_1}} \left(y_1 - y_2^2\right)^{-1/2} dy_2.$$

*From the hint,*

$$\int \frac{1}{\sqrt{y_1 - y_2^2}} dy_2 = c + \arcsin\left(\frac{y_2}{\sqrt{y_1}}\right)$$

*and, clearly,* $\arcsin(1) = \pi/2$ *and* $\arcsin(-1) = -\pi/2$, *so that*

$$f_{Y_1}(y_1) = \frac{1}{2} \exp\left(-y_1/2\right) \mathbb{I}_{(0,\infty)}(y_1)$$

*where the indicator follows from the definition of* $Y_1 = \sum_{i=1}^{2} X_i^2$. *Thus,* $Y_1 \sim \chi_2^2$.

2. *Simplify the following integral, which is used in the general case:*

$$J = \int_0^{y_0} u^{m/2-1} \left(y_0 - u\right)^{-1/2} du.$$

*This integral is, using* $v = (y_0 - u)/y_0, u = (1 - v)y_0, du = -y_0 dv$,

$$\begin{aligned} J &= \int_0^{y_0} u^{m/2-1} \left(y_0 - u\right)^{-1/2} du = -\int_1^0 \left((1 - v)y_0\right)^{m/2-1} \left(vy_0\right)^{-1/2} y_0 dv \\ &= y_0^{(m-1)/2} \int_0^1 v^{(1/2)-1}(1 - v)^{m/2-1} dv \\ &= y_0^{(m-1)/2} B\left(\frac{1}{2}, \frac{m}{2}\right) = y_0^{(m-1)/2} \frac{\Gamma(1/2)\Gamma(m/2)}{\Gamma((m+1)/2)}. \end{aligned}$$

*3. For the general case, conduct the multivariate transformation to show that*

$$f_{Y_1}(y_1) = \frac{1}{(2\pi)^{n/2}} \mathrm{e}^{-\frac{1}{2}y_1} \int \cdots \int_{\mathcal{S}} \left( y_1 - \sum_{i=2}^{n} y_i^2 \right)^{-\frac{1}{2}} dy_2 \cdots \mathrm{d}y_n$$

*where*

$$\mathcal{S} = \left\{ (y_2, \ldots, y_n) \in \mathbb{R}^{n-1} : 0 < \sum_{i=2}^{n} y_i^2 < y_1 \right\}.$$

*Then use the following identity (due to Joseph Liouville (1809-1882) in 1839, which extended a result from Dirichlet):*

*Let $\mathcal{V}$ be a volume consisting of (i) $x_i \geq 0$ and (ii) $t_1 \leq \sum (x_i/a_i)^{p_i} \leq t_2$, and let $f$ be a continuous function on $(t_1, t_2)$. Then, with $r_i = b_i/p_i$ and $R = \sum r_i$,*

$$\begin{aligned} &\int \cdots \int_{\mathcal{V}} x_1^{b_1-1} \cdots x_n^{b_n-1} f \left[ \left( \frac{x_1}{a_1} \right)^{p_1} + \cdots + \left( \frac{x_n}{a_n} \right)^{p_n} \right] \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &= \frac{\prod a_i^{b_i} p_i^{-1} \Gamma(r_i)}{\Gamma(R)} \int_{t_1}^{t_2} u^{R-1} f(u) \mathrm{d}u \end{aligned} \tag{6.46}$$

*For details on this and similar results, see Andrews, Askey and Roy (1999, Section 1.8) and Jones (2001, Chapter 9).*

*From the result using $n = 2$, one might guess that the general case might lead to $Y_1 \sim \chi_n^2$, which is true. Now $X_i = g_i^{-1}(\mathbf{Y}) = Y_i, i = 2, \ldots, n$ and $X_1 = g_1^{-1}(\mathbf{Y}) = \pm\sqrt{Y_1 - \sum_{i=2}^{n} Y_i^2}$. We again need to split the support of $\mathbf{X}$ into two regions as before, but we have seen for the $n = 2$ case that the components are the same. Thus,*

$$f_{\mathbf{Y}}(\mathbf{y}) = 2 \left| \mathbf{J}^{-1} \right|^{-1} f_{\mathbf{X}}(\mathbf{x}),$$

*where we use $\mathbf{J}^{-1}$ instead of $\mathbf{J}$ because it is algebraically more convenient. We have*

$$\mathbf{J}^{-1} = \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \cdots & \partial y_1/\partial x_n \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 & \cdots & \partial y_2/\partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial y_n/\partial x_1 & \partial y_n/\partial x_2 & \cdots & \partial y_n/\partial x_n \end{bmatrix} = \begin{bmatrix} 2x_1 & 2x_2 & \cdots & 2x_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

*with determinant $|\mathbf{J}^{-1}| = 2x_1 = 2\sqrt{Y_1 - \sum_{i=2}^{n} Y_i^2}$. Then, defining*

$$D = y_1 - \sum_{i=2}^{n} y_i^2$$

*for convenience, $\left|\mathbf{J}^{-1}\right|^{-1} = D^{-1/2}/2$ and*

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{y}) &= 2 \cdot \frac{1}{2} D^{-1/2} \frac{1}{(2\pi)^{n/2}} \exp\left\{ -\frac{1}{2} \left( D + \sum_{i=2}^{n} y_i^2 \right) \right\} \mathbb{I}_{\mathcal{S}_{\mathbf{Y}}}(\mathbf{y}) \\ &= D^{-1/2} \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}y_1} \mathbb{I}_{\mathcal{S}_{\mathbf{Y}}}(\mathbf{y}). \end{aligned} \tag{6.47}$$

*(To check, with $n = 2$ and no indicator functions, this reduces to*

$$f_{Y_1,Y_2}(y_1, y_2) = \left(y_1 - y_2^2\right)^{-1/2} \frac{1}{2\pi} e^{-\frac{1}{2}y_1},$$

*which agrees with the direct derivation above.) Inserting $D$ into (6.47) and setting up the integral,*

$$f_{Y_1}(y_1) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}y_1} \int \cdots \int_{\mathcal{S}} \left(y_1 - \sum_{i=2}^{n} y_i^2\right)^{-\frac{1}{2}} dy_2 \cdots dy_n, \tag{6.48}$$

*where*

$$\mathcal{S} = \left\{ (y_2, \ldots, y_n) \in \mathbb{R}^{n-1} : 0 < \sum_{i=2}^{n} y_i^2 < y_1 \right\}.$$

*We wish to apply Liouville's result to the integral*

$$I = \int \cdots \int_{\mathcal{S}} \left(y_1 - \sum_{i=2}^{n} y_i^2\right)^{-\frac{1}{2}} dy_2 \cdots dy_n,$$

*which we rewrite as*

$$I = \int \cdots \int_{\mathcal{S}} \left(y_0 - \sum_{i=1}^{m} y_i^2\right)^{-\frac{1}{2}} dy_1 \cdots dy_m,$$

*where*

$$\mathcal{S} = \left\{ (y_1, \ldots, y_m) \in \mathbb{R}^{m} : 0 < \sum_{i=1}^{m} y_i^2 < y_0 \right\}$$

*and $m = n - 1$. This is almost in the form of (9.13) when taking $p_i = 2$ and $a_i = b_i = 1, i = 1, \ldots, m$, (so that $r_i = 1/2$ and $R = m/2$) as well as $t_1 = 0, t_2 = y_0$ and $f(u) = (y_0 - u)^{-1/2}$.*

*The problem is that the condition $x_i \geq 0$ in (6.46) is not fulfilled. However, what we can compute is*

$$I' = \int \cdots \int_{\mathcal{S}'} \left(y_0 - \sum_{i=1}^{m} y_i^2\right)^{-\frac{1}{2}} dy_1 \cdots dy_m,$$

*where*

$$\mathcal{S}' = \left\{ (y_1, \ldots, y_m) \in \mathbb{R}_+^{m} : 0 < \sum_{i=1}^{m} y_i^2 < y_0 \right\},$$

*i.e., each $y_i$ is restricted to be positive. Then, via the symmetry of the standard normal distribution about zero and the fact that each $y_i$ enters the function $f$ as $y_i^2$, we see that $I = 2^m I'$. Now, using (6.46),*

$$I = 2^m I' = 2^m \frac{(1/2)^m \pi^{m/2}}{\Gamma(m/2)} \int_0^{y_0} u^{m/2-1} (y_0 - u)^{-1/2} \, du = \frac{\pi^{m/2}}{\Gamma(m/2)} J,$$

*where the integral $J$ was shown above to be*

$$J = y_0^{(m-1)/2} \frac{\Gamma(1/2)\Gamma(m/2)}{\Gamma((m+1)/2)}.$$

*Then, recalling that* $m = n - 1$,

$$I = \frac{\pi^{m/2}}{\Gamma(m/2)} y_0^{(m-1)/2} \frac{\Gamma(1/2)\Gamma(m/2)}{\Gamma((m+1)/2)} = \frac{\pi^{n/2}}{\Gamma(n/2)} y_0^{n/2-1}.$$

*Renaming* $y_0$ *back to* $y_1$, (6.48) *gives*

$$f_{Y_1}(y_1) = \frac{e^{-y_1/2}}{(2\pi)^{n/2}} \frac{\pi^{n/2}}{\Gamma(n/2)} y_1^{n/2-1} = \frac{1}{2^{n/2}\Gamma(n/2)} e^{-y_1/2} y_1^{n/2-1},$$

*which is the* $\chi^2_n$ *density.* ■

Our last example shows how a symbolic computing package (Maple) can be used to assist in some tedious but rudimentary calculations to expedite obtaining the solution.

**Example 6.27** *Let* $Y_i \overset{\text{ind}}{\sim} \chi^2(d_i)$, $i = 1, \ldots, n$, *and define*

$$S_k = \frac{\sum_{i=1}^k Y_i}{\sum_{i=1}^k d_i}, \quad k = 1, \ldots, n-1, \quad \text{and} \quad F_k = \frac{Y_k/d_k}{S_{k-1}}, \quad k = 2, \ldots, n.$$

*First consider the marginal distribution of* $F_k$. *Recall that a* $\chi^2(d)$ *distribution is just a gamma distribution with shape* $d/2$ *and (inverse) scale 1/2. Thus,* $\sum_{i=1}^k Y_i \sim \chi^2\left(\sum_{i=1}^k d_i\right)$. *Further, recall that an F distribution arises as ratio of independent* $\chi^2$ *r.v.s each divided by their degrees of freedom, so that* $F_k \sim F\left(d_k, \sum_{i=1}^{k-1} d_i\right)$.

*Next, and more challenging, for* $n = 3$, *we wish to show that* $C$, $F_2$ *and* $F_3$ *are independent, where* $C = \sum_{i=1}^n Y_i$. *This is true for general* $n$, *i.e., random variables*

$$\begin{aligned} C &= Y_1 + \cdots Y_n, \quad F_2 = \frac{Y_2/d_2}{Y_1/d_1}, \quad F_3 = \frac{Y_3/d_3}{(Y_1+Y_2)/(d_1+d_2)}, \ldots, \\ F_n &= \frac{Y_n/d_n}{(Y_1+Y_2+\cdots+Y_{n-1})/(d_1+d_2+\cdots+d_{n-1})}, \end{aligned}$$

*are independent.*[37]

*For* $n = 3$, *with Maple's assistance, we find*

$$Y_1 = \frac{C(d_1+d_2)d_1}{T_2T_3}, \quad Y_2 = \frac{CF_2d_2(d_1+d_2)}{T_2T_3}, \quad Y_3 = \frac{CF_3d_3}{T_3},$$

*where* $T_2 = (F_2d_2 + d_1)$ *and* $T_3 = F_3d_3 + d_1 + d_2$, *and*

$$\begin{bmatrix} \frac{\partial Y_1}{\partial C} & \frac{\partial Y_1}{\partial F_2} & \frac{\partial Y_1}{\partial F_3} \\ \frac{\partial Y_2}{\partial C} & \frac{\partial Y_2}{\partial F_2} & \frac{\partial Y_2}{\partial F_3} \\ \frac{\partial Y_3}{\partial C} & \frac{\partial Y_3}{\partial F_2} & \frac{\partial Y_3}{\partial F_3} \end{bmatrix} = \begin{bmatrix} \frac{d_1(d_1+d_2)}{T_3T_2} & -\frac{d_1d_2(d_1+d_2)C}{T_3T_2^2} & -\frac{d_1d_3(d_1+d_2)C}{T_3^2T_2} \\ \frac{F_2d_2(d_1+d_2)}{T_3T_2} & \frac{d_1d_2(d_1+d_2)C}{T_3T_2^2} & -\frac{F_2d_2d_3(d_1+d_2)C}{T_3^2T_2} \\ \frac{d_3F_3}{T_3} & 0 & \frac{d_3(d_1+d_2)C}{T_3^2} \end{bmatrix},$$

[37] The result is mentioned, for example, in Hogg and Tanis (1963, p. 436), who state that "this result is, in essence, well known and its proof, which is a rather easy exercise, is omitted". They use it in the context of sequential, or iterative, testing of the equality of exponential distributions. It is also used by Phillips and McCabe (1983) in the context of testing for structural change in the linear regression model.

*with*

$$\det \mathbf{J} = \frac{d_1 d_2 d_3 (d_1+d_2)^2 C^2}{T_3^3 T_2^2}.$$

*Thus, with* $d = d_1 + d_2 + d_3$ *(and, for simplicity, using* $C, F_2$ *and* $F_3$ *as both the names of the r.v.s and their arguments in the pdf), the joint density* $f_{C,F_2,F_3}(C, F_2, F_3)$ *is given by*

$$\begin{aligned}
&|\det \mathbf{J}| \prod_{i=1}^{3} \frac{1}{2^{d_i/2}\Gamma(d_i/2)} y_i^{d_i/2-1} e^{-y_i/2} \\
=\; & \frac{d_1 d_2 d_3 (d_1+d_2)^2 C^2}{T_3^3 T_2^2} \frac{\left(\frac{C(d_1+d_2)d_1}{T_2T_3}\right)^{d_1/2-1} \left(\frac{CF_2 d_2 (d_1+d_2)}{T_2T_3}\right)^{d_2/2-1} \left(\frac{CF_3 d_3}{T_3}\right)^{d_3/2-1}}{2^{d/2}\Gamma\left(\frac{d_1}{2}\right)\Gamma\left(\frac{d_2}{2}\right)\Gamma\left(\frac{d_3}{2}\right)} \\
& \times \exp\left\{-\frac{1}{2}\left[\frac{C(d_1+d_2)d_1}{T_2T_3} + \frac{CF_2 d_2 (d_1+d_2)}{T_2T_3} + \frac{CF_3 d_3}{T_3}\right]\right\} \\
=\; & \Gamma(d/2) \frac{d_1 d_2 d_3 (d_1+d_2)^2}{T_3^3 T_2^2} \frac{\left(\frac{(d_1+d_2)d_1}{T_2T_3}\right)^{d_1/2-1} \left(\frac{F_2 d_2 (d_1+d_2)}{T_2T_3}\right)^{d_2/2-1} \left(\frac{F_3 d_3}{T_3}\right)^{d_3/2-1}}{\Gamma\left(\frac{d_1}{2}\right)\Gamma\left(\frac{d_2}{2}\right)\Gamma\left(\frac{d_3}{2}\right)} \\
& \times \frac{1}{2^{d/2}\Gamma(d/2)} C^{d/2-1} e^{-C/2}.
\end{aligned} \tag{6.49}$$

*The pdf of* $C$ *has been separated from the joint density in (6.49), showing that* $C \sim \chi^2(d)$ *and that* $C$ *is independent of* $F_2$ *and* $F_3$*. It remains to simplify the joint pdf of* $F_2$ *and* $F_3$*. In this case, we have been told that* $F_2$ *and* $F_3$ *are independent, with* $F_k \sim F\left(d_k, \sum_{i=1}^{k-1} d_i\right)$*, so that we wish to confirm that*

$$\begin{aligned}
&\frac{\Gamma\left(\frac{d_1+d_2+d_3}{2}\right) \frac{d_3}{d_1+d_2}}{\Gamma\left(\frac{d_3}{2}\right)\Gamma\left(\frac{d_1+d_2}{2}\right)} \frac{\left(\frac{d_3}{d_1+d_2}F_3\right)^{d_3/2-1}}{\left(1+\frac{d_3}{d_1+d_2}F_3\right)^{(d_1+d_2+d_3)/2}} \times \frac{\Gamma\left(\frac{d_1+d_2}{2}\right)\left(\frac{d_2}{d_1}\right)}{\Gamma\left(\frac{d_2}{2}\right)\Gamma\left(\frac{d_1}{2}\right)} \frac{\left(\frac{d_2}{d_1}F_2\right)^{d_2/2-1}}{\left(1+\frac{d_2}{d_1}F_2\right)^{(d_1+d_2)/2}} \\
\stackrel{?}{=}\; & \Gamma(d/2) \frac{d_1 d_2 d_3 (d_1+d_2)^2}{T_3^3 T_2^2} \frac{\left(\frac{(d_1+d_2)d_1}{T_2T_3}\right)^{d_1/2-1} \left(\frac{F_2 d_2 (d_1+d_2)}{T_2T_3}\right)^{d_2/2-1} \left(\frac{F_3 d_3}{T_3}\right)^{d_3/2-1}}{\Gamma\left(\frac{d_1}{2}\right)\Gamma\left(\frac{d_2}{2}\right)\Gamma\left(\frac{d_3}{2}\right)}.
\end{aligned}$$

*The reader should check that, as* $d = d_1 + d_2 + d_3$*, all the gamma terms can be cancelled from both sides. Verifying the equality of the remaining equation just entails simple algebra, and Maple indeed confirms that both sides are equal.* ■

# 7 Appendices

## 7.1 Further Material on the Gamma Function

Recall the two definitions of the gamma function $\Gamma(x)$ given in (1.51) and (1.55), which we repeat here:

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt, \quad x \in \mathbb{R}_{>0}, \tag{7.1}$$

and the Gauss product formula

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)}, \quad x > 0. \tag{7.2}$$

We wish to prove their equivalence. We will require the beta function (1.62), i.e.,

$$B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}dx, \quad a,b \in \mathbb{R}_{>0}, \tag{7.3}$$

and relationship (1.64), namely

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{7.4}$$

We will also require that $e^\lambda = \lim_{n\to\infty}(1+\lambda/n)^n$ from (2.137), and the result, shown in Example 2.65, that sequence $s_n = (1+1/n)^n$ is monotone increasing and bounded (which implies, it converges). The last tool we need is Dini's theorem (2.282) for sequences of functions $f_n : D \to \mathbb{R}$: If (i) the $f_n : D \to \mathbb{R}$ are continuous, (ii) the $f_n$ are monotone, (iii) $D$ is a closed, bounded interval, and (iv) $f_n \to f$ to $f \in \mathcal{C}^0$, then $f_n \rightrightarrows f$ on $D$.

Theorem: Integral expression (7.1) and product formula (7.2) are equivalent.

Proof: (Duren, Invitation to Classical Analysis, 2012, p. 255.) Observe that

$$\begin{aligned}\int_0^1 t^{x-1}(1-t)^n dt = B(x,n+1) &= \frac{\Gamma(x)\Gamma(n+1)}{\Gamma(x+n+1)} \\ &= \frac{n!\Gamma(x)}{(x+n)(x+n-1)\cdots x\Gamma(x)} = \frac{n!}{x(x+1)\ldots(x+n)}.\end{aligned}$$

In the above integral, let $s = tn$, $t = s/n$, and $dt = (1/n)\,ds$, so that

$$\begin{aligned}\frac{n!}{x(x+1)\cdots(x+n)} &= \int_0^1 t^{x-1}(1-t)^n dt = \frac{1}{n}\int_0^n \left(\frac{s}{n}\right)^{x-1}\left(1-\frac{s}{n}\right)^n ds \\ &= \frac{1}{n^x}\int_0^n s^{x-1}\left(1-\frac{s}{n}\right)^n ds,\end{aligned}$$

or, just replacing $s$ with $t$,

$$\frac{n!n^x}{x(x+1)\cdots(x+n)} = \int_0^n t^{x-1}\left(1-\frac{t}{n}\right)^n dt = \int_0^\infty g_n(t)t^{x-1}dt,$$

where

$$g_n(t) = \begin{cases}\left(1-\frac{t}{n}\right)^n, & \text{for } 0 \le t \le n \\ 0, & \text{for } n < t < \infty.\end{cases}$$

Because $g_n(t)$ increases to the limit $e^{-t}$ as $n \to \infty$, Dini's theorem ensures that the integrals converge and

$$\lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)} = \int_0^\infty e^{-t}t^{x-1}dt = \Gamma(x).$$

Recall that the gamma function provides a generalisation of the factorial function, i.e., from (1.53), for $n \in \mathbb{N}$, $\Gamma(n+1) = n!$. We wish to show that, if some function $f$ satisfies the functional equation $f(1) = 1$ and $f(x+1) = xf(x)$ for $x > 0$, then in fact $f(x) = \Gamma(x)$, for $x > 0$. We will also show some related results for the beta function.

This material comes from Binmore, Mathematical Analysis: A Straightforward Approach, 2nd ed., 1982, predominantly §17.4-17.8. The desired result is shown below, after having proven some preliminary results.

Proposition (Binmore #12.21(6)): Let $f$ be continuous on an interval $I$ and satisfy

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}, \quad \forall x, y \in I.$$

Prove that, for any $x_1, x_2, \ldots, x_n$ in the interval $I$,

$$f\left(\frac{x_1+x_2+\ldots+x_n}{n}\right) \leqslant \frac{1}{n}\left\{f\left(x_1\right)+f\left(x_2\right)+\ldots+f\left(x_n\right)\right\}. \tag{7.5}$$

Proof: Let $P(n)$ be the assertion that

$$f\left(\frac{x_1+x_2+\ldots+x_n}{n}\right) \leqslant \frac{1}{n}\left\{f\left(x_1\right)+\cdots+f\left(x_n\right)\right\} \tag{7.6}$$

for any $x_1, x_2, \ldots, x_n$ in the interval $I$. That $P(2)$ holds is given. We assume that $P\left(2^n\right)$ holds and seek to establish $P\left(2^{n+1}\right)$. Write $m = 2^n$. Then $2m = 2^{n+1}$ and

$$\begin{aligned}
f\left(\frac{x_1+\ldots+x_{2m}}{2m}\right) &= f\left(\frac{1}{2}\left(\frac{1}{m}\left(x_1+\ldots+x_m\right)+\frac{1}{m}\left(x_{m+1}+\ldots+x_{2m}\right)\right\}\right) \\
&\leqslant \frac{1}{2}f\left(\frac{x_1+\ldots+x_m}{m}\right)+\frac{1}{2}f\left(\frac{x_{m+1}+\ldots+x_{2m}}{m}\right) \\
&\leqslant \frac{1}{2m}\left\{f\left(x_1\right)+\ldots+f\left(x_m\right)\right\}+\frac{1}{2m}\left\{f\left(x_{m+1}\right)+\ldots+f\left(x_{2m}\right)\right\} \\
&= \frac{1}{2m}\left\{f\left(x_1\right)+\ldots+f\left(x_{2m}\right)\right\}.
\end{aligned}$$

We now assume that $P(n)$ holds and seek to deduce that $P(n-1)$ holds. Write

$$X = \frac{x_1+x_2+\ldots+x_{n-1}}{n-1}$$

Then

$$\begin{aligned}
f\left(\frac{x_1+x_2+\ldots+x_{n-1}}{n-1}\right) = f(X) &= f\left(\frac{(n-1)X+X}{n}\right) \\
&= f\left(\frac{x_1+x_2+\ldots+x_{n-1}+X}{n}\right) \\
&\leqslant \frac{1}{n}\left\{f\left(x_1\right)+\ldots+f\left(x_{n-1}\right)+f(X)\right\}.
\end{aligned}$$

Hence

$$\left(1-\frac{1}{n}\right) f(X) \leqslant \frac{1}{n}\left\{f\left(x_1\right)+f\left(x_2\right)+\ldots+f\left(x_{n-1}\right)\right\}$$

and $P(n-1)$ follows.

Let $\alpha$ be a rational number satisfying $0<\alpha<1$. We may write $\alpha=m/n$. If $\beta=1-\alpha$, then $\beta=(n-m)/n$. Apply inequality (7.6) with $x_1=x_2=\ldots=x_m=x$ and $x_{m+1}=x_{m+2}=\ldots=x_n=y$. Then

$$f(\alpha x+\beta y) \leqslant \alpha f(x)+\beta f(y). \tag{7.7}$$

If $\alpha$ is irrational, consider a sequence $\langle\alpha_n\rangle$ of rational numbers such that $\alpha_n \rightarrow \alpha$ as $n \rightarrow \infty$. Then $\beta_n=1-\alpha_n \rightarrow 1-\alpha=\beta$ as $n \rightarrow \infty$. We have

$$f\left(\alpha_n x+\beta_n y\right) \leqslant \alpha_n f(x)+\beta_n f(y) \quad(n=1,2, \ldots).$$

Since $f$ is continuous, consideration of the limit shows that (7.7) holds even when $\alpha$ is irrational.

Proposition (Binmore, §17.5, p. 159, #4): Show that $\Gamma(x+1)=x\Gamma(x)(x>0)$. Deduce that, for $n=1,2,3,\ldots$,

$$\Gamma(n+1)=n!. \tag{7.8}$$

Proof: Consider, for $0<\delta<\Delta$, the value of

$$\begin{aligned}
\int_\delta^\Delta t^{x-1} e^{-t} d t & =\left[\frac{t^x}{x} e^{-t}\right]_\delta^\Delta+\int_\delta^\Delta \frac{t^x}{x} e^{-t} d t \\
& =\frac{1}{x}\left\{\Delta^x e^{-\Delta}-\delta^x e^{-\delta}\right\}+\frac{1}{x} \int_\delta^\Delta t^x e^{-t} d t .
\end{aligned}$$

Observe that $\Delta^x e^{-\Delta} \rightarrow 0$ as $\Delta \rightarrow+\infty$ (exponentials drown powers) and $\delta^x e^{-\delta} \rightarrow 0$ as $\delta \rightarrow 0+$. It follows that

$$\Gamma(x)=\frac{1}{x} \Gamma(x+1),$$

as required.

Proposition (Binmore, §17.5, p. 159, #5): Prove that the gamma function is continuous on $(0, \infty)$. [Hint: If $0<\alpha<a \leq x \leqslant y \leq b<\beta$, prove that, for some constant $H$ which does not depend on $x$ or $y$, $|\Gamma(x)-\Gamma(y)| \leq H|x-y|\{\Gamma(\alpha)+\Gamma(\beta)\}$.]

Proof: Let $0<\alpha<a \leq x \leq y \leqslant b<\beta$ and let $0<\delta<\Delta$. Then

$$\left|\int_\delta^\Delta t^{x-1} e^{-t} d t-\int_\delta^\Delta t^{y-1} e^{-t} d t\right| \leq \int_\delta^\Delta\left|t^{x-1}-t^{y-1}\right| e^{-t} d t. \tag{7.9}$$

The Mean Value Theorem implies

$$\exists \xi \in(x, y) \quad \text{such that} \quad \frac{t^{x-1}-t^{y-1}}{x-y}=(\log t) t^{\xi-1}. \tag{7.10}$$

For any $r>0, t^{-r} \log t \rightarrow 0$ as $t \rightarrow+\infty$ and $t^r \log t \rightarrow 0$ as $t \rightarrow 0+$. It follows from (7.10) that we can find an $H$ such that

$$\left|t^{x-1}-t^{y-1}\right| \leqslant H\left\{t^{\alpha-1}+t^{\beta-1}\right\}|x-y|$$

and hence it follows from (7.9) that

$$|\Gamma(x)-\Gamma(y)| \leqslant H|x-y|\{\Gamma(\alpha)+\Gamma(\beta)\}$$

and the continuity of the gamma function then follows from the sandwich theorem.

Proposition (Binmore, §17.5, p. 159, #6): Prove that the logarithm of the gamma function is convex on $(0,\infty)$.

Proof: We show that $\log\Gamma\left(\frac{1}{2}x+\frac{1}{2}y\right) \leqslant \frac{1}{2}\log\Gamma(x)+\frac{1}{2}\log\Gamma(y)$ and appeal to (7.5). By the Schwarz (Cauchy-Schwarz, Bunyakovsky-Schwarz) inequality (2.170), if $0<\delta<\Delta$, then

$$\begin{aligned}\left\{\int_\delta^\Delta t^{(x+y-2)/2}e^{-t}dt\right\}^2 &= \left\{\int_\delta^\Delta \left(t^{(x-1)/2}e^{-t/2}\right)\left(t^{(y-1)/2}e^{-t/2}\right)dt\right\}^2\\ &\leqslant \left\{\int_\delta^\Delta t^{x-1}e^{-t}dt\right\}\left\{\int_\delta^\Delta t^{y-1}e^{-t}dt\right\}.\end{aligned}$$

Thus

$$\left\{\Gamma\left(\frac{x+y}{2}\right)\right\}^2 \leqslant \{\Gamma(x)\}\{\Gamma(y)\},$$

and the result follows.

We are now in a position to prove the uniqueness result of the gamma function stated at the beginning.

Proposition (Binmore, p. 160): Let $f$ be positive and continuous on $(0,\infty)$ and let its logarithm be convex on $(0,\infty)$. If $f$ satisfies the functional equation $f(1)=1$ and $f(x+1)=xf(x)$ for $x>0$, then $f(x)=\Gamma(x)$, for $x>0$.

Proof: The proof consists of showing that, under the hypotheses of the theorem, for each $x>0$,

$$f(x)=\lim_{n\to\infty}\frac{n^x n!}{x(x+1)\dots(x+n)}. \tag{7.11}$$

It follows from exercise 17.5(4, 5 and 6) that the gamma function satisfies the hypotheses of the theorem. Since a sequence can have at most one limit, we can therefore conclude from (7.11) that $f(x)=\Gamma(x)(x>0)$.

The proof of (7.11) uses the convexity of $\log f$. Suppose that $s\leqslant t\leqslant s+1$. Then we may write $t=\alpha s+\beta(s+1)$ where $\alpha\geqslant 0,\beta\geqslant 0$ and $\alpha+\beta=1$. Now $t=(\alpha+\beta)s+\beta=s+\beta$ and so $\beta=t-s$. From the convexity of $\log f$, it follows that

$$\begin{aligned}\log f(t) &\leq \alpha\log f(s)+\beta\log f(s+1)\\ f(t) &\leq \{f(s)\}^\alpha\{f(s+1)\}^\beta\\ &= \{f(s)\}^\alpha\{sf(s)\}^\beta\\ &= s^\beta f(s)=s^{t-s}f(s).\end{aligned} \tag{7.12}$$

Since $s\leqslant t\leqslant s+1$, we also have $t-1\leq s\leq t$. Making appropriate substitutions in (7.12), we obtain

$$f(s)\leqslant(t-1)^{s-t+1}f(t-1)=(t-1)^{s-t}f(t). \tag{7.13}$$

Combining (7.12) and (7.13) yields the inequality

$$(t-1)^{t-s} f(s) \leqslant f(t) \leqslant s^{t-s} f(s).$$

Now suppose that $0 \leqslant x < 1$ and that $n$ is a natural number. We may take $s = n+1$ and $t = x+n+1$. Then

$$(x+n)^x f(n+1) \leqslant f(x+n+1) \leqslant (n+1)^x f(n+1). \tag{7.14}$$

From this inequality it follows that

$$(x+n)^x n! \leqslant (x+n)(x+n-1)\dots x f(x) \leqslant (n+1)^x n!$$

or

$$\left(1+\frac{x}{n}\right)^x \leqslant \frac{(x+n)(x+n-1)\dots x f(x)}{n^x n!} \leqslant \left(1+\frac{1}{n}\right)^x.$$

This completes the proof of the formula (7.11) in the case when $0 < x \leq 1$. The general case is easily deduced with the help of the functional equation $f(x+1) = xf(x)$.

We now present some further results related to the gamma function, as well as some for the beta function.

Proposition (Binmore #17.8(1)): If $x > 0$, prove that

$$\Gamma(x) = \int_0^1 \left\{\log \frac{1}{t}\right\}^{x-1} dt$$

Proof: Make the change of variable $-t = \log u$ in the integral

$$\int_\delta^\Delta t^{x-1} e^{-t} dt.$$

Proposition (Binmore #17.8(3)): Use L'Hôpital's rule to show that

$$\lim_{z\to 0}\left\{\frac{\log(1+z)-z}{z^2}\right\} = -\frac{1}{2}.$$

Proof: We have

$$\begin{aligned}\lim_{z\to 0}\left\{\frac{\log(1+z)-z}{z^2}\right\} &= \lim_{z\to 0}\left\{\frac{(1+z)^{-1}-1}{2z}\right\}\\ &= \lim_{z\to 0}\left\{\frac{-(1+z)^{-2}}{2}\right\} = -\frac{1}{2}.\end{aligned}$$

Proposition (Binmore #17.8(4)): For the beta function $B(x, y)$ in (1.62) given by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt,$$

confirm that the improper integral exists provided that $x > 0$ and $y > 0$. Prove that, for a given fixed value of $y > 0, B(x, y)$ is a positive, continuous function of $x$ on $(0, \infty)$ whose logarithm is convex on $(0, \infty)$.

Proof: We use the Comparison Test for Improper Integrals, (2.232), to check that the improper integral exists for $x > 0$ and $y > 0$. The inequalities

$$t^{x-1}(1-t)^{y-1} < t^{x-1} \quad (0 < t < 1)$$
$$t^{x-1}(1-t)^{y-1} < (1-t)^{y-1} \quad (0 < t < 1)$$

suffice for this purpose. The fact that $B(x, y)$ is a continuous function of $x$ (and of $y$) is proved in the same manner as the above proposition (§17.5, #5). The fact that its logarithm is convex is proved in the same manner as the above proposition (§17.5, #6).

## 7.2 The Digamma and Polygamma Functions

This material comes mostly from the math appendix in Paolella, Fundamental Probability: A Computational Approach.

The digamma function is given by

$$\psi(s) := \frac{d}{ds} \ln \Gamma(s) = \frac{\Gamma'(s)}{\Gamma(s)} = \int_0^\infty \left[ \frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}} \right] dt, \tag{7.15}$$

with the latter, well-known integral representation proven in, e.g., Andrews, Askey and Roy (1999, p. 26). Higher-order derivatives are denoted as

$$\psi^{(n)}(s) = \frac{d^n}{ds^n}\psi(s) = \frac{d^{n+1}}{ds^{n+1}} \ln \Gamma(s) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-st}}{1-e^{-t}} dt, \quad n \in \mathbb{N},$$

also known as the polygamma function. Numeric methods exist for their evaluation; see Abramowitz and Stegun (1972, Section 6.3).

A result of general interest (and required, e.g., to compute the variance of a Gumbel random variable) is that

$$\psi'(1) = \int_0^\infty \frac{te^{-t}}{1-e^{-t}} dt = \frac{\pi^2}{6};$$

see, e.g., Andrews, Askey and Roy (1999, p. 51 and 55) for proof.

**Example 7.1** *We wish to determine the expectation of* $\ln X$, *where* $X$ *is a chi-squared random variable, i.e., compute* $\mathbb{E}[\ln X]$ *when* $X \sim \chi^2_v$. *We could try to directly integrate based on the definition of expectation, i.e.,*

$$\mathbb{E}[\ln X] = \frac{1}{2^{v/2}\Gamma(v/2)} \int_0^\infty (\ln x) x^{v/2-1} e^{-x/2} dx, \tag{7.16}$$

*but this seems to lead nowhere. Note instead that the m.g.f. of* $Z = \ln X$ *is*

$$\mathbb{M}_Z(t) = \mathbb{E}\left[e^{tZ}\right] = \mathbb{E}\left[X^t\right] = \frac{1}{2^{v/2}\Gamma(v/2)} \int_0^\infty x^{t+v/2-1} e^{-x/2} dx,$$

*or, with* $y = x/2$,

$$\mathbb{M}_Z(t) = \frac{2^{t+v/2-1+1}}{2^{v/2}\Gamma(v/2)} \int_0^\infty y^{t+v/2-1} e^{-y} dy = 2^t \frac{\Gamma(t+v/2)}{\Gamma(v/2)}.$$

*Then, with $d2^t/dt = 2^t \ln 2$ (see Example 2.18),*

$$\frac{d}{dt}\mathbb{M}_Z(t) = \frac{1}{\Gamma(v/2)}\left(2^t\Gamma'(t+v/2) + 2^t \ln 2\Gamma(t+v/2)\right)$$

*and*

$$\mathbb{E}[\ln X] = \frac{d}{dt}\mathbb{M}_Z(t)\bigg|_{t=0} = \frac{\Gamma'(v/2)}{\Gamma(v/2)} + \ln 2 = \psi(v/2) + \ln 2.$$

*Having seen the answer, the integral (7.16) is easy; differentiating $\Gamma(v/2)$ with respect to $v/2$, using (2.129), and setting $y = 2x$,*

$$\begin{aligned}
\Gamma'\left(\frac{v}{2}\right) &= \int_0^\infty \frac{d}{d(v/2)} x^{v/2-1}e^{-x}dx = \int_0^\infty x^{v/2-1}(\ln x)e^{-x}dx \\
&= \int_0^\infty \left(\frac{y}{2}\right)^{v/2-1}\left(\ln\frac{y}{2}\right)e^{-y/2}\frac{dy}{2} \\
&= \frac{1}{2^{v/2}}\int_0^\infty y^{v/2-1}(\ln y)e^{-y/2}dy - \frac{\ln 2}{2^{v/2}}\int_0^\infty y^{v/2-1}e^{-y/2}dy \\
&= \Gamma(v/2)\mathbb{E}[\ln X] - (\ln 2)\Gamma(v/2),
\end{aligned}$$

*giving $\mathbb{E}[\ln X] = \Gamma'(v/2)/\Gamma(v/2) + \ln 2$.* ■

**Example 7.2** *Computation of the polygamma function is of course available in numerical and symbolic algebra computer packages such as Matlab and Maple. Still, it is of interest to know how one could compute them without the use of these optimized routines. We need only consider $s \in [1,2]$, because, for $s$ outside $[1,2]$, the recursion (Abramowitz and Stegun, 1972, eqs 6.3.5 and 6.4.6)*

$$\psi^{(n)}(s+1) = \psi^{(n)}(s) + (-1)^n n! s^{-(n+1)} \tag{7.17}$$

*can be used. We begin with the expansions*

$$\psi(s) = -\gamma + (s-1)\sum_{k=1}^{\infty}\frac{1}{k(k+s-1)}, \quad s \neq 0, -1, -2, \ldots, \tag{7.18}$$

*and, for $n \in \mathbb{N}$,*

$$\psi^{(n)}(s) = (-1)^{n+1} n! \sum_{k=0}^{\infty}(s+k)^{-(n+1)}, \quad s \neq 0, -1, -2, \tag{7.19}$$

*which can be found in, e.g., Abramowitz and Stegun (1972, eqs 6.3.16 and 6.4.10), where $\gamma$ is Euler's constant,*

*We will truncate these infinite expansions and approximate the tail sum by its continuity corrected integral. For example, for (7.19),*

$$\psi^{(n)}(s) \simeq (-1)^{n+1} n! \left[\sum_{k=0}^{N_n}(s+k)^{-n-1} + \int_{N_n+\frac{1}{2}}^{\infty}\frac{dt}{(s+t)^{n+1}}\right].$$

*This yields, for $1 \leq s \leq 2$ and $n \geq 1$,*

$$\psi^{(n)}(s) \approx (-1)^{n+1}\left[n! \sum_{k=0}^{N_n}(s+k)^{-n-1} + \frac{1}{n}\left(s + N_n + \frac{1}{2}\right)^{-n}\right]. \tag{7.20}$$

*For* (7.18)*, we get*

$$\psi(s) \approx -\gamma + (s-1)\sum_{k=1}^{N_0} k^{-1}(k+s-1)^{-1} + \ln\left|\frac{N_0+s-0.5}{N_0+0.5}\right|. \tag{7.21}$$

*These approximations* (7.20) *and* (7.21) *are derived as follows. For* (7.19)*, the integral is*

$$\int_{N_n+\frac{1}{2}}^{\infty} \frac{dt}{(s+t)^{n+1}} = \int_{s+N_n+\frac{1}{2}}^{\infty} u^{-n-1}du = -\left.\frac{1}{n}u^{-n}\right|_{s+N_n+\frac{1}{2}}^{\infty} = \frac{1}{n}\left(s+N_n+\frac{1}{2}\right)^{-n}.$$

*Similarly, for* (7.18)*, substituting* $u = 1+(s-1)/t$ *leads to*

$$\int_{N_0+\frac{1}{2}}^{\infty} \frac{s-1}{t(t+s-1)}dt = \int_{N_0+\frac{1}{2}}^{\infty} \frac{(s-1)/t^2}{1+(s-1)/t}dt = \ln\left(1+\frac{s-1}{N_0+\frac{1}{2}}\right),$$

*so that*

$$\psi(s) \approx -\gamma + (s-1)\sum_{k=1}^{N_0} k^{-1}(k+s-1)^{-1} + \ln\left|\frac{N_0+s-0.5}{N_0+0.5}\right|.$$

*This technique is of use in general for approximating certain functions based on truncation of infinite expansions.* ■

**Example 7.3** *We wish to relate the digamma function* (7.15) *to the harmonic numbers* $H_n = \sum_{k=1}^n k^{-1}$*, and show that* $\psi(1) = -\gamma$*, where* $\gamma = \lim_{n\to\infty}\left(\sum_1^n 1/n - \log n\right)$ *is the Euler-Mascheroni constant, discussed in Example 2.70.*

*Beginning with the second task, as in https://math.stackexchange.com/questions/4524968, from* (1.55)*,*

$$\Gamma(1+x) = x\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{(1+x)\cdots(n+x)} = \lim_{n\to\infty} n^x \prod_{k=1}^{n} \frac{k}{k+x}.$$

*Thus, for* $|x| < 1$*,*

$$\begin{aligned}
\log\Gamma(1+x) &= \lim_{n\to\infty}\left[x\log n - \sum_{k=1}^{n}\log\left(1+\frac{x}{k}\right)\right] \\
&= \lim_{n\to\infty}\left[x\log n + \sum_{k=1}^{n}\sum_{j=1}^{\infty}\frac{1}{j}\left(-\frac{x}{k}\right)^j\right] \\
&= \lim_{n\to\infty}\left[x\left(\log n - \sum_{k=1}^{n}\frac{1}{k}\right) + \sum_{j=2}^{\infty}\frac{(-x)^j}{j}\sum_{k=1}^{n}\frac{1}{k^j}\right] \\
&= -\gamma x + \sum_{j=2}^{\infty}\frac{(-x)^j}{j}\zeta(j),
\end{aligned}$$

*where the zeta function is given in Example 2.63; and the exchange of limit and infinite sum follows from Tannery's theorem* (2.258)*. Taking the derivative of both sides and evaluating at* $x=0$ *gives the result* $\psi(1) = -\gamma$*.*

*Next, as* $\Gamma(z+1) = z\Gamma(z)$*, taking logs gives* $\ln(\Gamma(z+1)) = \ln(z) + \ln(\Gamma(z))$*. Differentiating both sides with respect to* $z$ *gives*

$$\psi(z+1) = \psi(z) + \frac{1}{z},$$

*from which it follows that* $\psi(n) = H_{n-1} - \gamma$*, where* $H_0 = 0$*.* ■

## 7.3 Banach's Matchbox Problem

This section is not as relevant as the material in the main text, but it is instructive and interesting. It details the well-known "Banach's Matchbox Problem" from probability theory. It also shows an application of Wallis' product.

As the story goes, the mathematician Stefan Banach (1892–1945) kept two match boxes, one in each pocket, each originally containing $N$ matches. Whenever he wanted a match, he randomly chose between the boxes (with equal probability) and took one out. Upon discovering an empty box, what is the probability that the other box contains $K = k$ matches, $k = 0, 1, \ldots, N$? (See Feller, 1968, p. 166).

Assume that he discovers the right hand pocket to be empty (rhpe). Because trials were random, $X$, the number of matches that were drawn from the left, can be thought of as "failures" from a negative binomial-type experiment with $p = 1/2$ (and support only $\{0, 1, \ldots, N\}$ instead of $\{0 \cup \mathbb{N}\}$), where sampling continues until $r = N + 1$ "successes" (draws from the right pocket) occur. Thus $\Pr(K = k \cap \text{rhpe}) = \Pr(X = x \cap \text{rhpe})$, where $X = N - K$, $x = N - k$ and

$$\Pr(X = x \cap \text{rhpe}) = \binom{r + x - 1}{x} p^r (1 - p)^x = \binom{2N - k}{N} \left(\frac{1}{2}\right)^{2N+1-k}.$$

With $\Pr(X = x) = \Pr(X = x \cap \text{rhpe}) + \Pr(X = x \cap \text{lhpe})$ and from the symmetry of the problem,

$$f(k; N) = \Pr(K = k \mid N) = 2 \Pr(X = x \cap \text{rhpe}) = \binom{2N - k}{N} \left(\frac{1}{2}\right)^{2N-k}. \tag{7.22}$$

From (1.31) in Example 1.8, this mass function indeed sums to one.

The pmf (7.22) can also be expressed recursively as

$$\begin{aligned}
\Pr(K = k) &= \binom{2N - k}{N} \left(\frac{1}{2}\right)^{2N-k} \\
&= 2 \frac{N - (k-1)}{2N - (k-1)} \binom{2N - (k-1)}{N} \left(\frac{1}{2}\right)^{2N-(k-1)} \\
&= \frac{N - (k-1)}{N - \frac{k-1}{2}} \Pr(K = k - 1),
\end{aligned} \tag{7.23}$$

from which it is directly seen that $\Pr(K = 0) = \Pr(K = 1)$ and that $\Pr(K = k)$ decreases in $k$, $k \geq 1$. Recursion (7.23) also provides a way of calculating $\Pr(K = k)$ avoiding the computation of the gamma function.

One natural generalization of the original Banach matchbox problem is to allow for different numbers of matches in the left and right pockets, say $N_1$ and $N_2$, and probability $p$ not necessarily 1/2 of drawing from the left side. Derive the mass function $f(k; N_1, N_2, p)$ and construct a computer program, say `banach(n1,n2,p,sim)` that computes it, simulates the process `sim` times, and finally plots the true and simulated mass functions overlaid. As an example, the first panel in Figure 48 was produced with the Matlab code and function

```
vec=banach(30,10,1/2,10000);
text(3,0.07,'N_1=30, N_2=10, p=1/2','fontsize',14)
```

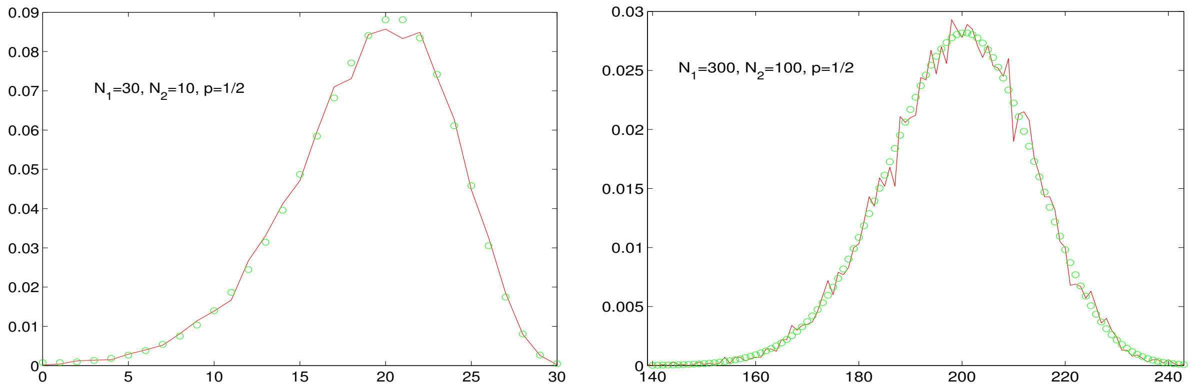



Figure 48: True (circles) and simulated (lines) mass function for the generalized Banach matchbox problem

For the mass function, let $X_i = N_i - K$, $x_i = N_i - k$, $i = 1, 2$, and $N = N_1 + N_2$, giving

$$\begin{aligned} f(k; N_1, N_2, p) &= \Pr(K = k \mid N_1, N_2, p) \\ &= \Pr(X_2 = x \cap \text{ lhpe }) + \Pr(X_1 = x \cap \text{ rhpe }) \\ &= \binom{N-k}{N_1} p^{N_1+1}(1-p)^{N_2-k}\mathbb{I}_{\{0,1,\dots,N_2\}}(k) \\ &+ \binom{N-k}{N_2}(1-p)^{N_2+1}p^{N_1-k}\mathbb{I}_{\{0,1,\dots,N_1\}}(k). \end{aligned}$$

Matlab code for the desired function is given in Listing 3. It imposes, without loss of generality, that $N_1 \geq N_2$.

Further extensions to the matchbox problem could allow for more than two matchboxes; see, for example, Cacoullos (1989, p. 80(317,318)) for some analytic results. Assume that Banach has $n \geq 2$ matchboxes distributed throughout his many pockets, each of which initially contains $N_i > 0$ matches, $i = 1, \dots, n$. Associated with each box is a probability $p_i$, with $\sum_i p_i = 1$.

Listing 4 gives a Matlab program that simulates the process and, once an empty box is discovered, reports the minimum and maximum number of matches in the remaining matchboxes. This can be used to simulate the process, and, for example, plot (approximations to the exact) marginal mass functions, overlaid together on a single plot.

We wish to compute the expected value of the random variable $K$ associated with Banach's matchbox problem. With $N = 1$, it is clear that $\mathbb{E}[K] = 1/2$. For general $N$, determining $\mathbb{E}_N[K]$ will exercise our combinatoric skills. Of use will be (1.29). The answer is

$$\mathbb{E}_N[K] = \binom{N+1/2}{N} - 1, \tag{7.24}$$

as was found by Feller (1957, p. 212) using a different method of derivation than what we show here.

```
function vec=banach(n1,n2,p,sim)
vec=zeros(sim,1);
for i=1:sim
  vec(i)=simul(n1,n2,p); if mod(i,100)==0, i, end
end
tt=tabulate(vec+1); a=tt(:,2); mx=tt(end,1)-1; b=0:mx; a=a./sim;
true=echt(n1,n2,p); plot(b,a,'r-',0:mx,true(1:mx+1),'go')
mn=min(vec); ax=axis; axis([mn mx 0 ax(4)]), set(gca,'fontsize',14)

function echte=echt(n1,n2,p) % true mass function
mx=max(n1,n2); echte=zeros(1,mx); n=n1+n2; d=0:mx;
d1=0:n2; h1=c(n-d1,n1);
k1=p^(n1+1).*(1-p).^(n2-d1); h1=h1.*k1;
d2=0:n1; h2=c(n-d,n2);
k2=(1-p)^(n2+1).*p.^(n1-d2); h2=h2.*k2;
if n1>n2
  g=zeros(1,n1-n2); h1=[h1,g];
else
  g=zeros(1,n2-n1); h2=[h2,g];
end
echte=h1+h2;

function output=simul(n1,n2,p);
ok=1; zaehlerone=0; zaehlertwo=0;
while ok
  y=unifrnd(0,1,1,1);
  if y<p % box 1
    if n1==0, x=n2; ok=0; end
    if n1>0, n1=n1-1; zaehlerone=zaehlerone+1; end
  end
  if y>p
    if n2==0, x=n1; ok=0; end
    if n2>0, n2=n2-1; zaehlertwo=zaehlertwo+1; end
  end
end
output=x;
```

**Program Listing 3:** Code to accomplish the matchbox problem with different numbers of matches in the two pockets.

```
function minmax=banachmultisim(N,p,sim)
if any(p<=0) | any(p>=1) | sum(p)~=1, error('bad p'), end
w=max(N); y=zeros(2,sim);
for i=1:sim
  [s,d]=banachmulti(N,p);
  y(1:2,i)=[s,d]';
end
tt1=tabulate(y(1,1:sim)+1); tt2=tabulate(y(2,1:sim)+1);
a=tt1(:,2); b=tt2(:,2); mx=tt1(end,1)-1; mi=tt2(end,1)-1;
a=a./sim; b=b./sim;
if length(N)==2
  plot(0:mx,a,'r-'), title('Mass Function of Remaining Matches')
else
  plot(0:mx,a,'r-',0:mi,b,'b--')
  title('Marginal Mass Function of Minimum and Maximum Remaining Matches')
end
minmax=y';

function [ma,mi]=banachmulti(x,p);
n=length(x); ok=1; mi=min(x);
while ok==1
  s=0; zaehler=0; r=unifrnd(0,1,1,1);
  for i=1:n, if zaehler==0
      s=s+p(i);
      if r<s, zaehler=i; end
  end, end
  x(zaehler)=x(zaehler)-1; mi=min(x); ma=max(x);
  if mi<1, ok=0; end
end
help=find(x==0); x(help)=max(x)+1; mi=min(x);
```

**Program Listing 4:** Code to accomplish the generalized matchbox problem.

Expectation $\mathbb{E}_N[K]$ is given by

$$\begin{aligned}
&\sum_{j=0}^{N} j \left( \begin{array}{c} 2N-j \\ N \end{array} \right) \left(\frac{1}{2}\right)^{2N-j} = \sum_{k=0}^{N-1} (k+1) \left( \begin{array}{c} 2N-k-1 \\ N \end{array} \right) \left(\frac{1}{2}\right)^{2N-k-1} \\
&= \sum_{k=0}^{N-1} (k+1) \left( \begin{array}{c} 2N-k-2 \\ N-1 \end{array} \right) \left(\frac{1}{2}\right)^{2N-k-1} + \sum_{k=0}^{N-1} (k+1) \left( \begin{array}{c} 2N-k-2 \\ N \end{array} \right) \left(\frac{1}{2}\right)^{2N-k-1} \\
&=: \ G + H.
\end{aligned}$$

Thus, $G$ is given by

$$\begin{aligned}
&\frac{1}{2} \left\{ \sum_{k=0}^{N-1} k \left( \begin{array}{c} 2(N-1)-k \\ N-1 \end{array} \right) \left(\frac{1}{2}\right)^{2(N-1)-k} + \sum_{k=0}^{N-1} \left( \begin{array}{c} 2(N-1)-k \\ N-1 \end{array} \right) \left(\frac{1}{2}\right)^{2(N-1)-k} \right\} \\
&= \frac{1}{2}\mathbb{E}_{N-1}[K] + \frac{1}{2}
\end{aligned}$$

and, with $j = k+2, H$ is

$$\begin{aligned}
&\sum_{j=2}^{N} (j-1) \left( \begin{array}{c} 2N-j \\ N \end{array} \right) \left(\frac{1}{2}\right)^{2N-j+1} \\
&= \frac{1}{2} \left\{ \sum_{j=2}^{N} j \left( \begin{array}{c} 2N-j \\ N \end{array} \right) \left(\frac{1}{2}\right)^{2N-j} - \sum_{j=2}^{N} \left( \begin{array}{c} 2N-j \\ N \end{array} \right) \left(\frac{1}{2}\right)^{2N-j} \right\} \\
&= \frac{1}{2} \left\{ \mathbb{E}_N[K] - \left( \begin{array}{c} 2N-1 \\ N \end{array} \right) \left(\frac{1}{2}\right)^{2N-1} \right\} - \frac{1}{2} \left\{ 1 - \left( \begin{array}{c} 2N \\ N \end{array} \right) \left(\frac{1}{2}\right)^{2N} - \left( \begin{array}{c} 2N-1 \\ N \end{array} \right) \left(\frac{1}{2}\right)^{2N-1} \right\} \\
&= \frac{1}{2}\mathbb{E}_N[K] - \frac{1}{2} + \left( \begin{array}{c} 2N \\ N \end{array} \right) \left(\frac{1}{2}\right)^{2N+1}.
\end{aligned}$$

Simplifying $G + H$ yields the recursion

$$\mathbb{E}_N[K] = \mathbb{E}_{N-1}[K] + \left( \begin{array}{c} 2N \\ N \end{array} \right) \left(\frac{1}{2}\right)^{2N}, \quad \mathbb{E}_1[K] = \frac{1}{2},$$

which resolves to

$$\mathbb{E}_N[K] = \sum_{i=1}^{N} \left( \begin{array}{c} 2i \\ i \end{array} \right) \left(\frac{1}{2}\right)^{2i}.$$

This can be further simplified using several previous results, namely (1.42), (1.39), and (1.33), in that order. We list these three results, in that order, for convenience:

$$\begin{aligned}
\binom{2n}{n} &= (-1)^n \, 2^{2n} \binom{-\frac{1}{2}}{n}. \\
\binom{-n}{k} &= (-1)^k \binom{n+k-1}{k}. \\
\sum_{i=0}^{n} \binom{i+r-1}{i} &= \binom{n+r}{n}.
\end{aligned}$$

We then obtain, noting from (1.38) that $\binom{-1/2}{0} = 1$,

$$\begin{aligned}
\mathbb{E}_N[K] &= \sum_{i=1}^{N} \binom{2i}{i} \left(\frac{1}{2}\right)^{2i} \\
&= \sum_{i=1}^{N} (-1)^i \binom{-\frac{1}{2}}{i} \\
&= \sum_{i=1}^{N} \binom{i-1/2}{i} + 1 - 1 = \sum_{i=0}^{N} \binom{i-1/2}{i} - 1 \\
&= \binom{N+1/2}{N} - 1.
\end{aligned} \tag{7.25}$$

This result can be extended to expressions for all moments of $K$. For example, some more work reveals that

$$\mathbb{V}(K) = [N - \mathbb{E}_N(K)] - \mathbb{E}_N(K)\,[1 + \mathbb{E}_N(K)] \approx \left(2 - \frac{4}{\pi}\right) N - \frac{2}{\sqrt{\pi}}\sqrt{N} + 2$$

and

$$\mathbb{E}_N\left(K^3\right) = 6\binom{N+\frac{3}{2}}{N} + 7\binom{N+\frac{1}{2}}{N} - 12N - 13,$$

with higher moments similarly obtained.[38]

**Example 7.4** *We wish to show that the expectation (7.25) for Banach's matchbox problem in (7.24) can be well approximated by*

$$\mathbb{E}_N[K] \approx -1 + 2\sqrt{N/\pi}.$$

*Let*

$$w_n = \frac{2^n\, n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \approx \sqrt{n\pi}$$

*from (2.266). Then, using the generalized binomial coefficient,*

$$\begin{aligned}
\binom{N+1/2}{N} &= \frac{\left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right)\cdots\frac{3}{2}}{N!} \\
&= \frac{(2N+1)\,(2N-1)\cdots 3}{2^N\, N!} = \frac{2N+1}{w_N} \approx \frac{2N+1}{\sqrt{n\pi}}.
\end{aligned} \tag{7.26}$$

*From Figure 49, we see that this approximation is very good even for modest values of $N$. Finally,*

$$\mathbb{E}_N[K] \approx -1 + \frac{2N+1}{\sqrt{N\pi}} \approx -1 + 2\sqrt{N/\pi},$$

*yielding the desired approximation.* ■

[38] This example appears as Exercise 4.13 in Paolella, Fundamental Probability, and was kindly contributed by my friend and professor colleague Markus Haas.

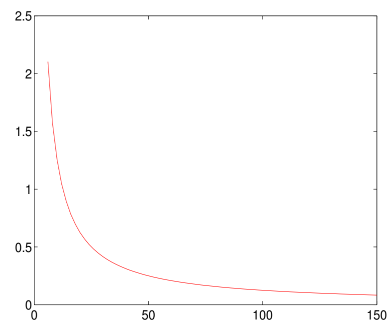



Figure 49: The relative percentage error, $100\,(\text{Approx} - \text{True})\,/\text{True}$, for the approximation in (7.26) as a function of $N$.